ELSEVIER

3-manifolds which are orbit spaces of diffeomorphisms

C. Bonatti[*], L. Paoluzzi

I.M.B., UMR 5584 du CNRS, B.P. 47 870, 21078 Dijon Cedex, France

Abstract

In a very general setting, we show that a 3-manifold obtained as the orbit space of the basin of a topological attractor is either $S^2 \times S^1$ or irreducible.

We then study in more detail the topology of a class of 3-manifolds which are also orbit spaces and arise as invariants of gradient-like diffeomorphisms (in dimension 3). Up to a finite number of exceptions, which we explicitly describe, all these manifolds are Haken and, by changing the diffeomorphism by a finite power, all the Seifert components of the Jaco–Shalen–Johannson decomposition of these manifolds are made into product circle bundles.

© 2007 Elsevier Ltd. All rights reserved.

1. Introduction

Consider an orientation preserving diffeomorphism f of a closed orientable 3-manifold, A a topological attractor of f, and $\mathcal{B}(A)$ its basin. The space of orbits of f in $\mathcal{B}(A) \setminus A$ is a closed orientable manifold V of a special type. Indeed, Theorem 3.3 shows that each connected component of V is either $S^2 \times S^1$ or is an irreducible Haken manifold.

---

[*] Corresponding address: Université de Bourgogne, Institut de Mathematiques de Bourgogne, UMR 5584 du CNRS, 9 Av Alain Savary, B.P.47 870, 21078 Dijon, France. Tel.: +33 3 80 39 58 39; fax: +33 3 80 39 58 69.

E-mail addresses: bonatti@u-bourgogne.fr (C. Bonatti), paoluzzi@u-bourgogne.fr (L. Paoluzzi).

This is a by-product of our main concern which consists in understanding the topology of a class of 3-manifolds appearing in the classification of gradient-like diffeomorphisms, i.e. Morse–Smale diffeomorphisms satisfying some extra requirements.

Dynamical systems of Morse–Smale type are *structurally stable* (i.e. topologically conjugated to any C^1-near diffeomorphism) and their dynamics are extremely simple. It seems thus natural to be able to classify them up to topological conjugation, that is to find a complete invariant which allows to decide whether two systems are conjugate. The classification of Morse–Smale vector fields of surfaces was obtained by Peixoto (see [17]). The classification of Morse–Smale diffeomorphisms of compact surfaces has been the object of several papers, see in particular [6,13], and was finally achieved in [4, 1] where, in fact, the classification of a larger class of surface diffeomorphisms was established, namely C^1-structurally stable diffeomorphisms.

For *gradient-like* vector fields (i.e. Morse–Smale vector fields without regular periodic orbits) of 3-manifolds there is a simple complete invariant for the topological equivalence (see [5] and Section 6.3.1): a compact surface transverse to the flow endowed with two families of simple closed curves induced by the stable and unstable manifolds of dimension 2 of saddle-type singularities. The curves of each family are pairwise disjoint and transverse to the curves of the other family. The classification of Morse–Smale vector fields in dimension 3 with regular periodic orbits is much more complex and is still the object of various papers.

The classification of Morse–Smale diffeomorphisms of compact 3-manifolds presents some new difficulties, and the study was initially restricted to the set of *gradient-like diffeomorphisms*, that is of Morse–Smale diffeomorphisms whose heteroclinic intersections always involve saddles of distinct Morse indices (i.e. dimensions of the unstable manifolds). In a recent article [3], a complete invariant for gradient-like diffeomorphisms of 3-manifolds, called a *global scheme*, was constructed. A global scheme is a 4-tuple $\mathcal{S}(f) = (V_f, \alpha_f, \Gamma_f^s, \Gamma_f^u)$, where V_f is a compact, connected manifold of dimension 3, canonically associated to f, α_f is an indivisible class of integer cohomology in $H^1(V_f, \mathbb{Z})$, and Γ_f^s and Γ_f^u are two families whose elements are tori or Klein bottles embedded in V_f, such that the restriction of α_f to any of these elements is non-zero; moreover, the elements of each family are pairwise disjoint and transverse to the elements of the other family. The 4-tuples $(V, \alpha, \Gamma^s, \Gamma^u)$ verifying these properties are called *abstract schemes*. In [3] a characterisation of the abstract schemes which are global schemes of a gradient-like diffeomorphism is given: these are called *perfect schemes*. The construction of a global scheme, as well as the characterisation of perfect schemes, will be recalled in more detail in Section 2.1.1.

The global scheme of a gradient-like diffeomorphism f is a complicated object; in particular, the manifold V_f may have some *hyperbolic components* and *Seifert fibred components*. However, if the diffeomorphism f is the time-1 of a gradient-like vector field X, the manifold V_f is the product of the circle S^1 times the surface S transverse to X, the class α_f being dual to the surface $S \subset S \times S^1 = V_f$. This intuitively says that the complexity of the topology of V_f reflects how far f from being the time-1 of a vector field. In other terms, we would like to read the dynamic information contained in V_f and in the couple (V_f, α_f).

In this spirit, we study here the topology of V_f and the couple (V_f, α_f). We shall start by showing that (apart from the case where V_f is diffeomorphic to $S^2 \times S^1$), V_f is an *irreducible manifold* (i.e. every embedded sphere bounds a ball), and in this case V_f is Haken.[1] Any such manifold admits a minimal

---

[1] A compact 3-manifold is said to be Haken if it is irreducible and contains an embedded *incompressible* surface, i.e. its fundamental group injects inside that of the ambient manifold.

family of embedded, incompressible, pairwise disjoint tori, which is unique up to isotopy, and which decomposes the manifold in geometric pieces (see [10,11]), called the *Jaco–Shalen–Johannson family*. We shall show that V_f cannot be a *Sol* manifold; thus V_f admits a decomposition into hyperbolic and Seifert fibred components.

Notice that, for each non-zero integer n, f^n is again a gradient-like diffeomorphism; moreover, the manifold V_{f^n} is the n-fold cyclic cover of V_f, associated with the cohomology class α_f, and the cohomology class α_{f^n} is $\frac{1}{n}$ times the lift to V_{f^n} of the class α_f. The geometric components of V_{f^n} are then the lifts of the geometric components of V_f. Our main result is then summarised by the following:

Theorem 1.1. *Let f be an orientation preserving gradient-like diffeomorphism of a compact orientable 3-manifold.*

1. *Either V_f is a circle bundle of non-trivial Euler class over the torus T^2 or the Klein bottle;*
2. *or there exists an integer n such that V_{f^n} verifies the following properties:*
    (a) *the Seifert components of V_{f^n} are trivial bundles and the value of α_{f^n} on the fibres is 1;*
    (b) *the hyperbolic components of V_{f^n} are link complements in $S^2 \times S^1$, such that each component of the link is freely homotopic to the fibre $\{x\} \times S^1$, and α_{f^n} induces on $S^2 \times S^1$ the form dual to the fibre $S^2 \times \{t\}$.*

In Section 6.2 we construct a gradient-like diffeomorphism f such that the associated manifold V_f is a circle bundle on which the form α_f vanishes on the fibre of any fibration of V_f; in particular, case 1 of the above theorem cannot be avoided. We also illustrate with a concrete example (see Section 6.3) the fact that V_f may be a Seifert fibration with singular fibres, which shows that the integer n in part 2 of the theorem need not be 1.

With a finite number of exceptions (see the example constructed in Section 6.1), where the manifold V_f is a Seifert fibred manifold admitting more than one fibration (e.g. if V_f is the 3-torus), the tori of the families Γ^s and Γ^u that are incompressible either belong to the Jaco–Shalen–Johannson family or, up to isotopy, they are vertical (i.e. fibred) tori inside some Seifert component of V_f.

Remark that, if V_f is a trivial circle bundle in which the tori of Γ^s and Γ^u are vertical, then f is conjugate to the time-1 map of the flow of a gradient-like vector field.

In view of the above observations and of Theorem 1.1, the Seifert parts of V_f seem to correspond to domains on which f behaves like the time-1 map of the flow of a gradient-like vector field. In a future work, we shall construct a gradient-like vector field associated with a gradient-like diffeomorphism f using the Seifert components of V_f and we shall show how to recover the diffeomorphism f from the flow by elementary operations.

2. Definitions and precise statements of results

2.1. Gradient-like diffeomorphisms

Let f be a diffeomorphism on a compact manifold M. A periodic point x of period k of f is *hyperbolic* if the differential $Df^k(x): T_x M \to T_x M$ has no eigenvalue of modulus equal to 1. The *stable manifold* of a point x is the set of points y such that $d(f^i(x), f^i(y))$ decreases to 0 when $i \to +\infty$. The stable manifold of a hyperbolic periodic point is a smooth manifold injectively immersed in M and tangent at x to the subspace of $T_x M$ corresponding to the eigenvalues of $Df^k(x)$ with modulus less than 1. In this

text we will denote by $W^s(x)$ the *stable manifold of the orbit of* x, that is, the union of the iterates of the stable manifold of x. The *unstable manifold* $W^u(x)$ is by definition the stable manifold for f^{-1} of the orbit of x.

A point $x \in M$ is *non-wandering* if any neighbourhood U of x admits a positive iterate $f^k(U), k > 0$ such that $f^k(U) \cap U \neq \emptyset$.

A diffeomorphism f on a closed manifold M is called *a Morse–Smale* diffeomorphism if:

1. any non-wandering point x of f is a hyperbolic periodic point,
2. for any pair of periodic points x, y of f, the stable manifold $W^s(x)$ is transverse to the unstable manifold $W^u(y)$, at each intersection point.

Under these hypotheses, the set of periodic orbits of f is finite. For a Morse–Smale diffeomorphism, the relation defined by $\mathrm{orb}(x) \prec \mathrm{orb}(y)$ if $W^u(x) \cap W^s(y) \neq \emptyset$ is an order relation called *Smale's order*. Another characterisation of Smale's order is given by the following property:

Let f be a Morse–Smale diffeomorphism and A and B two sets of periodic points. Then the two following properties are equivalent:

- For any $x \in A$ and $y \in B$, one has $\mathrm{orb}(y) \nprec \mathrm{orb}(x)$.
- There is an open set $U \subset M$ such that $f(\bar{U}) \subset U$ and $B \subset U$ and $A \cap \bar{U} = \emptyset$.

A Morse–Smale diffeomorphism f is called *gradient-like* if for each pair of periodic points $x \neq y$ one has

$$\mathrm{orb}(x) \prec \mathrm{orb}(y) \implies \dim(W^u(x)) > \dim(W^u(y)).$$

For any hyperbolic periodic point x, the dimension $\dim(W^u(x))$ is called the *Morse index* of x.

As a consequence of this definition and of the property above, if f is a gradient-like diffeomorphism of a compact 3-manifold, then there is an open set U with $f(\bar{U}) \subset U$ such that U contains all the sinks and all the saddles of index 1 and $M \setminus \bar{U}$ contains all the sources and all the saddles of index 2. Furthermore, U may be chosen to be bounded by a smooth compact surface.

2.1.1. The global scheme of a gradient-like diffeomorphism

In [3] a complete topological invariant for gradient-like diffeomorphisms of 3-manifolds, analogous to the invariant for vector fields, was constructed. In the following we shall briefly recall the construction of the invariant. The reader is referred to [3] for a more complete explanation.

Let $f: M \to M$ be an orientation preserving gradient-like diffeomorphism of a closed oriented 3-manifold. Smale's theory (as recalled in the previous section) implies the existence of a surface S which cuts M into two manifolds with boundary, M_1 and M_2, such that $f(M_1)$ is included in the interior of M_1. Furthermore M_1 contains all the sinks and the set Σ_1 of all the saddles of Morse index (i.e. dimension of the unstable manifold) 1. In a similar way M_2 contains all the sources and the set Σ_2 of all the saddles of index 2.

For all $x \in M$, the orbit of x meets $M_1 \setminus f(M_1)$ in at most one point, and precisely one point if x is neither a periodic point nor a point on a one-dimensional invariant manifold of a saddle point (i.e. $x \notin \mathrm{Per}(f) \cup \bigcup_{y \in \Sigma_2} W^s(y) \cup \bigcup_{z \in \Sigma_1} W^u(z)$).

Let V_f be the space of orbits of f restricted to $M \setminus (\mathrm{Per}(f) \cup \bigcup_{y \in \Sigma_2} W^s(y) \cup \bigcup_{z \in \Sigma_1} W^u(z))$, i.e. to the complement of the union of the one-dimensional stable and unstable manifolds of the periodic orbits. Thanks to the above remark it is possible to see that the space V_f is canonically identified to

the manifold obtained by gluing $M_1 \setminus \text{Int}(f(M_1))$ via $f \colon S \to f(S)$. From this follows that V_f is a three-dimensional closed manifold; furthermore V_f is connected for $(\text{Per}(f) \cup \bigcup_{y \in \Sigma_2} W^s(y) \cup \bigcup_{z \in \Sigma_1} W^u(z))$ is of dimension 1 and does not disconnect M.

Remark that $M \setminus (\text{Per}(f) \cup \bigcup_{y \in \Sigma_2} W^s(y) \cup \bigcup_{z \in \Sigma_1} W^u(z))$ is the infinite cyclic cover of V_f whose group of deck transformations (automorphisms of the covering) is generated by f. This defines in a natural way a surjective morphism from $\pi_1(V_f)$ onto $\mathbb{Z}$, that is an integer indivisible cohomology class $\alpha \in H^1(V_f, \mathbb{Z})$.

