On groups with Schottky set boundary

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Abstract

We study relatively hyperbolic group pairs whose boundaries are Schottky sets. We characterize the groups that have boundaries where the Schottky sets have incidence graphs with 1 or 2 components.

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1 Introduction

Convergence group actions on the 2-sphere were introduced by Gehring and Martin in [12] and have been studied extensively since then. It was conjectured in [21] that every faithful convergence group action G on S^2 by orientation preserving homeomorphisms is *covered* by the induced action of a discrete group of $\mathrm{Isom}(\mathbb{H}^3)$ on S^2, i.e., there exist a Kleinian group K, an isomorphism $\rho : K \to G$ and a degree 1 map $\phi : \widehat{\mathbb{C}} \to S^2$ such that the following diagram commutes :

$$\begin{array}{ccccc} K & \curvearrowright & \widehat{\mathbb{C}} & \longrightarrow & \widehat{\mathbb{C}} \\ \big\downarrow\rho & & \big\downarrow\phi & & \big\downarrow\phi \\ G & \curvearrowright & S^2 & \longrightarrow & S^2 \end{array}$$

This remains open. This conjecture is closely related to Cannon's conjecture [8], which asserts that a hyperbolic group with 2-sphere boundary is virtually a discrete group of $\mathrm{Isom}(\mathbb{H}^3)$. Cannon's conjecture is a particular case of the previous one when the action is faithful on its boundary and orientation preserving. Here we are dealing with the case of relatively hyperbolic groups, and specifically those whose boundaries are *topological Schottky sets*. These are defined and described in Section 5. Some familiar examples are the Sierpiński carpet and the Apollonian gasket. Our motivation for studying those groups is essentially twofold. Firstly, there are many examples of groups that admit a peripheral structure for which their boundary is a topological Schottky set, cf. Theorem D. Secondly, the relatively hyperbolic groups with these boundaries are all conjectured to be virtually discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$, see [19]. Here we show that many relatively hyperbolic groups with boundaries that are topological Schottky sets are virtually discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$. Furthermore, we say which Kleinian groups arise when the boundaries are certain types of topological Schottky sets.

We say that $(G, \mathcal{P})$ is a *relatively hyperbolic group pair* if $(G, \mathcal{P})$ acts as a geometrically finite convergence group on a hyperbolic space X. See section 2 for the detailed definition. In this case, we say that the Gromov boundary of X, ∂X, is the Bowditch boundary of $(G, \mathcal{P})$, denoted $\partial(G, \mathcal{P})$. We also call $\partial(G, \mathcal{P})$ the *relatively hyperbolic boundary* or sometimes just "the boundary". Throughout, $(G, \mathcal{P})$ is a non-elementary relatively hyperbolic group pair (besides Proposition 3.3 where the classification of elementary convergence groups acting on S^2 is provided), which means that $\partial(G, \mathcal{P})$ has more than two points.

Following [1], we define a *Schottky set* as the complement of at least three disjoint open round balls in the n-sphere S^n, where S^n is equipped with the standard metric as a subset of Euclidean space. Throughout this paper, we will restrict ourselves to $n = 2$, so all our Schottky sets are planar. We actually work with the non-metric analog, which we call *topological* Schottky sets.

Due to the properties of a topological Schottky set $\mathcal{S}$ in Definition 5.1, every $\mathcal{S}$ produces an *incidence graph* $\Gamma(\mathcal{S})$, the simplicial graph whose vertices correspond to the open disks $\{D_i\}_{i\in I}$ of its complement in S^2, and whose edges correspond to (1-point) incidences between closures of the disks D_i.

Our main results are as follows:

Theorem A. *Let $\mathcal{S}$ be a topological Schottky set with $\mathcal{S} = \partial(G, \mathcal{P})$. Then the incidence graph $\Gamma(\mathcal{S})$ has 1, 2 or infinitely many components. Their stabilizers are virtual surface groups.*

Theorem B. *Let $\mathcal{S}$ be a topological Schottky set with $\mathcal{S} = \partial(G, \mathcal{P})$.*

When the incidence graph $\Gamma(\mathcal{S})$ has one component, then G is virtually a free product of a free group F_n of rank $n \geq 0$ and some finite index subgroups of groups in $\mathcal{P}$. Moreover, if G is finitely generated, its action is faithful and orientation preserving, then G is covered by a geometrically finite Kleinian group K.