For each index 2 saddle, $W^u(x)$ induces inside V_f either a torus or a Klein bottle, denoted as γ_x^u, according to whether or not the restriction of f to $W^u(x)$ preserves the orientation. We shall write Γ^u for the set of γ_x^u's, $x \in \Sigma_2$.

In a similar fashion, we shall write Γ^s for the set of tori or Klein bottles embedded in V_f, and induced by the stable manifolds of the periodic points of index 1.

It is easy to see that the cohomology class induced by α on each element of $\Gamma^s \cup \Gamma^u$ is non-zero.

Definition 2.1. A *formal scheme* is a 4-tuple $(V, \alpha, \Gamma^s, \Gamma^u)$ where:

1. V is a closed oriented 3-manifold,
2. $\alpha \in H^1(V, \mathbb{Z})$ is a non-trivial indivisible integer cohomology class,
3. Γ^s and Γ^u are two families of tori or Klein bottles embedded in V, on which the restriction of α is non-zero,
4. the elements of Γ^s (respectively Γ^u) are pairwise disjoint,
5. the elements of Γ^s are transverse to those of Γ^u.

Two formal schemes are said to be *equivalent* if there exists a homeomorphism from the manifold of the first to that of the second such that the image of the cohomology class and of the families of tori or Klein bottles of the first are those of the second.

For each gradient-like diffeomorphism f the equivalence class $\mathcal{S}_f$ of $(V_f, \alpha, \Gamma^s, \Gamma^u)$ is well defined (independent of choices) and is called the *global scheme* of f. In [3] the following result is proved:

Theorem 2.1. *Two gradient-like diffeomorphisms of* 3-*manifolds f and g are conjugate if and only if* $\mathcal{S}_f = \mathcal{S}_g$.

In [3], a surgery on V is defined for a formal scheme $\mathcal{S}$ as follows. For each element γ of Γ^s the class α has a non-trivial kernel which defines a notion of meridian on $\pi_1(\gamma)$. The surgery consists in cutting V along each element of Γ^s and gluing solid tori on each boundary component, in such a way that the meridian of the solid tori are identified with the meridians of the boundary components. The resulting manifold is denoted as $CS(V, \Gamma^s)$.

$CS(V, \Gamma^u)$ is defined in an analogous fashion. In [3] the following is proved:

Theorem 2.2. *A formal scheme $\mathcal{S}$ is the global scheme of a gradient-like diffeomorphism if and only if each component of $CS(V, \Gamma^s) \cup CS(V, \Gamma^u)$ is diffeomorphic to $S^2 \times S^1$.*

This results motivates the following definition:

Definition 2.2. A formal scheme $\mathcal{S} = (V, \alpha, \Gamma^s, \Gamma^u)$ is a *perfect scheme* if each component of $CS(V, \Gamma^s) \cup CS(V, \Gamma^u)$ is diffeomorphic to $S^2 \times S^1$.

2.2. Standard definitions and results in 3-manifold topology

A compact orientable 3-manifold V without boundary is called *irreducible* if any embedded sphere in V bounds a ball embedded in V.

A closed orientable surface of genus greater than 0 embedded in an orientable compact 3-manifold is *incompressible* if its fundamental group injects in the fundamental group of the ambient manifold.

A closed orientable manifold is called *Haken* if it is irreducible and if it contains an incompressible surface.

A compact 3-manifold (possibly with boundary) is *Seifert fibred* if it is a union of disjoint circles (called *fibres*) such that every fibre admits a neighbourhood which is a fibred solid torus whose fibration is given by the mapping torus (suspension for the dynamicists) of a rational rotation of the disc. The fibre is called *singular* if the rotation is not the identity.

A compact 3-manifold is called hyperbolic if its interior admits a complete Riemannian metric with sectional curvature equal to -1.

Every irreducible orientable 3-manifold V admits a minimal family (called the Jaco–Shalen–Johannson family) of embedded, incompressible, pairwise disjoint tori which is unique up to isotopy and which decomposes the manifold into geometric pieces. Here by geometric pieces we mean either Seifert fibred or atoroidal pieces (a manifold with boundary is called *atoroidal* if any incompressible torus is parallel to some component of the boundary) (see [10,11]). From now on we shall denote by JSJ(V) the Jaco–Shalen–Johannson family of V.

The reader is referred to [10,11] for the construction and the basic properties of this family.

As a consequence of Thurston hyperbolisation theorem (see [19, Theorem 2.3]), if V is a closed Haken 3-manifold, then the JSJ components of V are Seifert or hyperbolic.

2.3. Statements of results

Let f be a gradient-like (orientation preserving) diffeomorphism of a compact connected orientable 3-manifold M. Let $(V_f, \alpha_f, \Gamma_f^s, \Gamma_f^u)$ be the global scheme of f.
Then:

1. (Theorem 3.1 and Corollary 3.16) The manifold V_f is either irreducible or diffeomorphic to $S^2 \times S^1$. Moreover, if V_f is not $S^2 \times S^1$ then V_f is Haken and, in fact, toroidal (i.e. V_f contains an incompressible torus).
2. (Proposition 3.17 and Corollary 5.1) V_f cannot admit an S^3, Sol, $\widetilde{PSL_2}(\mathbb{R})$ or hyperbolic structure.[2]
3. (Corollary 3.12) Each element (either torus or Klein bottle) of $\Gamma_f^s \cup \Gamma_f^u$ is
   (a) either incompressible,
   (b) or the boundary of a solid torus, either embedded in V_f, in the case of a torus, or with interior embedded in V_f, in the case of a Klein bottle.
   Moreover, if $\Gamma_f^s \cup \Gamma_f^u$ contains a compressible Klein bottle, then V_f is $S^2 \times S^1$.
4. (Theorem 5.1)

---

[2] S^3, H^3 (i.e. hyperbolic structure), Sol and $\widetilde{PSL_2}(\mathbb{R})$ are four of the eight Thurston's geometries described in [18]. The remaining four geometries ($S^2 \times \mathbb{R}$, E^3, Nil, $H^2 \times \mathbb{R}$) uniformise V_f in the case where V_f is a Seifert manifold. Note that, if V_f has a non-trivial JSJ decomposition, some of its geometric pieces may be hyperbolic.

(a) Either V_f is a circle bundle with non-trivial Euler class and base the torus T^2 or the Klein bottle K,

(b) or there exists an integer $m > 0$ such that each of the Seifert components of V_{f^m} is a trivial circle bundle for which α_{f^m} is 1 on the fibres (where $(V_{f^m}, \alpha_{f^m}, \Gamma^s_{f^m}, \Gamma^u_{f^m})$ is the global scheme of f^m).

3. The topology of V_f and the tori contained in V_f

Let f be a gradient-like diffeomorphism of a compact connected orientable 3-manifold M and $\mathcal{S}(f) = (V_f, \alpha_f, \Gamma^s_f, \Gamma^u_f)$ its global scheme. By construction, $H^1(V_f, \mathbb{Z})$ contains a non-trivial element α_f. Denote $\pi \colon \tilde{V}_f \to V_f$ the infinite cyclic cover of V_f determined by α_f, that is, the epimorphism from $\pi_1(V_f)$ onto $\mathbb{Z}$, induced by α_f coincides with the natural morphism onto the group of deck transformations of the covering. By construction (see [3]) $\tilde{V}_f$ is identified with the open subset of the closed manifold M obtained by removing from M the closure of the one-dimensional stable and unstable manifolds of the periodic orbits of saddle type of f: The restriction of the diffeomorphism f to $\tilde{V}_f$ is the generator of the group of deck transformations of the covering π.

As the perfect schemes are precisely the global schemes of gradient-like diffeomorphisms (Theorem 2.2) from now on we shall omit the reference to the diffeomorphism f, and just consider a perfect scheme $\mathcal{S} = (V, \alpha, \Gamma^s, \Gamma^u)$. We shall denote $\pi \colon \tilde{V} \to V$ the infinite cyclic cover of V determined by α. When needed, a perfect scheme will be treated as the global scheme of a diffeomorphism f.

3.1. The topology of V

Our first result is

Theorem 3.1. *Let $\mathcal{S} = (V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme. The manifold V is either diffeomorphic to $S^2 \times S^1$ or irreducible, i.e. every smoothly embedded sphere in V bounds a ball.*

Moreover, if $V \neq S^2 \times S^1$ then V contains an orientable incompressible embedded surface S dual to α (i.e. α is the intersection form with S), and hence V is Haken.

This section will be devoted to the proof of this result which is based on the following proposition which, on its turn, was obtained by Kneser in [12] to prove the uniqueness of prime decompositions for closed 3-manifolds (see also the paper of Haken [7, page 42], and, for a proof, Hempel's book [8, 3.8, 3.10 and 3.14]).

Proposition 3.1. *Let $\{S_i\}_{i \in \mathbb{Z}}$ be a family of pairwise disjoint 2-spheres smoothly embedded in a closed orientable 3-manifold. Then either almost all spheres S_i bound balls in M or there exist two indices $i, j \in \mathbb{Z}$, $i \neq j$ such that the spheres S_i and S_j are parallel in M, i.e. there is an embedding of $S^2 \times [-1, 1]$ into M such that the image of $S^2 \times \{-1, 1\}$ is $S_i \cup S_j$.*

More precisely, let $\{S_{n_i}\}$ the sub-family of spheres which do not bound balls in M. Then $\{S_{n_i}\}$ has only a finite number of parallelism classes.

We are now able to prove the first part of Theorem 3.1:

Proof. We consider here $\mathcal{S}$ to be the global scheme of a gradient-like diffeomorphism f of a compact 3-manifold M.

Let S be a smoothly embedded sphere in V. The preimage $\pi^{-1}(S)$ of S in $\tilde{V}$ is a countable family of pairwise disjoint embedded spheres, which are images of one another by iterates of f. Denote by $\{S_i\}_{i \in \mathbb{Z}}$ such family of spheres considered as embedded in M (for $\tilde{V} \subset M$). According to Proposition 3.1, the spheres S_i which do not bound balls in M represent a finite number of parallelism classes in M.

Since there is only a finite number of connected components of $M \setminus \tilde{V}$, we can find either a sphere bounding a ball disjoint from $M \setminus \tilde{V}$ or two spheres, S_i and S_j, which bound a product which does not intersect $M \setminus \tilde{V}$.

In the first case, S_i bounds a ball in $\tilde{V}$ which implies that S bounds a ball in V.

In the second case, S_i and S_j are the boundary of an embedding of $S^2 \times I$ into $\tilde{V}$. Observe that S_j is an iteration of S_i via f. We deduce that $S^2 \times S^1$ is a covering of V. Recall that V is assumed to be orientable and that the only orientable manifolds which are covered by $S^2 \times S^1$ are $S^2 \times S^1$ and $\mathbb{RP}^3 \sharp \mathbb{RP}^3$. However, the fact that $H_1(\mathbb{RP}^3 \sharp \mathbb{RP}^3) = (\mathbb{Z}/2\mathbb{Z})^2$ implies that α cannot exist for this manifold and we conclude that $V = S^2 \times S^1$ in this case.

The second part of Theorem 3.1 follows from Lemma 3.4 below.   □

Remark 3.2. The proof of the first part of Theorem 3.1 seems to suggest that incompressible surfaces dual to α in V lift to families of parallel surfaces in $\tilde{V}$, however this may not be the case, since the lifted surfaces are not necessarily incompressible in M. If the lifts are parallel, then V is a mapping torus, and this is not always true (see the example given in Section 6.4).

Definition 3.3. Let M be a compact 3-manifold, perhaps with boundary. A compact surface S with empty boundary, embedded in M is *essential* if every connected component C of S is not boundary parallel (i.e. not parallel to some component of ∂M) and, either C is a sphere which does not bound a ball, or C is incompressible.

Lemma 3.4. *Let $(V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme. There exists an essential surface S on which α is identically zero.*

More precisely, there exists a compact (perhaps non-connected) surface S without boundary, embedded into V, which is Poincaré dual to α (i.e. α is given by the intersection form with S) and such that each component of S is either a sphere not bounding a ball or an incompressible surface of genus at least 1.

Proof. If V is $S^2 \times S^1$, then one can choose S to be $S^2 \times \{x\}$. So we can assume, from now on, that V is not diffeomorphic to $S^2 \times S^1$.

There exists a closed (compact, with empty boundary, perhaps non-connected) orientable (hence bicollared) surface smoothly embedded in V such that α is the intersection form $\gamma \mapsto \gamma \cdot S$. The following two remarks show how to simplify S:

Remarks 3.5. 1. The intersection form is invariant by the following surgery: let D be a disc whose interior is embedded in the complement of S and whose boundary lies on S. Cut S along the boundary of D to obtain a surface with boundary and glue a copy of D along each boundary component of the surface.

2. If a connected component S_0 of S is a sphere, then S_0 bounds a ball in V, for V is irreducible; it follows that α is the intersection form of $S \setminus S_0$.

We shall need the following corollary of the Dehn's lemma–loop theorem, that can be found in Hempel's book [8, Page 58, Lemma 6.1]:

Lemma 3.6. *Let S be a compact 2-manifold in a 3-manifold M such that each component of S is either properly embedded (i.e. $S \cap \partial M = \partial S$) and two-sided in M or is contained in ∂M. If for some component F of S one has $\ker(\pi_1(F) \longrightarrow \pi_1(M)) \neq 1$, then there is a 2-cell D in M such that $D \cap S = \partial D$ and ∂D is not contractible in S.*

If S is not incompressible, then Dehn's lemma assures the existence of a disc whose interior is embedded in $V \setminus S$ and whose boundary is an essential curve on S. The first part of the above remark allows us to construct a new surface $\tilde{S}_1$ dual to α and verifying one of the following properties:

- either the sum of the genera of the connected components of $\tilde{S}_1$ is strictly less than the sum of the genera of the connected components of S;
- or the sums of the genera of the connected components of $\tilde{S}_1$ and of S are equal but the number of components of genus different from 0 of $\tilde{S}_1$ is one more than that of S, while the number of components of genus 0 is the same; in particular the surgery was performed on a component of genus at least 2.

Let S_1 be the surface obtained from $\tilde{S}_1$ by eliminating all genus 0 components. According to the second part of the remark, S_1 is dual to α.

If some component of S_1 is compressible, we repeat the process, thus obtaining a sequence S_i of surfaces dual to α and without components of genus 0. Let us denote by (n_i, g_i) the number of connected components of S_i and the sum of their genera. There are three possibilities: $(n_{i+1}, g_{i+1}) = (n_i + 1, g_i)$ (corresponding to the second case above), $(n_{i+1}, g_{i+1}) = (n_i, g_i - 1)$ (corresponding to the first case when the component of S_i which is compressed has genus > 1) or $(n_{i+1}, g_{i+1}) = (n_i - 1, g_i - 1)$ (corresponding to the first case when the component of S_i which is compressed has genus 1; so a sphere is created which is then deleted). In any case, the difference between twice the sum of the genera of the components and the number of components is strictly decreasing. Since such difference is always positive, the sequence S_i is finite. The last term of the sequence is a surface dual to α and whose components are all incompressible. $\quad \square$

Remark 3.7. The proof of Theorem 3.1 does not use the fact that f is a gradient-like diffeomorphism so that the conclusion of Theorem 3.1 holds in a more general setting. Indeed, to prove the theorem we only need to find an open subset U of the closed orientable 3-manifold M, whose complement $M \setminus U$ consists of only finitely many connected components, such that $\pi : U \longrightarrow V$ is an infinite cyclic cover of an orientable compact manifold V.

In the next section we shall see a dynamical setting in which the hypotheses described in the remark above are verified.

3.2. A generalisation of Theorem 3.1 for any topological attractor

In this section, we shall exploit the proof of Theorem 3.1 to establish a more general result which, however, will not be needed in the rest of the paper.

Let f be a diffeomorphism of a closed manifold M, and U be a *trapping region of f* that is, U is an open set such that $f(\bar{U}) \subset U$ where $\bar{U}$ denotes the closure of U; then the intersection $A = \bigcap_{n \in \mathbb{N}} \bar{U}$ is a compact non-empty invariant set of f called *a (topological) attractor of f*, and the union $\bigcup_{n \in \mathbb{Z}} f^n(U)$ is an invariant open set called *the basin of the attractor A* and denoted by $W^s(A)$.

Notice that $V = M \setminus \bar{U}$ is a trapping region for f^{-1}, and the corresponding attractor $R = \bigcap_{n \in \mathbb{N}} f^{-n} \bar{V}$ (for f^{-1}) is called *a (topological) repellor of f* and its basin $\bigcup_{n \in \mathbb{Z}} f^n(V)$ is denoted by $W^u(R)$.

The pair (A, R) is the *attractor/repellor pair* induced by U. Notice that any open set $\tilde{U}$ containing $f(\bar{U})$ and contained in U is a trapping region which leads to the same attractor/repellor pair. In particular, one can choose the trapping region $\tilde{U}$ being the interior of a compact manifold (of the same dimension as M) with boundary.

Let first recall the (probably well known) following fact:

Theorem 3.2. *Let (A, R) be an attractor/repellor pair of a diffeomorphism of a closed manifold M. Then*

1. *The sets of connected components of A and of R are finite.*
2.

$$W^s(A) \cap W^u(R) = M \setminus (A \cup R) = W^s(A) \setminus A = W^u(R) \setminus R.$$

Denote by $\tilde{M}$ this set, which is an f-invariant open subset of M.

3. *The set of connected components of $\tilde{M}$ is finite. So f induces a permutation of these components and each component is periodic under this action of f.*
4. *Let V be the space of the orbits of f contained in $\tilde{M}$, endowed with the quotient topology. Then V is a closed manifold, and the canonical projection $\pi \colon \tilde{M} \to V$ is a covering.*
5. *Let V_0 be a connected component of V and M_0 be a connected component of $\tilde{M}$ whose projection is V_0. Then $\pi \colon M_0 \to V_0$ is a regular cover whose automorphism group is generated by f^k, where k is the period of M_0; in particular it is an infinite cyclic cover.*

Proof. Choose a trapping region U inducing (A, R) such that U is the interior of a compact manifold $\bar{U}$ with boundary. For any $N \in \mathbb{N}$, the intersection $\bigcap_0^N f^i(\bar{U})$ is equal to $f^N(\bar{U})$, so that it has the same number of connected components as $\bar{U}$. As a decreasing intersection of compact connected sets is connected, one deduces that the number of connected components of A is bounded by those of $\bar{U}$, hence is finite. The same argument holds for R replacing $\bar{U}$ with $M \setminus U$ and f with f^{-1}. This proves part 1. Part 2 is a very general remark which holds for any attractor/repellor pair of a homeomorphism on a compact metric space.

The rest of the theorem follows from the following argument: consider $\Delta = \bar{U} \setminus f(U)$. It is a compact manifold with boundary and $\partial \Delta = \partial U \cup \partial f(U)$. Then $\tilde{M} = \bigcup_{n \in \mathbb{Z}} f^n(\Delta)$ has the same number of connected components as Δ. Furthermore, $\Delta \setminus \partial f(U)$ is a fundamental domain of f for $\tilde{M}$: any orbit in $\tilde{M}$ has exactly one point in $\Delta \setminus \partial f(U)$. One deduces that the quotient V is canonically identified with the closed manifold obtained from Δ by gluing ∂U with $\partial f(U)$ along the diffeomorphism $f \colon \partial U \to \partial f(U)$. $\square$

We can now restate Theorem 3.1 in this general context:

Theorem 3.3. *Let f be an orientation preserving diffeomorphism of an oriented closed 3-manifold M, and let A be a topological attractor of f. Let V be the closed oriented manifold obtained as the orbit space of f in $W^s(A) \setminus A$. Then each connected component of V is either diffeomorphic to $S^2 \times S^1$ or is an irreducible Haken manifold.*

The proof of Theorem 3.3 is identical to the proof of Theorem 3.1.

3.3. Tori embedded in V

Ref. [2] shows that every embedded torus in $S^2 \times S^1$ whose fundamental group is not trivial inside $\pi_1(S^2 \times S^1)$ bounds a solid torus.

Proposition 3.8. *Suppose that $V \neq S^2 \times S^1$. Every torus embedded in V is either incompressible, or is the boundary of a solid torus, or is contained in a ball.*

The above proposition is a straightforward consequence of the following result:

Proposition 3.9. *Let T be a compressible torus embedded in a compact, orientable, irreducible 3-manifold with empty boundary. Then either T bounds a solid torus embedded in M or T is contained in a ball inside M.*

Proof. This is a consequence of Dehn's lemma–loop theorem (see Lemma 3.6).

If a torus $T \subset M$ is compressible, then there exists a simple closed curve $\gamma \subset T$ which is not null-homotopic in T but which bounds a disc D in M. Let Δ be a tubular neighbourhood of D whose boundary consists in the union of an annulus $A \subset T$ and two discs D_1 and D_2 parallel to D. Let $B \subset T$ the annulus which is the closure of $T \setminus A$. Then $B \cup D_1 \cup D_2$ is a sphere embedded in M. Since M is irreducible, such sphere bounds a ball $\tilde{\Delta}$ in M. Remark that the boundaries of the two spheres Δ and $\tilde{\Delta}$ meet precisely in $D_1 \cup D_2$. There are two possibilities:

1. either $\Delta \subset \tilde{\Delta}$: in this case $T \subset \tilde{\Delta}$ and T satisfies the conclusion of the proposition;
2. or $\Delta \cap \tilde{\Delta} = D_1 \cup D_2$ and $\Delta \cup \tilde{\Delta}$ is a solid torus bounded by T, which verifies again the conclusion of the proposition. $\square$

Remark 3.10. Let K be a Klein bottle embedded in a compact orientable 3-manifold, and let T be an embedded torus in M which is the boundary of a small tubular neighbourhood for K. In this case K is incompressible if and only if so is T. Indeed, in the tubular neighbourhood, the fundamental group of T injects into the fundamental group of K as a (normal) subgroup of index 2. If the canonical morphism from $\pi_1(T)$ to $\pi_1(M)$ is not injective, this must be the case also for K. Conversely if the canonical morphism from $\pi_1(K)$ to M has non-trivial kernel, such kernel must contain an infinite order element (for $\pi_1(K)$ is torsion-free). Its square is a non-trivial element of $\pi_1(T)$ which is null-homotopic in $\pi_1(M)$.

Remark 3.11. The Klein bottle embeds in $S^2 \times S^1$ in the following way. Let γ be the great circle of the unit sphere S^2 of $\mathbb{R}^3$ obtained by intersecting S^2 with the plane $x = 0$ and let R_θ be the diffeomorphism of S^2 induced by the rotation of angle θ about the z-axis. The union of the circles $R_{\pi \cdot t}(\gamma) \times \{t\} \subset S^2 \times S^1$ as t varies in $S^1 = \mathbb{R}/\mathbb{Z}$ is a Klein bottle K embedded in $S^2 \times S^1$. Moreover, the complement of a tubular neighbourhood of K is a solid torus whose meridian represents the null-homologous generator for the Klein bottle.

Corollary 3.12. *Let $\mathcal{S}(f) = (V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme. Every element in $\Gamma^s \cup \Gamma^u$ is either incompressible, or the boundary of a solid torus embedded in V in the case of a torus, and with interior embedded in V in the case of a Klein bottle.*
To get a compressible Klein bottle, V must be $S^2 \times S^1$.

Proof. Let $T \in \Gamma^s \cup \Gamma^u$ be a torus embedded in V. We know that the restriction to T of the class $\alpha \in H^1(V, \mathbb{Z})$ is not identically zero, so that T cannot be contained in a ball of V. If V is not $S^2 \times S^1$ then (Theorem 3.1) V is irreducible and Proposition 3.9 implies that either T bounds a solid torus or is incompressible.

Assume now that $K \in \Gamma^s \cup \Gamma^u$ is a Klein bottle and denote by T the boundary of a tubular neighbourhood of K in V. Notice that the restriction of α to K is not identically zero and thus neither is its restriction to T. According to the above argument, either T and K are both incompressible or T bounds a solid torus. Such solid torus must be one of the two connected components obtained by cutting V along T. Since K cannot embed in a solid torus, we deduce that V consists in a solid torus and the tubular neighbourhood of K glued along their boundaries. The gluing along T depends only on the choice of the image of the meridian of the solid torus on the boundary of the tubular neighbourhood of K. Note that the image of the meridian must be the generator of the kernel of the restriction to T of α. Recall now that $H^1(K, \mathbb{Z}) = \mathbb{Z}$, thus the kernel of α does not depend on the choice of the form α. We can deduce from Remark 3.11 that in this case V is $S^2 \times S^1$. $\square$

3.4. The tori of Γ^s

Lemma 3.13. *If $(V, \alpha, \Gamma^s, \Gamma^u)$ is a perfect scheme and if $T \in \Gamma^s$ is a compressible torus (or Klein bottle), then $(V, \alpha, \Gamma^s \setminus \{T\}, \Gamma^u)$ is also a perfect scheme.*

Proof. If $T \in \Gamma^s$ is compressible, then it bounds a solid torus whose meridian coincides with the kernel of $\alpha|_T$. We deduce that $CS(V, \Gamma^s) = S^2 \times S^1 \coprod CS(V, \Gamma^s \setminus \{T\})$. In particular, each component of $CS(V, \Gamma^s \setminus \{T\})$ is diffeomorphic to $S^2 \times S^1$ and thus, according to Theorem 2.2, $(V, \alpha, \Gamma^s \setminus \{T\}, \Gamma^u)$ is a perfect scheme. $\square$

Reasoning in a similar way one can prove:

Lemma 3.14. *Let $(V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme and let $K \in \Gamma^s$ be a Klein bottle. Denote by T the boundary of a small neighbourhood of K; one can choose T to be transverse to the elements of Γ^u. Then $(V, \alpha, (\Gamma^s \setminus \{K\}) \cup \{T\}, \Gamma^u)$ is a perfect scheme.*

Corollary 3.15. *If V is not $S^2 \times S^1$ then Γ^u contains at least one incompressible torus or Klein bottle.*

Proof. If Γ^u does not contain an incompressible torus or Klein bottle, then the formal scheme $(V, \alpha, \Gamma^s, \emptyset)$ is perfect. In particular, each connected component of $V = CS(V, \emptyset)$ is diffeomorphic to $S^2 \times S^1$, hence, since V is connected, $V = S^2 \times S^1$. $\square$

Thanks to the above fact, we can give a different proof of the second part of Theorem 3.1, showing moreover that V is toroidal:

Corollary 3.16. *If V is not $S^2 \times S^1$, then V is Haken.*

Proof. If V is not $S^2 \times S^1$, then there exists an incompressible torus embedded in V: according to Corollary 3.15 such torus is either an element of Γ^u or the boundary of a tubular neighbourhood of a Klein bottle of Γ^u. $\square$

3.5. No Sol manifolds

Proposition 3.17. *Let $(V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme. The manifold V cannot admit a Sol structure.*

Lemma 3.18. *If V is the mapping torus[3] of an Anosov diffeomorphism, then $H^1(V, \mathbb{Z})$ is $\mathbb{Z}$.*

Proof. Let V be the mapping torus of an Anosov diffeomorphism $\varphi : T^2 \longrightarrow T^2$. The fundamental group of V admits the following presentation:

$$\langle a, b, t \mid [a, b],\ ta^{-1}t^{-1}\varphi_*(a),\ tb^{-1}t^{-1}\varphi_*(b) \rangle.$$

Consider the abelianisation

$$\langle a, b, t \mid [a, b],\ [a, t],\ [b, t],\ \varphi_*(a) = a,\ \varphi_*(b) = b \rangle$$

where φ_* can be interpreted as a hyperbolic element of $SL_2(\mathbb{Z})$, and the latter two conditions are satisfied if and only if both a and b are trivial. Indeed, 1 cannot be an eigenvalue of a hyperbolic element of $SL_2(\mathbb{Z})$. □

We can now prove Proposition 3.17.