From a topological viewpoint, K contains a finite-index torsion-free subgroup that uniformizes a 3-manifold obtained by gluing together along compression disks a handlebody and I-bundles over surfaces.

Theorem C. *Let $\mathcal{S}$ be a topological Schottky set with $\mathcal{S} = \partial(G, \mathcal{P})$. When the incidence graph $\Gamma(\mathcal{S})$ has exactly 2 components, G is virtually a closed surface group.*

In contrast, when the incidence graph has infinitely many components, then the group is covered by a geometrically finite convergence group that may have a Sierpiński carpet boundary. Showing that these are essentially Kleinian is still a wide open question, even in the word hyperbolic case, cf. [20]. Note that Theorem D below enables us to construct examples of Schottky limit sets that have infinitely many components of their incidence graphs but that do not come from a Sierpiński carpet limit set. For example, apply the theorem to a geometrically finite Kleinian group that contains a rank-2 accidental parabolic fixed point (see for instance the first example in [6], illustrated by Figure 6 therein). So far, all the examples we know of with Sierpínski carpet boundary are virtually fundamental groups of hyperbolic 3-manifolds with totally geodesic boundary (which may have cusps), and this is consistent with conjectures in [20] and [19].

Theorem D. *Let K be a geometrically finite Kleinian group with non-empty domain of discontinuity. Then there is a peripheral structure $\mathcal{P}_{K'}$ on a finite index subgroup*

K' of K, such that $(K', \mathcal{P}_{K'})$ is a relatively hyperbolic group pair and $\partial(K', \mathcal{P}_{K'})$ is a topological Schottky set. Moreover, $\mathcal{P}_{K'}$ contains the natural peripheral structure of the Kleinian group $K' \subset K$.

In Section 2 we prove some general facts about relatively hyperbolic groups, generalizing some theorems about hyperbolic boundaries to relatively hyperbolic boundaries. In Section 3 we restrict to relatively hyperbolic groups that are geometrically finite convergence groups acting on S^2. Although the results in this section will be used later in the context of topological Schottky sets, they do not only apply to this specific context. In Section 4 we describe how to "blow up" 2-ended peripheral subgroups in geometrically finite groups acting on S^2. This will change the peripheral structure, but not the group; moreover the group with its new peripheral structure is shown to admit again a convergence action on the 2-sphere. In Sections 5 and 6 we introduce and discuss topological Schottky sets and their incidence graphs, and prove Theorem A. Finally in Section 7 we prove Theorem B, and in Section 8 we prove Theorems C and D.

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2 Relative hyperbolicity and relative quasiconvexity

Here we provide some results about general relatively hyperbolic groups and their boundaries. References on metric spaces in the sense of Gromov include [13, 5]. Let G be a finitely generated group and a family $\mathcal{P}$ of subgroups consisting of finitely many conjugacy classes.

Let us first recall that a *convergence group* G is a group of homeomorphisms of a compact metric space Z such that any sequence $(g_n)_n$ of distinct elements contains a convergent subsequence, i.e., up to a subsequence, there are two points a and b in Z so that (g_n) tends uniformly to the constant map a on compact subsets of $Z \setminus \{b\}$. One may then define the limit set Λ_G as the set of limit points a of all convergence sequences in G. It is a compact invariant subset of Z. Its complement, Ω_G, is the ordinary set: the action of G on Ω_G is properly discontinuous, see [12] for more properties. Note that any discrete group of isometries on a geodesic, proper, hyperbolic space X admits a convergence action on $X \cup \partial X$.

Definition 2.1 ([4])**.** The pair $(G, \mathcal{P})$ is *relatively hyperbolic* if G acts on X properly discontinuously and by isometries, where X is a proper hyperbolic geodesic metric space such that:

1. each point of ∂X is either a *conical limit point* or a *bounded parabolic point*.
2. $\mathcal{P}$ is exactly the collection of maximal *parabolic subgroups*.