Proof. Assume that V is a Sol manifold. Then (see [18, Theorem 4.17]), the manifold V admits a finite cover $\tilde{V}$ which is the mapping torus of an Anosov diffeomorphism.

Let T be an incompressible torus embedded in V, on which α is not identically zero. According to Corollary 3.15 such torus exists since V is not $S^2 \times S^1$ (T is either a torus in Γ^s or Γ^u, or the boundary of a tubular neighbourhood of a Klein bottle in Γ^s or Γ^u).

The torus T lifts in $\tilde{V}$ to an incompressible torus $\tilde{T}$. According to Lemma 3.18, $H_2(\tilde{V}, \mathbb{R}) = \mathbb{R}$ is generated by the fibre (note that $\tilde{V}$ is a fibration with base S^1 and fibre T^2) which is dual to the generator of $H_1(\tilde{V}, \mathbb{R})$ (corresponding to the base).

However, every class in $H^1(\tilde{V}, \mathbb{Z})$ is trivial on the fibre, in particular $\tilde{T}$ cannot lift to a fibre for the restriction to $\tilde{T}$ of the lift $\tilde{\alpha}$ is not identically zero.

This shows that T meets each fibre in at least one non-null-homologous curve. Such homology class is well defined (up to sign) and thus must be left invariant by the monodromy: a contradiction.   □

Corollary 3.19. *The manifold V is either $S^2 \times S^1$, or a Seifert fibration, or admits a non-trivial decomposition in pieces which admit a Seifert fibred or hyperbolic structure.*

Proof. We saw that V cannot be a Sol manifold. V cannot be hyperbolic either. In fact, a hyperbolic manifold does not contain incompressible tori, whilst, by Corollary 3.15, the family Γ^u must contain an incompressible torus unless V is $S^2 \times S^1$.   □

The following remark shows that the space of orbits V_0, as defined in Theorem 3.2, of a non-gradient-like diffeomorphism may admit a Sol structure or a hyperbolic structure.

Remark 3.20. Let F be a compact, closed, orientable surface of genus at least 1. Let ϕ be an Anosov (respectively pseudo-Anosov) diffeomorphism of F if its genus is 1 (respectively > 1). Let ψ be the "north–south" diffeomorphism of the circle S^1 with attractor a and repellor r. Consider the

---

[3] The mapping torus is called *suspension* by the dynamicists.

diffeomorphism $f = (\phi, \psi)$ defined on the manifold $M = F \times S^1$. Observe that $(F \times \{a\}, F \times \{r\})$ is an attractor–repellor pair of f. Each of the two connected components V_0 of the corresponding orbit space V is diffeomorphic to the mapping torus of ϕ (or of ϕ^{-1}), hence admits either a Sol structure (if the genus of F is 1) or a hyperbolic structure (if the genus is > 1).

3.6. The Jaco–Shalen–Johannson family

A compact manifold (possibly with boundary) is *Seifert fibred* if it is the union of disjoint circles (called *fibres*) such that every fibre admits a neighbourhood which is a solid torus where the fibration is given by the mapping torus (suspension for the dynamicists) of a rational rotation of the disc. The fibre is called *singular* if the rotation is not the identity.

Recall that every irreducible orientable 3-manifold admits a minimal family of embedded, incompressible, pairwise disjoint tori which is unique up to isotopy and which decomposes the manifold into geometric pieces, called the Jaco–Shalen–Johannson family. Here by geometric pieces we mean either Seifert fibred or atoroidal pieces (a manifold with boundary is called *atoroidal* if any incompressible torus is parallel to some component of the boundary) (see [10,11]).

The reader is referred to [10,11] for the construction and the basic properties of this family. Among the properties of the JSJ family we shall use the following:

Proposition 3.21. *Let V be an irreducible manifold. For every collection Γ of incompressible, pairwise disjoint tori embedded in V, the families $\mathrm{JSJ}(V)$ and Γ are disjoint up to isotopy.*

As a consequence of Thurston hyperbolisation theorem (see [19, Theorem 2.3]), if V is a closed Haken 3-manifold, the components of V are either Seifert fibred of hyperbolic.

The aim of this section is to proof:

Theorem 3.4. *Let $(V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme such that the Jaco–Shalen–Johannson family for V is non-empty, and let T be a torus of the family. The restriction of α to T is not identically zero.*

Proof. Let us denote by Γ^s_{inc} the set of incompressible tori of Γ^s. We have seen that $(V, \alpha, \Gamma^s_{\mathrm{inc}}, \Gamma^u_{\mathrm{inc}})$ is a perfect scheme. Because of Proposition 3.21, we can assume that T is disjoint from the tori in Γ^s_{inc}. This means that T induces a torus $\tilde{T}$ in $CS(V, \Gamma^s_{\mathrm{inc}})$. Each connected component of $CS(V, \Gamma^s_{\mathrm{inc}})$ is diffeomorphic to $S^2 \times S^1$ and the class α induces a non-trivial class on each of these components. Moreover, any such component contains a torus or a Klein bottle, corresponding to an element of Γ^s_{inc}, on which the class α is non-identically zero. As a consequence, $\tilde{T}$ is disjoint from a non-null-homologous circle embedded in $S^2 \times S^1$. This implies that the homology class of $\tilde{T}$ inside $H_2(S^2 \times S^1)$ is zero.

A torus embedded in $S^2 \times S^1$ on which the class α is trivial is either homologous to the fibre S^2 or contained in a ball. The former case is impossible because of the above argument. Thus $\tilde{T}$ is contained in a ball inside $S^2 \times S^1$ that can be chosen so that it does not intersect the tori coming from Γ^s_{inc}. We deduce that T is also contained in a ball inside V against the hypothesis that T is incompressible. $\square$

4. Reduced schemes

We shall say that $(V, \alpha, \Gamma^s, \Gamma^u)$ is a *reduced scheme* if it is a perfect scheme and if

- for each element $T \in \Gamma^s$, there is a component of $CS(V, \Gamma^s \setminus \{T\})$ which is not diffeomorphic to $S^2 \times S^1$ (in other words, the scheme obtained by omitting T from the family Γ^s is no more perfect),

- for each element $T \in \Gamma^u$, there is a component of $CS(V, \Gamma^u \setminus \{T\})$ which is not diffeomorphic to $S^2 \times S^1$,
- the number of connected components of $\Gamma^s \cap \Gamma^u$ is minimal among all numbers of connected components of the intersections $\Gamma^s \cap \varphi(\Gamma^u)$, where φ is a diffeomorphism of V isotopic to the identity and such that $\varphi(\Gamma^u)$ is transverse to Γ^s.

Lemma 4.1. *If* $\mathcal{S} = (V, \alpha, \Gamma^s, \Gamma^u)$ *is a reduced scheme then the elements of* Γ^s *and* Γ^u *are all incompressible in* V. *Moreover, for each pair of distinct elements* T_1, T_2 *in* Γ^s *(respectively* Γ^u*),* T_1 *is not isotopic to* T_2.

Proof. If $T \in \Gamma^s$ is not incompressible, then Corollary 3.12 implies that T bounds a solid torus. The surgery (which preserves the meridian) along T is thus trivial. One deduces that the scheme obtained by omitting T from Γ^s is still perfect, which contradicts the fact that $\mathcal{S}$ is reduced. In conclusion, we have shown by contradiction that every element of $\Gamma^s \cup \Gamma^u$ is incompressible.

In a similar way one can prove that, if two elements T_1, T_2 of Γ^s (or Γ^u) are isotopic, then the scheme obtained by ignoring T_2 is still perfect, which contradicts the fact that $\mathcal{S}$ is reduced. $\square$

We shall say that a perfect scheme $(V, \alpha, \tilde{\Gamma}^s, \tilde{\Gamma}^u)$ is a *reduction* of $(V, \alpha, \Gamma^s, \Gamma^u)$ if $\tilde{\Gamma}^s$ and $\tilde{\Gamma}^u$ are isotopic to a part of Γ^s and Γ^u, respectively. A simple induction argument allows to see that, for each scheme $\mathcal{S}$ there exists a (non-unique) reduced scheme $\tilde{\mathcal{S}}$ obtained from $\mathcal{S}$ by a sequence of reductions.

The relation *being a reduction of* is, by construction, reflexive and transitive. We shall say that two perfect schemes are $\mathcal{R}$-equivalent (or equivalent via reduction) if they are connected by a finite sequence of perfect schemes, such that for any two successive schemes of the sequence there is one which is the reduction of the other; in other words, the equivalence via reduction, denoted as $\sim_{\mathcal{R}}$, is the equivalence relation generated by the reduction relation.

Definition 4.2. We shall say that $(V, \alpha, \Gamma^s, \Gamma^u)$ is a *prepared scheme* if it is perfect and if

- every element of $\Gamma^s \cup \Gamma^u$ is a torus (i.e. no element is a Klein bottle),
- both $\tilde{\Gamma}^s$ and $\tilde{\Gamma}^u$ contain sub-families, denoted as JSJ^s and JSJ^u respectively, isotopic to the Jaco–Shalen–Johannson family,
- for each element $T \in \Gamma^s \setminus JSJ^s$, there exists a component of the manifold $CS(V, \Gamma^s \setminus \{T\})$ (obtained by surgery along T) which is not diffeomorphic to $S^2 \times S^1$ (in other words, the scheme is no longer perfect once T is deleted from the family Γ^s),
- for each element $T \in \Gamma^u \setminus JSJ^u$, there exists a component of the manifold $CS(V, \Gamma^u \setminus \{T\})$ which is not diffeomorphic to $S^2 \times S^1$,
- for each pair of distinct elements T_1, T_2 of Γ^s (respectively Γ^u), T_1 is not isotopic to T_2,
- the number of connected components of $\Gamma^s \cap \Gamma^u$ is minimal among all the numbers of connected components of intersections $\Gamma^s \cap \varphi(\Gamma^u)$ where φ is a diffeomorphism of V isotopic to the identity and such that $\varphi(\Gamma^u)$ is transverse to Γ^s.

Lemma 4.3. *Let* $\mathcal{S} = (V, \alpha, \Gamma^s, \Gamma^u)$ *be a reduced scheme. Then there exists a prepared scheme* $\tilde{\mathcal{S}} = (V, \alpha, \tilde{\Gamma}^s, \tilde{\Gamma}^u)$ *which is* $\mathcal{R}$-equivalent to $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a perfect scheme and $K \in \Gamma^s$ a Klein bottle. Let T be the boundary of a tubular neighbourhood of K: T can be chosen to be disjoint from every element of Γ^s and transverse to every element of Γ^u. The scheme $\mathcal{S}_1$ obtained by adding T to Γ^s is still perfect (and consequently $\mathcal{S}$ is a

reduction of $\mathcal{S}_1$). Indeed, denote $\Gamma_1^s = \{T\} \cup \Gamma^s$. In this case $CS(V, \Gamma_1^s)$ consists of the components of $CS(V, \Gamma^s)$ (all diffeomorphic to $S^2 \times S^1$) plus an extra component obtained in the following way: cut the tubular neighbourhood of K along K (thus obtaining a manifold diffeomorphic to $T^2 \times [0, 1]$) and glue two solid tori on the boundary components in such a way that meridians are preserved. This new component is, on its turn, diffeomorphic to $S^2 \times S^1$, which shows that $\mathcal{S}_1$ is a perfect scheme.

Let now $\mathcal{S}_2$ be the scheme obtained from $\mathcal{S}_1$ by omitting the element K from Γ_1^s, i.e. $\Gamma_2^s = \Gamma_1^s \setminus \{K\}$. The scheme $\mathcal{S}_2$ is perfect (in particular is a reduction of $\mathcal{S}_1$). Indeed, $CS(V, \Gamma_2^s)$ consists of the components of $CS(V, \Gamma^s)$ plus a component obtained by gluing a solid torus along T on the tubular neighbourhood of K, in such a way that meridians are preserved. This component is once more diffeomorphic to $S^2 \times S^1$, which proves that $\mathcal{S}_2$ is perfect.

The scheme $\mathcal{S}_2$ is equivalent to $\mathcal{S}$ and was obtained by replacing the Klein bottle K with the torus T. By repeating this operation (a finite number of times) for each Klein bottle of $\Gamma^s \cup \Gamma^u$, we get a perfect scheme $\bar{\mathcal{S}}$, $\mathcal{R}$-equivalent to $\mathcal{S}$ and without Klein bottles.

Starting from $\bar{\mathcal{S}}$, by a sequence of reductions, it is possible to construct a reduced scheme $\hat{\mathcal{S}} = (V, \alpha, \hat{\Gamma}^s, \hat{\Gamma}^u)$, with no Klein bottles. According to Lemma 4.1 every torus of $\hat{\Gamma}^s \cup \hat{\Gamma}^u$ is incompressible.

Suppose that T is a torus of the Jaco–Shalen–Johannson family which is not isotopic to any torus of $\hat{\Gamma}^s$. In this case, since $\hat{\Gamma}^s$ is a family of disjoint incompressible tori, T is isotopic to a torus disjoint from the elements of $\hat{\Gamma}^s$ (see Proposition 3.21). Moreover, the restriction of the form α to T is non-trivial, according to Theorem 3.4. We deduce that T induces a torus, denoted as again T, contained in one of the components V_i of $CS(V, \hat{\Gamma}^s)$ and such that this torus is not null-homologous. Since $\hat{\mathcal{S}}$ is a perfect scheme, V_i is diffeomorphic to $S^2 \times S^1$, and T bounds a solid torus in V_i. Moreover, the cohomology class α induces a non-zero class on V_i, and this class determines a notion of meridian on T. Thus $CS(V_i, T)$ is the union of two copies of $S^2 \times S^1$. We deduce that $CS(V, \hat{\Gamma}^s \cup \{T\})$ is the disjoint union of $CS(V, \hat{\Gamma}^s)$ and of $S^2 \times S^1$. In particular, the scheme obtained by adding T to $\hat{\Gamma}^s$ is still perfect. One can repeat this operation to include the entire JSJ family inside $\hat{\Gamma}^s$ and, similarly, inside $\hat{\Gamma}^u$. The resulting scheme $\tilde{\mathcal{S}}$ satisfies the requirements of the lemma. $\square$

In a future work, we shall see that $\mathcal{R}$-equivalent schemes correspond to gradient-like diffeomorphisms which differ by a sequence of elementary bifurcations (of codimension 1), moreover the elementary bifurcations preserve gradient-like diffeomorphisms.

5. Seifert components of V

Let $(V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme, associated with a gradient-like diffeomorphism $f : M \to M$. For every integer $m > 0$ let us denote $\pi_m : V_{\alpha,m} \to V$ the m-fold cyclic branched cover of V associated with the morphism from $\pi_1(V)$ onto $\mathbb{Z}/m\mathbb{Z}$ obtained by composing α with the surjection of $\mathbb{Z}$ onto $\mathbb{Z}/m\mathbb{Z}$. Note that the lift of α on $V_{\alpha,m}$ takes values in $m\mathbb{Z}$. We shall denote by α_m the product $1/m$ times the lift of α on $V_{\alpha,m}$.

It is straightforward to verify that $V_{\alpha,m}$ is the underlying manifold of the scheme associated with f^m, and that the diffeomorphism induced by f on $V_{\alpha,m}$ generates the group of deck transformations of the covering.

The aim of Section 5 is to prove:

Theorem 5.1. *Let $(V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme. Then one of the following statements holds:*
1. *either V is a circle bundle, with non-zero Euler class, on the torus T^2 or the Klein bottle K;*

2. *or there exists m such that $V_{\alpha,m}$ is a trivial circle bundle such that the image of the fibre under α_m is 1;*
3. *or V admits a non-trivial Jaco–Shalen–Johannson decomposition and there exists m such that each Seifert component of the decomposition of $V_{\alpha,m}$ is a trivial circle bundle such that the image of its fibre under α_m is 1.*

This result has the following straightforward consequence:

Corollary 5.1. *Let $(V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme. No piece of $\mathrm{JSJ}(V)$ admits a $\widetilde{PSL_2(\mathbb{R})}$-structure.*

Proof. It is enough to remark that each compact $\widetilde{PSL_2(\mathbb{R})}$-manifold is finitely covered by a circle bundle with non-trivial Euler class and with base a hyperbolic surface. $\square$

The proof of Theorem 5.1 will require an analysis of the Seifert fibred pieces according to the behaviour of α on the fibres. Notice that the statement of Theorem 5.1 does not concern Γ^s and Γ^u so that we can replace the scheme with an $\mathcal{R}$-equivalent one. In particular according to Lemma 4.3 the scheme can be chosen to be prepared.

5.1. Case $\alpha = 0$ on the fibre

The goal of Section 5.1 is to prove the following proposition:

Proposition 5.2. *Let V_0 be a Seifert component of V and assume that the form α is trivial on the fibre. Then V_0 is one of the following fibrations:*

1. *$V_0 = V$ is a circle bundle over T^2.*
2. *$V_0 = V$ is a circle bundle over the Klein bottle K with orientable total space.*
3. *V_0 is the trivial circle bundle over the annulus.*
4. *V_0 is the circle bundle over the Möbius band with orientable total space.*

Corollary 5.3. *Let V_0 be a Seifert component of V. Then, either V_0 admits a Seifert fibration for which α is not zero on the fibres, or $V = V_0$ and V is a Seifert manifold, with non-trivial Euler class and base the torus T^2 or the Klein bottle K.*

Proof of Corollary. Assume that $\alpha = 0$ on the fibre and V is not a Seifert fibred manifold with non-trivial Euler class and base the torus T^2 or the Klein bottle K. An analysis of the cases listed in Proposition 5.2 shows that V_0 admits another Seifert fibration for which α is not zero on the fibres. Indeed:

If V_0 is a trivial fibration over the annulus or the torus, α is not zero on some cycle of the base, and V_0 admits a fibration with fibres parallel to such cycle.

If $V = V_0$ is the fibration over the Klein bottle K with zero Euler class and orientable total space, then V is the quotient of T^3 by the relation $(x, y, z) \sim (x + \frac{1}{2}, -y, -z)$. The first homology group $H_1(V, \mathbb{R})$ is then generated by the image of $S^1 \times \{0, 0\}$. The projection $(x, y, z) \mapsto (y, z)$ is a Seifert fibration over the sphere (considered as the quotient of T^2 by the relation $(y, z) \sim (-y, -z)$), and the regular fibre is the generator of $H_1(V_0, \mathbb{R})$. In particular, α is not zero on the fibre.

If the base of V_0 is the Möbius band, then V_0 is the quotient of $S^1 \times [-1, 1] \times S^1$ by the relation $(x, y, z) \sim (x + \frac{1}{2}, -y, -z)$. The projection $(x, y, z) \mapsto (y, z)$ is a Seifert fibration over the disc D^2

(seen as the quotient of the annulus $[-1, 1] \times S^1$ via the relation $(y, z) \sim (-y, -z)$), and the regular fibre of this fibration generates $H_1(V_0, \mathbb{R})$; again this implies that α is not zero on the fibre. $\square$

The proof of Proposition 5.2 will be the object of the entire Section 5.1. If the manifold V is $S^2 \times S^1$ there is nothing to prove so we shall assume, from now on, that V is not diffeomorphic to $S^2 \times S^1$.

As we assumed $(V, \alpha, \Gamma^s, \Gamma^u)$ to be a prepared scheme, the boundary of V_0 is parallel to certain tori in Γ^s and in Γ^u.

Lemma 3.4 assures the existence of an incompressible embedded surface S, dual to α. As consequence, S meets all the tori of Γ^s and of Γ^u in such a way that the intersection is homologically not trivial on the tori, for α is non-zero on the elements of Γ^s and of Γ^u. Such intersection can be assumed to be transverse. In particular, $S_0 = V_0 \cap S$ is non-empty.

Recall the following result which is due to Waldhausen (see [9, Theorem VI.34.]):

Theorem 5.2. *Up to isotopy, every essential (see Definition 3.3) surface in a Seifert manifold is either transverse to the fibration or fibred.*

Denote $\Gamma = \Gamma^s$. If $T \in \Gamma$ does not belong to the JSJ family, T is contained in a geometric piece of the decomposition, and is not boundary parallel. It follows that T is an essential surface in the piece that contains it. By definition of the Jaco–Shalen–Johannson decomposition, such piece must be Seifert fibred (as the hyperbolic components are atoroidal, see Section 3.6). According to Theorem 5.2, T can be isotoped inside the Seifert piece either to a Seifert torus or to a torus which is transverse to the fibres.

In the second case, T meets transversally all the fibres and thus meets transversally the boundary of the Seifert piece. Since the torus is disjoint from the boundary, we deduce that the boundary is empty, and the whole manifold V admits a Seifert fibration. In this case, the union of the tori in Γ forms an essential surface which, up to isotopy, is transverse to all the fibres.

As a consequence, we have shown:

Lemma 5.4. *Up to isotopy, Γ verifies one of the following two properties:*

- *either each element of Γ contained in a Seifert piece is fibred,*
- *or JSJ $= \emptyset$ and each element of Γ is transverse to the fibration.*

The proof of Proposition 5.2 uses a presentation for the fundamental group of the total space of a Seifert fibration, which is given in the following proposition (see [9, page 91]):

Proposition 5.5. *Let M be a compact orientable 3-manifold (perhaps with boundary) admitting a Seifert fibration with base B. A presentation for its fundamental group is given by:*

- *if the fibration is orientable, the fundamental group $\pi_1(M)$ admits a presentation whose generators are $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_q, d_1, \ldots, d_p, h$ satisfying the relations:*

$$[a_i, h] = 1 \quad \forall i, \qquad [b_i, h] = 1 \quad \forall i, \qquad [c_j, h] = 1 \quad \forall j, \qquad [d_k, h] = 1 \quad \forall k,$$

$$c_j^{\alpha_j} = h^{\beta_j} \quad \forall j, \quad and \quad h^e = \prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{q} c_j \cdot \prod_{k=1}^{p} d_k,$$

- *if the fibration is not orientable, the fundamental group $\pi_1(M)$ admits a presentation whose generators are $a_1, \ldots, a_g, c_1, \ldots, c_q, d_1, \ldots, d_p, h$ satisfying the relations:*

$$[a_i, h] = h^2 \quad \forall i, \qquad [c_j, h] = 1 \quad \forall j, \qquad [d_k, h] = 1 \quad \forall k,$$

$$c_j^{\alpha_j} = h^{\beta_j} \quad \forall j, \quad and \quad h^e = \prod_{i=1}^{g} a_i^2 \cdot \prod_{j=1}^{q} c_j \cdot \prod_{k=1}^{p} d_k$$

where g is the genus of the base,[4] h represents the regular fibre, $d_k, k \in \{1, \ldots, p\}$, correspond to the p boundary components, and the fibration has q exceptional fibres, with invariants (α_j, β_j). Finally e is the Euler class of the fibration.

We shall distinguish two cases:

5.1.1. The case where the tori of Γ are transverse to the fibres

Notice that in this case the Jaco–Shalen–Johannson family is empty (by Lemma 5.4). In particular, $V_0 = V$ is a Seifert manifold (without boundary). The tori of Γ being transverse to the fibres, the torus T^2 is a branched cover of the base B. The list of the bases of such covers is finite: the torus (unramified cover), the Klein bottle (unramified cover), the sphere with four branch points of order 2 (*the pillow*), the sphere with three branch points of orders $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$ (in orbifold terms, these are the Euclidean *turnovers*) and the projective plane with two branch points of order 2 (see [19, Chapter 13.21]).

Remark, moreover, that the existence of tori transverse to the fibres implies that the Euler class of these fibrations is zero.

Lemma 5.6. *Assume that V is not a circle bundle with base T^2 or the Klein bottle. Under this hypothesis, $H^1(V, \mathbb{R})$ is generated by the fibre.*

Proof. We shall exploit Proposition 5.5. Note that V has no boundary component ($p = 0$). We have seen that $e = 0$.

Assume, first of all, that the fibration is orientable. Since V is not a circle bundle with base T^2, the genus of the base is then 0. The quotient of $\pi_1(V)$ obtained by requiring that the fibre h is trivial is then generated by $c_1, \ldots, c_q$ verifying the relations $c_j^{\alpha_j} = 1$ and $\prod_1^q c_j = 1$. The abelianisation of this quotient is thus finite and the conclusion follows.