A conical limit point is a point $y \in \partial X$ such that there exists a sequence (g_i) in G and distinct points $a, b \in \partial X$, such that $g_i(y) \to a$ and $g_i(z) \to b$, for all $z \in \partial X \setminus \{y\}$. A parabolic point y_P is a point with an infinite stabilizer that fixes no other point, i.e., the fixed point of a parabolic subgroup P. It is bounded if $(\partial X \setminus \{y_P\})/P$ is compact. Whenever we have a properly discontinuous action by isometries and these two conditions are satisfied, we say $(G, \mathcal{P})$ acts *geometrically finitely* on X. If $(G, \mathcal{P})$ is a relatively hyperbolic pair, then $\partial(G, \mathcal{P}) = \partial X$ is its *Bowditch boundary*, or *relatively hyperbolic boundary*. This depends on $\mathcal{P}$, but is well-defined for the pair $(G, \mathcal{P})$.

As we will be using topological properties of Bowditch boundaries, we recall two topological notions that will be used several times.

Definition 2.2 (Null sequences and E-sets)**.** Given a compact metric space Z, a *null-sequence* is a collection of subsets $\mathcal{C}$ such that, for any $\delta > 0$, the collection $\mathcal{C}$ contains at most finitely many elements of diameter at least δ.

An *E-set* is a connected compact subset of the sphere S^2 such that the collection of connected components of its complement is a null-sequence.

Proposition 2.3. *Let $(G, \mathcal{P})$ be relatively hyperbolic.*

1. *If K is the limit set of a relatively quasiconvex subgroup, then the set of elements in the orbit GK forms a null-sequence.*
2. *Let $\mathcal{C}$ be a G-invariant collection of compact subsets of $\partial(G, \mathcal{P})$ which defines a null-sequence, where each element of $\mathcal{C}$ contains more than one point. Then $\mathcal{C}/G$ is finite and, for any perfect set $K \in \mathcal{C}$, Stab(K) is a relatively quasiconvex subgroup with limit set K.*

Proof. Let us first consider a geometrically finite action of the group G on a proper geodesic hyperbolic metric space X so that the stabilizers of the parabolic points are the elements of $\mathcal{P}$. We may then identify ∂X with $\partial(G, \mathcal{P})$ and endow it with a visual distance seen from a base point $o \in X$.

Let H be a relatively quasiconvex subgroup of the relatively hyperbolic group $(G, \mathcal{P})$. We will prove that the orbit of its limit set $\Lambda_H = K$ forms a null sequence. See [14, Corollary 2.5] for the hyperbolic case.

Fix $\delta > 0$ and let $R > 0$ denote the upper bound on the distances from the origin o to any geodesic joining points δ-apart in the boundary. Let us pick a G-invariant collection of horoballs $\mathcal{H}$ in X centered at the set of parabolic points in such a way that they are pairwise disjoint and that their distance to o is at least $R+1$ (by shrinking if necessary). By abuse of notation, we will also let $\mathcal{H}$ denote the union of the horoballs of the collection. Let $\mathcal{C}$ denote the set of translates $g(K)$ of diameter at least δ and assume that $K \in \mathcal{C}$. Since H is relatively quasiconvex, there is some $q > 0$ so that, for any geodesic γ joining two points in K, $\gamma \cap (X \setminus \mathcal{H})$ is contained in the q-neighborhood of Ho, [18, Definition 6.6]. If $L = g(K) \in \mathcal{C}$, then we may find a geodesic γ at distance at most R from o and such that $g^{-1}(\gamma)$ is in the q-neighborhood of Ho outside $\mathcal{H}$. Since the horoballs are at distance at least $R+1$ from o, we may find a point of γ at distance at most R from the origin and at distance at most q from gHo. Thus, there exists $g_L \in gH$ such that $g_L(o) \in B(o, R+q)$. Since the action of G is properly discontinuous, there are finitely many elements $g \in G$ with $g(o) \in B(o, R+q)$, hence finitely many $L \in \mathcal{C}$. This shows that GK is a null-sequence.

We now establish point 2. Let $m > 0$ be such that any distinct pair of points of $\partial(G, \mathcal{P})$ can be m-separated by an element of G, i.e., for any $x, y \in \partial(G, \mathcal{P})$, $x \neq y$, there is some $g \in G$ such that $d(g(x), g(y)) \geq m$. Such m exists since the action on the set of distinct pairs is co-compact, see [31]. Given $\delta > 0$, we let $\mathcal{C}_\delta$ denote the subset of elements K of $\mathcal{C}$ such that $\operatorname{diam} K \geq \delta$; this set is finite since $\mathcal{C}$ is a null-sequence and non-empty for small enough $0 < \delta \leq m$.