Assume now that the fibration is not orientable. Since V is not a circle bundle with base the Klein bottle, the genus g is 1, $q = 2$ and the α_j's are equal to 2. The quotient of $\pi_1(V)$ by h admits the presentation $\langle a_1, c_1, c_2 \mid c_j^2 = 1, a_1^2 c_1 c_2 = 1 \rangle$. The abelianisation of this quotient is again finite (isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$) and the conclusion follows. $\square$

Under the hypotheses of Lemma 5.6, the first homology group of each torus of Γ is trivial inside the first homology group of V which is impossible as α is non-trivial on the tori of Γ.

To summarise, in this case we have proved:

Corollary 5.7. *Let V_0 be a Seifert component of V. Assume that the form α is zero on the fibres of the given fibration and that the tori of Γ are transverse to the fibres. Then, $V = V_0$ and the fibration is either the trivial bundle on T^2 (i.e. $V = T^3$) or the regular fibration with base the Klein bottle, trivial Euler class and orientable total space. In the latter case, the cover of the orientations of the fibres is the torus T^3 endowed with the trivial fibration of base T^2.*

---

[4] In the non-orientable case, the base is (by definition of the genus of non-orientable surfaces) a connected sum of g copies of $\mathbb{RP}^2$.

5.1.2. The case where the tori of Γ are fibred

Let us consider a connected component V_1 obtained by cutting V_0 along Γ. By definition of perfect scheme, the surgery killing the meridians of the boundary components transforms V_1 into $S^2 \times S^1$. Since α is assumed to be trivial on the fibre and that the tori of Γ are fibred, the fibre is a meridian for the tori. The surgery consists then in killing the fibre, and the fundamental group of the resulting manifold, which must be $\mathbb{Z}$, coincides with the quotient of $\pi_1(V_1)$ by the normal group generated by the fibre. Note that the boundary of V_1 is non-empty if we suppose that V is not $S^2 \times S^1$.

Lemma 5.8. *With the above notation, V_1 is the regular fibration, with orientable total space and base the annulus or the Möbius band.*

Proof. Using the notation of Proposition 5.5, the quotient of the fundamental group of V_1 by the fibre admits the following presentation:

- $$\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_q, d_1, \ldots, d_p \mid c_j^{\alpha_j} \quad \forall j, [a_1, b_1] \ldots [a_g, b_g] c_1 \ldots c_q d_1 \ldots d_p \rangle$$
  if the fibration is orientable,

- $$\langle a_1, \ldots, a_g, c_1, \ldots, c_q, d_1, \ldots, d_p \mid c_j^{\alpha_j} \quad \forall j, a_1^2 \ldots a_g^2 c_1 \ldots c_q d_1 \ldots d_p \rangle$$
  if the fibration is not orientable.

We have seen that the boundary of V_1 is non-empty ($p \geq 1$) so that the generator d_p can be eliminated by means of the last relation. We thus obtain the following presentations:

$$\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_q, d_1, \ldots, d_{p-1} \mid c_j^{\alpha_j} \forall j \rangle \quad \text{(in the orientable case)}$$

$$\langle a_1, \ldots, a_g, c_1, \ldots, c_q, d_1, \ldots, d_{p-1} \mid c_j^{\alpha_j} \forall j \rangle \quad \text{(in the non-orientable case)}.$$

If $q > 0$ the group contains torsion elements and cannot be $\mathbb{Z}$. This means that $q = 0$ and the fibration has no singular fibres. We thus obtain the following presentations:

$$\langle a_1, b_1, \ldots, a_g, b_g, d_1, \ldots, d_{p-1} \mid \rangle \quad \text{(in the orientable case)}$$

$$\langle a_1, \ldots, a_g, d_1, \ldots, d_{p-1} \mid \rangle \quad \text{(in the non-orientable case)}.$$

It is now possible to verify that for the group to be $\mathbb{Z}$, one must have $g = 0$, $p = 2$ in the orientable case, and $g = 1$, $p = 1$ in the non-orientable one. $\square$

To achieve the proof of the proposition in this case, one remarks that V_0 is obtained by gluing some components V_1 along their boundary in such a way that the fibrations are preserved.

This finishes the proof of Proposition 5.2. $\square$

Remark 5.9. In the case where V is a regular fibration with base the torus or the Klein bottle and all the tori of Γ are fibred, the Euler class of the fibration can be non-trivial.

5.2. The case $\alpha \neq 0$ on the fibre

Lemma 5.10. *Assume that the manifold V is not $S^2 \times S^1$. Let V_0 be a Seifert component of V endowed with a Seifert fibration such that α is non-trivial on the fibre. Then the Euler class of the fibration of V_0 is trivial.*

Proof. We only need to consider the case where $V = V_0$ since the Euler class of Seifert manifolds with non-empty boundary is always 0. Let us consider the surface S, dual to α. S is an incompressible surface with empty boundary, hence an essential surface sitting inside the irreducible manifold V.

Such surface cannot be isotopic to a vertical (i.e. fibred) surface, for it is dual to the form α which is non-zero on the fibres. This fact has two consequences:

- the base of the fibration admits each connected component of S as a branched cover,
- the Euler class of the Seifert fibration for V is trivial. □

Lemma 5.11. *The Seifert fibrations of $S^2 \times S^1$ are quotients of the trivial one.*

Proof. The reader can find in [15] the exhaustive list of all 3-manifolds admitting (up to isotopy) non-unique Seifert fibrations and a list of the fibrations involved. These manifolds are called *small Seifert manifolds*. It is easy to check that $S^2 \times S^1$ can be seen, besides as the product fibration, as the Seifert fibration obtained as mapping torus of the rotation of angle $\frac{\beta}{\alpha}$. This fibration has two singular fibres with invariants (α, β) and $(\alpha, \alpha - \beta)$, where $0 < \beta < \alpha$ are coprime integers.

The list in [15] shows that $S^2 \times S^1$ does not admit other Seifert fibrations apart from the ones just seen. □

End of the proof of Theorem 5.1. Let $(V, \alpha, \Gamma^s, \Gamma^u)$ be a perfect scheme. Suppose that V is not a circle bundle with non-trivial Euler class and base the torus or the Klein bottle. We have seen (Corollary 5.3) that each Seifert component of V admits a Seifert fibration such that α is non-trivial on the fibres.

Let $r > 0$ be an integer and let γ be an element in $\pi_1(V)$. Let $s = \alpha(\gamma)$. The lift $\gamma_{\alpha,r}$ of γ on the cover $V_{\alpha,r}$ consists of k connected components, where $k = r \wedge s$. Suppose now that γ is a singular fibre of order a inside a Seifert component V_i of V. In this case $\alpha(\gamma') = a\alpha(\gamma)$ where γ' denotes the regular fibre. The lift $\gamma_{\alpha,r}$ of the singular fibre is a regular fibre of the lift of V_i if the lift of γ' consists of ak connected components. This can be rephrased as $r \wedge (as) = a(r \wedge s)$, that is a divides $\frac{r}{r \wedge s}$. It suffices then for r to be a multiple of $as = \alpha(\gamma')$, where γ' is the regular fibre of V_i.

Let m be the lowest common multiple of the values of α on the regular fibres of all the Seifert pieces of V. In this case, the lifts of the given fibrations to $V_{\alpha,m}$ are non-singular Seifert fibrations. To see that they are trivial, it suffices to show that their Euler classes are all 0 which follows from Lemmas 5.10 and 5.11.

To finish the proof, we are left to show that the value of α_m on the regular fibres of the Seifert pieces of $V_{\alpha,m}$ is 1. Consider the following commutative diagram of group morphisms:

$$
\begin{array}{ccc}
\pi_1(V_{\alpha,m}) & \xrightarrow{(\pi_m)_*} & \pi_1(V) \\
\downarrow{\scriptstyle \alpha_m} & & \downarrow{\scriptstyle \alpha} \\
\mathbb{Z} & \xrightarrow{\times m} & \mathbb{Z} \\
& & \downarrow{\scriptstyle p_m} \\
& & \mathbb{Z}/m\mathbb{Z}
\end{array}
$$

where $(\pi_m)_*$ is the morphism induced by the cover $\pi_m \colon V_{\alpha,m} \to V$, p_m is the canonical surjection and $\times m$ denotes multiplication times m. Note that horizontal arrows represent injections while vertical ones surjections.

Let $\tilde{\gamma}_i$ be the regular fibre of the fibration on the lift of the Seifert piece V_i, the regular fibre of V_i being γ_i'.

The choice of m implies that $\alpha(\gamma_i')$ divides m, that is $m = \ell_i \alpha(\gamma_i')$. Then the loop $\ell_i \gamma_i'$ lifts to a loop $\hat{\gamma}_i$ of V_{f^m}, on which the lift of α is equal to m, that is α_m is equal to 1. However $\hat{\gamma}_i$ is a multiple of the regular fibre of $V_{\alpha,m}$, so that $\alpha_m(\tilde{\gamma}_i) = 1$. $\quad\square$

6. Examples

6.1. Non-fibred tori, when $V = T^3$

Proposition 6.1. *There is a diffeomorphism f of $S^2 \times S^1$ such that V_f is the torus T^3 (with coordinates x, y, z), the cohomology class α_f is the one induced by the form dz, the family Γ^s consists of the torus $y = 0$ and the family Γ^u consists of the torus $y - z = 0$.*

This example shows that one cannot endow T^3 with a circle fibration such that α_f is non-zero on the fibres and the tori of $\Gamma^u \cup \Gamma^s$ are fibred.

Proof. The diffeomorphism f will be constructed as the composition $Z_1 \circ g$, where Z_1 is the time-1 of the flow of a gradient-like vector field Z, and where g is a diffeomorphism of "Dehn twist" type on the fundamental domain of Z_1.

Let S^2 be the sphere viewed as $\mathbb{R}^2 \cup \{\infty\}$. On the manifold $S^2 \times S^1$, we can thus choose coordinates (x, y, z), with $x, y \in \mathbb{R}$ and $z \in \mathbb{R}/\mathbb{Z}$. We shall write Z for the vector field which, in these coordinates, is given by

$$Z(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \sin(2\pi z)\frac{\partial}{\partial z}.$$

Z is a gradient-like vector field of $S^2 \times S^1$, with a repellor (a source) in $(0, 0, 0)$, an attractor (a sink) in $(\infty, \frac{1}{2})$, a saddle of Morse index (i.e. dimension of the unstable manifold) 2 in $\sigma_2 = (0, 0, \frac{1}{2})$ and a saddle of index 1 in $\sigma_1 = (\infty, 0)$.

For each $r \in \,]0, \infty[$ we shall denote by $T_r \subset S^2 \times S^1$ the torus defined by the equation $x^2 + y^2 = r^2$. Notice that, for all r, the image of T_r under the diffeomorphism Z_1 is the torus $T_{e \cdot r}$.

Consider the union

$$\Sigma = \bigcup_{r=1}^{e} T_r$$

Σ is a fundamental domain for the restriction of Z_1 to the complement of $\{(0, 0)\} \times S^1 \cup \{\infty\} \times S^1$, that is to the complement of the closures of the one-dimensional invariant manifolds $W^s(\sigma_2)$ and $W^u(\sigma_1)$. Note that Σ is diffeomorphic to the thickened torus $T^2 \times [1, e]$ (with coordinates (α, β, r) where $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$ and $r \in [1, e]$): indeed it suffices to consider the map $(x, y, z) \mapsto (\alpha, z, r)$, where $2\pi\alpha$ is the argument of the complex number $x + iy$, and where $r = \sqrt{x^2 + y^2}$.

Let $h \colon [1, e] \longrightarrow [0, 1]$ be an increasing smooth function, which is 0 in a neighbourhood of 1 and 1 in a neighbourhood of e and whose derivative is strictly positive in each point r such that $h(r) \notin \{0, 1\}$. Let g be the diffeomorphism of $S^2 \times S^1$ which coincides with the identity outside Σ and which, on Σ and in coordinates (α, β, r), is written as

$$g(\alpha, \beta, r) = (\alpha, \beta + h(r), r).$$

Let f be the diffeomorphism obtained by composing Z_1 and g; more precisely $f = Z_1 \circ g$. Remark that $\Omega(f) = \Omega(Z_1)$, for f coincides with Z_1 outside the interior of Σ, which is disjoint from all its images. In particular, f has exactly two saddles, of Morse index 1 and 2, and the closure of their one-dimensional invariant manifolds is the union of two circles $\{(0,0), \infty\} \times S^1 \subset S^2 \times S^1$.

Let V_f be the quotient of $M_f = S^2 \times S^1 \setminus \{(0,0), \infty\}$ by the action of f. Remark that V_f is canonically identified with the manifold obtained from Σ by gluing the two connected components of its boundary T_1 and T_2 via the diffeomorphism:

$$(\alpha, \beta, 1) \mapsto (\alpha, \varphi(\beta), e)$$

(obtained by restriction of f to T_1), where φ is the time-1 of the flow associated with $\sin(2\pi z)\frac{\partial}{\partial z}$. One now deduces that V_f is diffeomorphic to the torus T^3. Moreover, the form $\frac{1}{e}dr$, which is well defined over M_f, is invariant by f and thus induces a closed form α_f on the quotient V_f (we shall denote again α_f its cohomology class).