For all $K \in \mathcal{C}$, we can find two points $x_1, x_2 \in K$ and a group element $g \in G$ such that $\{g(x_1), g(x_2)\}$ is m-separated: this implies that $g(K) \in \mathcal{C}_m$, so that $\mathcal{C}$ is composed of finitely many orbits.

Let $K \in \mathcal{C}$ be a perfect compact set. Since $G_K = \mathrm{Stab}\ (K)$ is a subgroup of G, its action on the set of distinct triples of K is automatically properly discontinuous. Let us prove that it is also geometrically finite.

Let $x, y \in K$, $x \neq y$, and assume that x is conical for G. Let (g_n) be a sequence of G such that $(g_n(x))_n$ tends to a and $(g_n(y))_n$ tends to $b \neq a$. This means that for all n large enough $\mathrm{diam}\, g_n(K)$ is larger than some constant $\delta > 0$ (for instance $\delta = d(a,b)/2$) so belongs to a finite subcollection of $\mathcal{C}$. Extracting a subsequence if necessary, we may assume that $g_n(K) = L$ for some $L \in \mathcal{C}$. It follows that $h_n = g_1^{-1} g_n$ defines a sequence of G_K such that $(h_n(x))$ tends to $g_1^{-1}(a)$ and $(h_n(y))$ tends to $g_1^{-1}(b)$ for all other points y. This means that x is conical for G_K.

If $x \in K$ is parabolic, denote by G_x its stabilizer and let L be a compact fundamental domain for the action of G_x on $\partial(G, \mathcal{P}) \setminus \{x\}$. We first prove that $G_x \cap G_K$ is infinite, establishing that x is a also a parabolic point for G_K. Since x is non-isolated in K, we may find a sequence $(x_n)_n$ in K which tends to x and a sequence (g_n) in G_x so that $g_n(x_n) \in L$. The collection $(g_n)_n$ is infinite and $\mathrm{diam}\, g_n(K)$ is at least $d(x, L) > 0$ so belongs to a finite subcollection $\mathcal{C}_L$. Extracting a subsequence if necessary, we may assume that $g_n(K)$ is a fixed compact subset so that $(g_1^{-1} g_n)_n$ is an infinite sequence in $G_x \cap G_K$. We will now prove that x is also bounded as a parabolic point for G_K. Let us label the elements of $\mathcal{C}_L$ by $\{K_1, \dots, K_N\}$ and let us fix, for each index $j \in \{1, \dots, N\}$, an element $h_j \in G_x$ such that $h_j(K) = K_j$. Set $L_K = \cup_{1 \leq j \leq N} h_j^{-1}(L)$ that is compact in $\partial(G, \mathcal{P}) \setminus \{x\}$. For any $y \in K \setminus \{x\}$, we may find $g \in G_x$ so that $g(y) \in L$; note that $g(y) \in K_j$ for some $j \in \{1, \dots, N\}$, implying that $h_j^{-1} g(y) \in L_K$. This shows that x is a bounded parabolic point. Thus, any point in K is either conical or bounded parabolic for G_K, so that G_K is geometrically finite with limit set K. $\square$

We observe now that a collection of compact sets forms a null-sequence if it is finite so, in particular, if it contains a single element. If the Bowditch boundary of a relatively hyperbolic group consists of more than one component, then Bowditch showed that the group must split. More precisely the following holds.

Theorem 2.4. *[4, Theorem 10.1] The boundary $\partial\Gamma$ of a relatively hyperbolic group, Γ, is connected if and only if Γ does not split non-trivially over any finite subgroup relative to peripheral subgroups.*

In the case where the group splits, we have the following description which is again due to Bowditch.

Theorem 2.5. *[4, Theorem 10.3] Suppose a relatively hyperbolic group pair splits as a graph of groups with finite edge groups and relative to the peripheral subgroups. Then each vertex group is hyperbolic relative to the peripheral subgroups that it contains and its boundary is naturally identified as a closed subset of the boundary of the whole group.*

The following proposition is an immediate consequence of our Proposition 2.3 and the above discussion and results by Bowditch.

Proposition 2.6. *The set of components of the Bowditch boundary of a relatively hyperbolic group $(G, \mathcal{P})$ forms a null-sequence. Moreover, for each non-trivial component, the stabilizer is hyperbolic relative to conjugates of the original peripheral subgroups $\mathcal{P}$.*

While the boundary of a relatively hyperbolic group is not always connected and sometimes contains cut points, the structure of cut points allows us to rule out a dendrite boundary:

Lemma 2.7. *Let $(G, \mathcal{P})$ be a geometrically finite convergence group. Then $\partial(G, \mathcal{P})$ is not a dendrite.*

Recall that a *dendrite* is a connected, locally connected, compact metric space containing at least two points that admits no simple closed curve.

Proof. According to [10, Theorem 1.1], every cut point of $\partial(G, \mathcal{P})$ is a parabolic point. This readily implies that there are at most countably many cut points in $\partial(G, \mathcal{P})$. To reach the desired conclusion, we shall show that a dendrite contains an open path of cut points. To see this, let L be a dendrite. Then L is path connected according to [33, II.5.1], since it is a locally connected complete metric space. Let x and y be distinct points in L and p a path between them. Remove a point z on p. If $z \in L$ is a not a cut point, $L \setminus \{z\}$ is connected. Since $L \setminus \{z\}$ is connected and locally compact, i.e. a *generalized continuum*, the fact that it is locally connected implies that it is path-wise connected [33, II.5.2]. Thus there is another path p' from x to y that misses z. Then the set $\{r \in [0, 1] \mid p'(r) \in p([0, 1])\}$ is closed in $[0, 1]$ and not all of $[0, 1]$ so its complement contains an open interval, and this gives us a loop in L, which is absurd. $\square$

Lemma 2.7 can also be derived using Theorem 1.2 of [10].

The statement of the next proposition is due to Susskind and Swarup for geometrically finite Kleinian groups [29, Thm 3]. The same argument applies verbatim to relatively hyperbolic groups.

Proposition 2.8 (Susskind and Swarup)**.** *Let $(G, \mathcal{P})$ be relatively hyperbolic and H, K be two relatively quasiconvex subgroups. Then $H \cap K$ is relatively quasiconvex and $\Lambda_H \cap \Lambda_K = \Lambda_{H \cap K} \cup P$ where P is a (possibly empty) discrete set of common parabolic points.*

Together with Theorem 2.5 above, we will rely on one more result regarding splittings, again by Bowditch.

Theorem 2.9. *[4, Theorem 10.2] Any relatively hyperbolic group pair can be expressed as the fundamental group of a finite graph of groups with finite edge groups and with every peripheral subgroup conjugate into a vertex group, with the property that no vertex group splits non-trivially over any finite subgroup relative to the peripheral subgroups.*

We obtain in this way

Corollary 2.10. *Suppose $(G, \mathcal{P})$ is a relatively hyperbolic group pair and $\partial(G, \mathcal{P})$ is a Cantor set. The group G is the fundamental group of a finite graph of groups where all the edge groups are finite, and each vertex group is either finite or a peripheral group.*

Proof. We apply Theorem 2.9 to express G as the fundamental group of a finite graph of groups with finite edge groups and with every peripheral subgroup conjugate into a vertex group, with the property that no vertex group splits non-trivially over any finite subgroup relative to the peripheral subgroups. Theorem 2.4 tells us that this graph of groups is non-trivial since the boundary is disconnected. Since it is totally disconnected, Theorem 2.5 implies that a vertex group is the stabilizer of a point or trivial, so each vertex group is either conjugate to a peripheral subgroup or finite. $\square$

Theorem 2.11. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group pair with $\partial(G, \mathcal{P})$ homeomorphic to a Cantor set. Assume that for all $P \in \mathcal{P}$, P is residually finite. Then G is virtually $F * (*_1^n P_i)$ where F is free (possibly of rank 0) and each P_i is a finite index subgroup of some $P \in \mathcal{P}$.*

Theorem 2.11 follows from Corollary 2.10 and

Theorem 2.12. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group pair, such that G can be written as a finite graph of groups, where every edge group is finite and each vertex group is either finite or a peripheral group. Assume that each peripheral group is residually finite. Then G is virtually the free product of a free group and finite index subgroups of peripheral groups.*

Before proving Theorem 2.12 we set some notation.

We will express a splitting of a group in terms of an action of the group on a simplicial tree with finite edge stabilizers and without edge inversions. A splitting is said to be *relative* to a certain collection of subgroups if every subgroup in this collection fixes a vertex of the tree. It is *non-trivial* if no vertex of the tree is fixed by the whole group.