Notice that

$$W^u(\sigma_2, f) \cap \Sigma = W^u(\sigma_2, Z_1) \cap \Sigma = \left\{ (\alpha, \beta, r) \in \Sigma \,\Big|\, \beta = \frac{1}{2} \right\}.$$

In fact, for each point p of Σ, the negative f-orbit of p coincides with its negative Z_1-orbit. The projection of $W^u(\sigma_2, f)$ to V_f is then a torus T^u, and we set $\Gamma^u = \{T^u\}$.

In the same way one shows that

$$W^s(\sigma_1, f) \cap \Sigma = g^{-1}(W^s(\sigma_1, Z_1)) = g^{-1}(\{(\alpha, \beta, r) \in \Sigma \mid \beta = 0\})$$
$$= \{(\alpha, \beta, r) \in \Sigma \mid \beta = -h(r)\}.$$

It follows that the projection of $W^s(\sigma_1, f)$ to V_f is then a torus T^s, and we set $\Gamma^s = \{T^s\}$.

Note that the tori T^u and T^s intersect along the circle

$$T^u \cap T^s = \left\{ (\alpha, \beta, r) \in \Sigma \,\Big|\, \beta = \frac{1}{2} = \left(-\frac{1}{2}\right) \text{ and } h(r) = \frac{1}{2} \right\}.$$

Recall that the derivative of h is non-zero in $h^{-1}(\frac{1}{2})$. We can thus deduce that these tori are transverse, and so that $W^u(\sigma_2, f)$ and $W^s(\sigma_1, f)$ are transverse. It follows that f is a gradient-like diffeomorphism whose global scheme is $\mathcal{S}(f) = (V_f, \alpha_f, \Gamma^s, \Gamma^u)$.

On the other hand, $T^u \cap T^s$ is a circle on which the form α_f is zero. As a consequence, there is no circle (Seifert) fibration on $V_f = T^3$ such that α_f is non-zero on the fibres and the tori of $\Gamma^s \cap \Gamma^u$ are fibred. $\quad\square$

6.2. A diffeomorphism for which α is zero on the fibre of V

Proposition 6.2. *There is a gradient-like diffeomorphism of $S^2 \times S^1$ whose global scheme $(V_f, \alpha_f, \Gamma^s, \Gamma^u)$ verifies the following properties:*

1. *the manifold V_f is a circle bundle with non-trivial Euler class and base T^2;*
2. *for each Seifert fibration of V_f, the form α_f is zero on the fibres.*

Proof. Just like in the previous example, we shall identify $S^2 \times S^1$ with $(\mathbb{R}^2 \cup \{\infty\}) \times S^1$, with coordinates (x, y, z), and we shall write Z for the vector field which, in this coordinates, is given by

$$Z(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \sin(2\pi z)\frac{\partial}{\partial z}.$$

Call $\tilde{g}$ the diffeomorphism of $S^2 \times S^1$ defined by

$$\tilde{g}(x, y, z) = (R_{2\pi z}(x, y), z)$$

where $R_{2\pi z}$ is the rotation of angle $2\pi z$ of the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

Let $\tilde{f}$ be $Z_1 \circ \tilde{g}$, where Z_1 is the time-1 of the vector field Z. It is easy to verify that $\Omega(\tilde{f}) = \Omega(Z_1)$, and that the invariant manifolds of the saddles σ_1 and σ_2 of $\tilde{f}$ are disjoint (and thus transverse). This means that $\tilde{f}$ is a gradient-like diffeomorphism.

Moreover, $\Sigma = \bigcup_{r=1}^{e} T_r$ is again a fundamental domain for the restriction of $\tilde{f}$ to the complement $M_{\tilde{f}}$ of the closure of the one-dimensional invariant manifolds of the saddles σ_1 and σ_2. One deduces that $V_{\tilde{f}}$ is the manifold obtained by gluing the two boundary components T_1 and T_2 of its boundary by means of the map induced by $\tilde{f}$:

$$(\alpha, \beta, 1) \mapsto (\alpha + \beta, \varphi(\beta), e).$$

This map is isotopic to $(\alpha, \beta, 1) \mapsto (\alpha + \beta, \beta, e)$. It follows that $V_{\tilde{f}}$ is a circle bundle with base the torus T^2 and non-zero Euler class. Moreover, the fibres are generated by the circles $S^1 \times \{(\beta, r)\}$ of Σ.

Finally, $\alpha_{\tilde{f}}$ is the cohomology class of the closed form induced on $V_{\tilde{f}}$ by the (exact) form $\frac{1}{e}dr$ of $M_{\tilde{f}}$ which is zero on the fibres. $\quad\square$

6.3. A diffeomorphism of the sphere S^3 for which V is a Seifert manifold with singular fibres

In this section we shall build a gradient-like diffeomorphism f such that the underlying manifold V admits a Seifert fibration with singular fibres. This example will be the composition of the time-1 map of the flow of a gradient-like diffeomorphism with some periodic diffeomorphism commuting with the flow. We shall start by explaining abstractly the construction (for this we need first to describe briefly the topological invariant associated with a gradient-like vector field) before building an explicit example.

6.3.1. The global scheme of the time-1 map X_1 of the flow of a gradient-like vector field

Assume that X is a gradient-like vector field on a closed orientable connected 3-manifold M. There is a compact connected surface embedded in M, transverse to X and dividing M into two handlebodies M_1 and M_2, M_1 containing the sinks and the saddles with one-dimensional unstable manifolds, M_2 containing the sources and the saddles with two-dimensional unstable manifolds. Then the two-dimensional unstable manifolds of the saddles in M_2 cut S along a family $\mathcal{C}^u$ of disjoint simple loops embedded in S. Moreover the complement in S of these circles consists of orbits coming from the sources so that each component of $S \setminus \bigcup_{c \in \mathcal{C}^u} c$ is a sphere minus a finite number of points. In the same way the two-dimensional stable manifolds of the saddles in M_1 induce a family $\mathcal{C}^s$ of disjoint circles in S cutting S in surfaces of genus 0 (i.e. planar). Finally, the circles in $\mathcal{C}^s$ are transverse to the circles in $\mathcal{C}^u$. The conjugacy class of the triple $(S, \mathcal{C}^s, \mathcal{C}^u)$ is the global scheme of X, and it is a complete invariant for the topological equivalence.

Consider the time-1 map X_1 of the flow of X. It is a gradient-like diffeomorphism whose global scheme $(V_1, \alpha_1, \Gamma_1^s, \Gamma_1^u)$ is obtained as follows: V_1 is the product of the surface S by the circle S^1; the cohomology class α_1 is the dual of $S \times \{0\} \subset S \times S^1$; the family Γ_1^s consists of the product of the circles in $\mathcal{C}^s$ by S^1; the family Γ_1^u consists of the product of the circles in $\mathcal{C}^u$ by S^1.

6.3.2. The global scheme of the composition of X_1 by a periodic diffeomorphism commuting with X

Assume now that X is invariant by some periodic (orientation preserving) diffeomorphism $\Phi \colon M \to M$. Let k be the period of f, that is the smallest integer k such that $f^k = id_M$. So Φ acts on the orbits of X, therefore Φ induces a periodic (orientation preserving) diffeomorphism φ on the surface S. The diffeomorphism φ leaves invariant the curves in $\mathcal{C}^s \cup \mathcal{C}^u$. Consider now the composition $f = \Phi \circ X_1$. It is a gradient-like diffeomorphism, whose global scheme $(V, \alpha, \Gamma^s, \Gamma^u)$ is obtained as follows: V is the quotient of $S_1 = S \times S^1$ by the action of the diffeomorphism ϕ defined as $(x, t) \mapsto (\varphi(x), t + \frac{1}{k})$. Notice that ϕ induces a free action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ which preserves the circle-bundle structure of $S \times S^1$. As a consequence, the quotient V of V_1 by ϕ is a Seifert 3-manifold and the projection $\pi \colon V_1 \to V$ is a k-fold cyclic cover. The fibres of V are the projections of the fibres of $V_1 = S \times S^1$, and the singular fibres are the projections of the circles $\{x\} \times S^1$ where the period of x by φ is strictly less than k. The families Γ_1^s and Γ_1^u are invariant by this action and induce the families Γ^s and Γ^u. The surface S projects to a surface embedded in V and α is its dual cohomology class.

6.3.3. An explicit example

Consider the sphere S^3 as being $\mathbb{R}^3 \cup \infty$. Denote by R_θ, $\theta \in \mathbb{R}$ the diffeomorphism of S^3 whose expression in these coordinates is the rotation of angle $2\pi\theta$ around the vertical axis, that is, the linear map whose matrix is

$$\begin{pmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) & 0 \\ -\sin(2\pi\theta) & \cos(2\pi\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote by Z the radial vector field on S^3, that is

$$Z(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Notice that Z is invariant under the rotation R_θ, for every θ.

For a fixed θ denote by C_θ the (positive) cone

$$C_\theta = \left\{(x, y, z) \mid x \geq 0, |y| \leq \tan\left(\frac{\pi}{2}\theta\right) x, |z| \leq \tan\left(\frac{\pi}{2}\theta\right) x\right\}.$$

It is invariant by the flow of Z. Denote by Γ_θ the intersection of the cone C_θ with $\{x^2 + y^2 + z^2 \in [1, e^2]\}$ (a fundamental domain of the time one map of Z). Notice that, for all choices of $q \in \mathbb{N} \setminus \{0\}$ and $i \in \{1, \ldots, q - 1\}$, the box $\Gamma_{\frac{1}{q}}$ is disjoint from $R_{\frac{i}{q}}^i(\Gamma_{\frac{1}{q}})$.

Lemma 6.3. *For any fixed integer $q \in \mathbb{N} \setminus \{0\}$, there is a gradient-like vector field X on the sphere S^3 with the following properties:*
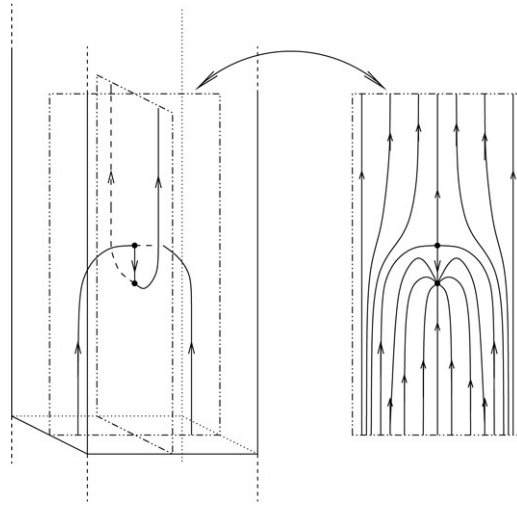
1. *X is invariant by $R_{\frac{1}{q}}$;*

Fig. 1. The vector field X inside the boxes.

2. X coincides with the radial vector field Z outside $\bigcup_{i=0}^{q-1} R_{\frac{1}{q}}^{i}(\Gamma_{\frac{1}{q}})$; as a consequence, X admits a source at $(0, 0, 0)$ and a sink at ∞:

3. for every $i \in \{0, \ldots, q - 1\}$, the vector field Z has exactly two zeros s_i and σ_i in $R_{\frac{1}{q}}^{i}(\Gamma_{\frac{1}{q}})$; moreover s_i and σ_i are saddle points with Morse index 1 and 2, respectively;

4. for every $i \in \{0, \ldots, q - 1\}$, the invariant manifolds $W^s(s_i)$ and $W^u(\sigma_i)$ intersect transversally along exactly one orbit.

Idea of the proof. Notice that the punctured cone $C_{\frac{1}{q}} \setminus \{(0, 0, 0)\}$ is diffeomorphic to $[-1, 1]^2 \times \mathbb{R}$, endowed with the coordinates (x, y, t), by a diffeomorphism whose derivative maps the radial vector field Z on the constant vector field $\frac{\partial}{\partial t}$. Replace this vector field in $\Gamma_{\frac{1}{q}}$ with a vector field of the form illustrated by the picture in Fig. 1. Replace then the vector field in the boxes $R_{\frac{1}{q}}^{i}(\Gamma_{\frac{1}{q}})$ in an $R_{\frac{1}{q}}$-equivariant way (that is, the vector field in the box $R_{\frac{1}{q}}^{i}(\Gamma_{\frac{1}{q}})$ is the image by $R_{\frac{1}{q}}^{i}$ of the vector field in the box $\Gamma_{\frac{1}{q}}$). □

Let $(S, \mathcal{C}^s, \mathcal{S}^u)$ be the global scheme of the vector field X given by Lemma 6.3. Let $c_i \in \mathcal{C}^s$ and $\gamma_i \in \mathcal{C}^u$ be the loops induced by $W^s(s_i)$ and $W^u(\sigma_i)$ respectively, for $i \in \{0, \ldots, q - 1\}$. Then c_i cuts γ_i transversally in exactly one point and $c_i \cap \gamma_j = \emptyset$ for $i \neq j$. This implies that $S \setminus \bigcup_i c_i$ and $S \setminus \bigcup_i \gamma_i$ are connected, and hence are diffeomorphic to the sphere minus $2q$-discs (because $(S, \mathcal{C}^s, \mathcal{C}^u)$ is the global scheme of a gradient-like vector field). One deduces that S is a genus q orientable closed surface. Notice that $R_{\frac{1}{q}}$ acts on S as a diffeomorphism $\varphi_{\frac{1}{q}}$ with exactly two fixed points, corresponding to the two orbits of X composing the rotation axis of $R_{\frac{1}{q}}$. Furthermore $\varphi_{\frac{1}{q}}(c_i) = c_{i+1}$ and $\varphi_{\frac{1}{q}}(\gamma_i) = \gamma_{i+1}$ (see Fig. 2).