Given a group G with an action on a simplicial tree T with no edge inversions, we let $\Gamma = T/G$ be the orbit space. For each vertex v of Γ we may consider a vertex group G_v defined as the stabilizer of a representative of the vertex in v. In the same manner, we define edge groups G_e for edges. The action of G on T provides us with injective maps $\phi_{0,e} : G_e \to G_v$, $\phi_{1,e} : G_e \to G_v$ defined whenever $e(0)$ or $e(1)$ is v.

We say the tuple $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, \{\phi_{\epsilon,e}\})$ is a graph of groups, and G is the *fundamental group of the graph of groups $\mathcal{G}$*. The set of generators of G is the union of the sets of generators for all the G_e and the G_v, together with a set contaning a generator t_e for each edge of Γ. The relations are all the relations in each G_e and G_v, $t_e = 1$ if e is in a fixed maximal tree, $t_e^{-1} = t_{\bar{e}}$ and $t_e \phi_{0,e}(x) t_e^{-1} = \phi_{1,e}(x)$ for all $x \in G_e$.

Let G be the fundamental group of a graph of groups $\mathcal{G}$ with underlying graph Γ. Suppose further, as in the hypotheses of Theorem 2.12, that $\mathcal{P}$ is a collection of subgroups of G where each subgroup is residually finite, each edge group is finite, and each vertex group is either finite or a subgroup in $\mathcal{P}$.

Proof of Theorem 2.12. We will map G to the fundamental group of a graph of groups G', over the same graph Γ but where the vertex groups and edge groups are all finite. For each infinite vertex group G_v, conjugate to some $P \in \mathcal{P}$, there are finitely many edges meeting the vertex v. Since P is residually finite, there is a map $\psi_v : G_v \to C_v$ onto a finite group C_v which is injective on the union of the images $\phi_{\epsilon,e}(G_e)$ where $e(\epsilon) = v$. We will define G' as the fundamental group of $\mathcal{G}' = (\Gamma, \{G'_v\}, \{G_e\}, \{f_{\epsilon,e}\})$ where

- $G'_v = G_v$ if G_v is finite, and $G'_v = C_v$ if G_v is infinite.
- $f_{\epsilon,e} = \psi_v \circ \phi_{\epsilon,e} : G_e \to C_v$ if $G_{e(\epsilon)}$ is infinite, and $f_{\epsilon,e} = \phi_{\epsilon,e}$ if $G_{e(\epsilon)}$ is finite.

The group G admits a natural surjection to G'. Furthermore, G' admits a surjection to a finite group which is injective on every edge and vertex group of G', by Scott and Wall [28, Chapter 7]. Then the composition of these two maps is a map from G to a finite group which is injective on every finite vertex group and every edge group. The kernel H of this composition is a finite index subgroup of G which acts on the same tree as G but with trivial edge groups. Thus, we see H as the fundamental group of a finite graph of groups where the edge groups are trivial and the vertex groups are either finite or finite index subgroups conjugate to peripheral subgroups of G. This implies that H is the free product of a free group and finite index subgroups of peripheral groups. $\square$

3 Geometrically finite convergence groups acting on S^2

A relatively hyperbolic group pair $(G, \mathcal{P})$ can have a planar boundary where the action does not extend to S^2; see, for example [20, Section 9], where the group G is hyperbolic and virtually Kleinian. The group G need not be virtually Kleinian for $\partial(G, \mathcal{P})$ to be planar, though, and its peripheral subgroups can be arbitrary [19]. Here we collect some general results on geometrically finite convergence groups on S^2, which will be used for the more specific case of Schottky sets which we study here.

Let G be a convergence group acting on S^2 with limit set $\Lambda = \Lambda_G \subset S^2$. A relatively hyperbolic group pair $(G, \mathcal{P})$ is a *geometrically finite convergence group on* S^2 if every point of Λ is either a bounded parabolic point (with maximal parabolic group in $\mathcal{P}$) or a conical limit point. We are not in general assuming that the action is faithful: there could be a finite normal subgroup of G which acts as the identity on S^2. When we know that the quotient by this finite normal subgroup is virtually a 2 or 3-manifold group, there is a finite index subgroup of G which acts as a subgroup of $Homeo(S^2)$, by [16, Theorem 1.3]. In what follows we will be analyzing the quotients by the finite normal subgroup, and the results in general will be virtual.