Now, choose an integer $p \in \{1, \ldots, q - 1\}$ with $p \wedge q = 1$ and define $\bar{f} = R_{\frac{p}{q}} \circ f$. Then $\bar{f}$ is a gradient-like diffeomorphism of S^3 whose scheme $(V, \alpha, \Gamma^s, \Gamma^u)$ verifies that V is the Seifert fibration over T^2 with two singular fibres having (q, p) and $(q, q - p)$ as invariants (if p and q are not mutually
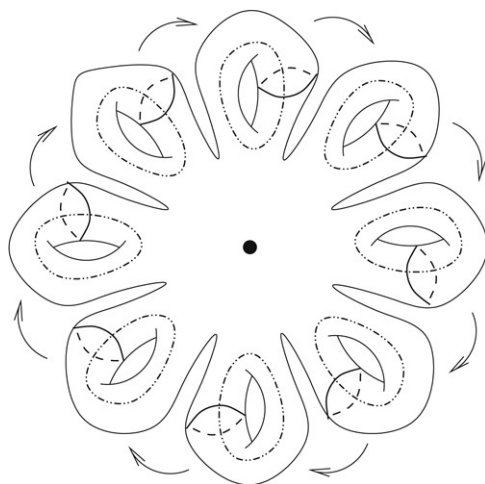
Fig. 2. The global scheme of the vector field X and the action of φ.

prime, then V will be a Seifert fibration over the closed oriented surface of genus $p \wedge q$ having two singular fibres whose invariants are $(\frac{q}{q \wedge p}, \frac{p}{q \wedge p})(\frac{q}{q \wedge p}, \frac{q-p}{q \wedge p}))$.

6.4. An example of a perfect scheme which is not a mapping torus

In this section, motivated by a question of the referee, we try to elucidate which is the relation between the underlying manifolds of perfect schemes and manifolds which are mapping tori.

It is possible to prove the following:

Proposition 6.4. *The mapping torus of a homeomorphism h is the underlying manifold of a perfect scheme if and only if each pseudo-Anosov component of h is planar.*

On the other hand, as a consequence of Theorem 1.1 we have:

Proposition 6.5. *If V is the underlying manifold of a perfect scheme whose JSJ pieces are all Seifert fibred, then V is a mapping torus.*

In the general case, we believe that one can show that *a manifold V is a mapping torus if and only if each hyperbolic piece of its* JSJ *decomposition is the complement of a closed hyperbolic braid in $S^2 \times S^1$.*

The rest of the section will be devoted to show:

Theorem 6.1. *There are infinitely many perfect schemes whose underlying manifolds cannot be mapping tori.*

We shall start by showing that a knot in $S^2 \times S^1$ freely homotopic to the generator of the fundamental group is fibred if and only if it is *trivial* (that is, isotopic to the S^1 fibre). Then we shall notice that the hyperbolic components of a mapping torus admit a surface fibration transverse to the boundary (which consists of a disjoint union of tori). These two remarks imply that each perfect scheme containing a hyperbolic component which is the complement of a non-fibred knot $\gamma \subset S^2 \times S^1$ cannot be a mapping torus. Finally we construct examples of perfect schemes which do contain a hyperbolic component

which is the complement of a non-fibred knot $\gamma \subset S^2 \times S^1$ (freely homotopic to the generator of the fundamental group) concluding the proof.

6.4.1. Knots in $S^2 \times S^1$

A knot in a closed manifold is *fibred* if its exterior admits a surface fibration transverse to the boundary.

Proposition 6.6. *Let $\gamma \subset S^2 \times S^1$ be a knot freely homotopic to $\{x\} \times S^1$, where x is a point of S^2. Then γ is fibred if and only if γ is isotopic to $\{x\} \times S^1$.*

Proof. Let T be the boundary of a small tubular neighbourhood of γ. A meridian of γ is a simple closed curve $\mu \subset T$, non-zero-homotopic in T, and bounding a disc in the tubular neighbourhood. We denote by V_γ the *exterior of γ*, i.e. the complement of the interior of the tubular neighbourhood.

First notice that μ is homologous to 0 in V_γ. For that consider an S^2 fibre S transverse to γ. As γ is freely homotopic to an S^1 fibre, the algebraic intersection $\gamma.S$ is 1. As a consequence, the boundary of $S \cap V_\gamma$ is homologous to μ.

Now assume that γ is fibred, and let $\{F_t\}_{t \in S^1}$ be a fibration of V_γ transverse to T. This fibration induces a circle fibration of T. If the circles of this fibration are not meridians, then the meridian μ can be chosen transverse to the fibration, thus cannot be homologically trivial in V_γ. We proved that $\{F_t\}_{t \in S^1}$ induces a fibration by meridians on T. Henceforth, $\{F_t\}_{t \in S^1}$ can be extended to a fibration of $S^2 \times S^1$. This fibration is the standard S^2 fibration, and γ is transverse to the fibres (hence is a braid). The homotopy of γ implies that this braid is trivial. $\square$

6.4.2. Hyperbolic components of a mapping torus

Given a closed surface S and an element h of the mapping class group of S we denote by M_h the manifold which is the *mapping torus of h* (i.e. the quotient of $S \times \mathbb{R}$ by the equivalence relation generated by $(x, t+1) \sim (\tilde{h}(x), t)$ where $\tilde{h}$ is a diffeomorphism of S in the isotopy class h).

Let h be an element of the mapping class group of a closed oriented surface S. We also denote by h a Nielsen–Thurston representative.

1. If S is the sphere S^2 then $M_h = S^2 \times S^1$.
2. If S is the torus T^2, then the mapping class group of S is canonically identified to $SL(2, \mathbb{Z})$ and we have to consider different cases:
   (a) If h is hyperbolic, then M_h is a Sol manifold and all incompressible tori in M_h are isotopic to the fibre T^2.
   (b) If h is elliptic, then there is an integer $n > 0$ such that $h^n = id$. Furthermore there is a Euclidean metric on T^2 such that h admits a representative which is an isometry of T^2. Hence M_h has a Euclidean metric, and the JSJ family is empty.
   (c) If h is parabolic there is $\mu \in \{-1, 1\}$ and $n \in \mathbb{N}^*$ such that h is conjugate to $\begin{pmatrix} \mu & n \\ 0 & \mu \end{pmatrix}$. In that case M_h is a circle bundle whose basis is either the torus T^2 (if $\mu = 1$) of the Klein bottle (in which case the bundle is not oriented). In both cases, M_h is a unique geometric piece (with Nil geometry).
   In all the above situations, M_h has no hyperbolic components.
3. We are now interested in the case $S \notin \{S^2, T^2\}$.
   (a) If h is pseudo-Anosov then M_h is hyperbolic (this is part of the hyperbolisation theorem due to Thurston; see [16]).

(b) If h is periodic, then M_h has an $\mathbb{H}^2 \times \mathbb{R}$ geometric structure: the JSJ decomposition is trivial.

(c) If h is reducible, consider a reduction system $\{c_i\}$ for h. This is a finite family of disjoint simple closed essential curves, invariant under the action of h and such that the first return map of h on each component of its complement is isotopic either to a pseudo-Anosov homeomorphism or to a periodic one. Each orbit of circle c_i induces an incompressible torus inside M_h. The complement $\hat{M}_h$ of these tori is a finite union of geometric pieces which are either hyperbolic or Seifert fibred. More precisely, each component M_j of $\hat{M}_h$ corresponds to an orbit of a component S_k of $S \setminus \bigcup_i c_i$. Then M_j is hyperbolic if S_k is a pseudo-Anosov component and Seifert fibred otherwise.

The uniqueness of the JSJ decomposition implies that the hyperbolic pieces of M_h are those described above. Hence they admit a surface fibration transverse to the boundary, induced by the canonical fibration of the mapping torus.

6.4.3. An obstruction for a perfect scheme to be a mapping torus

It is a straightforward consequence of the previous section that an irreducible manifold V containing a hyperbolic component C which is not fibred by surfaces cannot be a mapping torus. That is the case, according to the first section, if C is the exterior of a hyperbolic knot of $S^2 \times S^1$ homotopic to a generator of the fundamental group of $S^2 \times S^1$.

6.4.4. An example

Let $\gamma \subset S^2 \times S^1$ be a hyperbolic knot which is homotopic to the positive generator of the fundamental group of $S^2 \times S^1$. There are infinitely many such knots, according to a result of Myers [14].

We consider the manifold V obtained as the double of the exterior V_γ of γ: V_γ is a compact manifold whose boundary is a torus $T_\gamma \simeq T^2$; the manifold V is obtained by gluing two copies of V_γ along their boundary by the identity map.

Notice that the cohomology group $H^1(V, \mathbb{Z})$ contains a class α which induces on each copy of V_γ the class induced by restriction of the generator of $H^1(S^2 \times S^1, \mathbb{Z})$. We denote by T^s and T^u two disjoint tori parallel to $T_\gamma = \partial V_\gamma$. Notice that the class α does not induce the trivial class on T^s and T^u. Hence, $(V, \alpha, \{T^s\}, \{T^u\})$ is a formal scheme, according to Definition 2.1. Furthermore, the scheme $(V, \alpha, \{T^s\}, \{T^u\})$ is a perfect scheme (see Definition 2.2), by construction.

Finally notice that T_γ is incompressible in V and cuts V in two hyperbolic components diffeomorphic to the interior of V_γ. According to the previous section, V is not a mapping torus.

Let just show that T_γ is incompressible in V_γ (hence in V according to the Dehn lemma). Otherwise, using the Dehn lemma, there is a disc embedded in V_γ whose boundary is an essential simple curve of T_γ. Making a surgery along the disc, one obtains a sphere Σ contained in V_γ. As V_γ is hyperbolic, Σ bounds a ball. Hence V_γ is a solid torus, which is impossible.

Acknowledgements

References

[1] F. Béguin, Classification des difféomorphismes de Smale des surfaces: Types géométriques réalisables, Ann. Inst. Fourier (Grenoble) 52 (4) (2002) 1135–1185.

[2] C. Bonatti, V.Z. Grines, Knots as topological invariants for gradient-like diffeomorphisms of the sphere S^3, J. Dyn. Control Syst. 6 (4) (2000) 579–602.

[3] C. Bonatti, V. Grines, V. Medvedev, E. Pécou, Topological classification of gradient-like diffeomorphisms on 3-manifolds, Topology 43 (2) (2004) 369–391.

[4] C. Bonatti, R. Langevin, Difféomorphismes de Smale des surfaces, Astérisque 250 (1998) viii+235. With the collaboration of E. Jeandenans.

[5] G. Fleitas, Classification of gradient-like flows in dimensions two and three, Bol. Soc. Brasil. Mat. 6 (2) (1975) 155–183.

[6] V. Grines, Topological classification of Morse–Smale diffeomorphisms with a finite set of heteroclinic trajectories on surfaces, Mat. Zametki 54 (3) (1993) 3–17, 157; Translation in Math. Notes 54 (3–4) (1993–1994) 881–889 (in Russian).

[7] W. Haken, Some results on surfaces in 3-manifolds, in: Studies in Modern Topology, in: Math. Assoc. Amer., Prentice-Hall, Englewood Cliffs, NJ, 1968, pp. 39–98.

[8] J. Hempel, 3-manifolds, in: Ann. of Math. Studies, vol. 86, Princeton University Press, University of Tokyo Press, Princeton, NJ, Tokyo, 1976.

[9] W. Jaco, Lectures on three-manifold topology, in: Conference Board of the Mathematical Sciences, in: Regional Conference Series in Mathematics, vol. 43, AMS, Providence, Rhode Island, 1980.

[10] W. Jaco, P. Shalen, Seifert fibered spaces in 3-manifolds, Mem. Amer. Math. Soc. 21 (220) (1979) viii+192.

[11] K. Johannson, Homotopy equivalences of 3-manifolds with boundaries, in: Lecture Notes in Mathematics, vol. 761, Springer, Berlin, 1979, p. ii+303.

[12] H. Kneser, Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten, Jahresber. Deutsch. Math.-Verein. 38 (1929) 248–260.

[13] R. Langevin, Quelques nouveaux invariants des diffómorphismes Morse–Smale d'une surface, Ann. Inst. Fourier (Grenoble) 43 (1) (1993) 265–278.

[14] R. Myers, Excellent 1-manifolds in compact 3-manifolds, Topology Appl. 49 (2) (1993) 115–127.

[15] P. Orlik, Seifert manifolds, in: Lecture Notes in Mathematics, vol. 291, Springer-Verlag, Berlin, New York, 1972.

[16] J.-P. Otal, Le thèoréme d'hyperbolisation pour les variétés fibrées de dimension 3, Astérisque 235 (1996) x+159.

[17] M.M. Peixoto, On the classification of flows on 2-manifolds, in: Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, New York, 1973, pp. 389–419.

[18] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (5) (1983) 401–487.

[19] W. Thurston, Geometry and topology of 3-manifolds, in: Lecture Notes, Princeton University, 1978.