Lemma 3.1. *An infinite-order, orientation-preserving parabolic element of a geometrically finite convergence group on S^2 is conjugate to a translation.*

Proof. Let $g \in G$ be parabolic with fixed point $p \in S^2$. Its restriction to $S^2 \setminus \{p\}$ is fixed-point free and its action is properly discontinuous. Hence, $(S^2 \setminus \{p\})/\langle g \rangle$ is a surface with cyclic fundamental group and so homeomorphic to a cylinder. This implies that the action of g is conjugate to that of a translation. $\square$

Proposition 3.2. *Let $\partial(G, \mathcal{P})$ be a geometrically finite convergence group on S^2 with G finitely generated. Then each $P \in \mathcal{P}$ is a virtually finite type surface group, that is virtually free of rank at least 1 or virtually a closed surface group.*

Proof. Any maximal parabolic subgroup P is finitely generated, since we are assuming that G is finitely generated by [26, Prop. 2.29]. Since P also acts properly on $\mathbb{R}^2$, this is exactly [19, Cor. 3.2]. The proof uses [16, Thm 1.3] in the case that there is a finite normal subgroup. $\square$

Elementary action.— A convergence action on a compact metrizable space is *elementary* if its limit set is finite, i.e., contains at most two points. Such actions are classified on the sphere, cf. [21, Theorem 3.4, Lemma 4.2].

Proposition 3.3. *Let G be a finitely generated subgroup of $Homeo^+(S^2)$ (the orientation preserving homeomorphisms of S^2) which is an elementary convergence group.*

Then its action is conjugate to a subgroup of Möbius transformations. More precisely,

1. *If $\Lambda_G = \emptyset$, then its action is conjugate to that of a finite subgroup of $SO_3(\mathbb{R})$;*

2. *If Λ_G is a singleton, then G contains a finite index subgroup whose action is conjugate to that of a Fuchsian group that defines a surface of finite type. In particular, if G is two-ended, then G is either isomorphic to $\mathbb{Z}$ or to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.*
3. *If Λ_G is a pair of points, then G has an index 1 or 2 subgroup G' which is Abelian, and the action of G' is conjugate to that of $\langle z \mapsto 2z, z \mapsto \zeta z \rangle$, with $\zeta^n = 1$ for some $n \in \mathbb{N}$.*

Proof. If the limit set is empty, then the action of G on S^2 is properly discontinuous and cocompact so that S^2/G is naturally equipped with a good spherical orbifold structure. In other words, its action is conjugate to that of a finite subgroup of $SO_3(\mathbb{R})$.

Let us now assume that the limit set consists of a single point p. It must be parabolic so Proposition 3.2 implies that it is virtually a surface group of finite type.

Let us now assume furthermore that G is two-ended. Following [28, Theorem 5.12], we consider a finite normal subgroup F of G. Note that, as F is normal and finite, we can find an infinite order element g in G that centralizes it. This implies that F also fixes the point p, and, by the previous case, F has to be a finite cyclic group. Let us assume that F is non trivial and let q be the other fixed point under F. Since g has infinite order, it acts as a translation by Lemma 3.1. Thus we should have $g^n f g^{-n}(q) = f(q) = q$ for all $n \in \mathbb{Z}$ and $f \in F$. However, this shows that f fixes infinitely many points and cannot be non-trivial, so that we may conclude that F is trivial and that G is isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ by [28, Theorem 5.12 (iii)].

We now assume that Λ_G has two points, so G is two-ended. Now take the subgroup G' of G which fixes pointwise Λ_G. This is a subgroup of index at most two. As above, we consider a finite normal subgroup F of G'. Since $F < G'$ fixes $\Lambda_G = \{p, q\}$ pointwise and is finite, F has to be a rotation group, i.e., a finite (cyclic) subgroup of $SO_2(\mathbb{R})$. Since the action is properly discontinuous, cocompact and free on $S^2 \setminus \{p, q\}$, the quotient by G' is a torus. If $G' \neq G$, $G = G' \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts dihedrally on G'. The result follows.

□

The following is immediate from [10] and in the case when the peripheral groups are tame from previous work [2, Thm 0.1] and [3, Thm 0.2].

Corollary 3.4. *Let $\partial(G, \mathcal{P})$ be a geometrically finite convergence group acting on S^2. Then every cut point of a component is a parabolic point. Furthermore, the components of the limit set are all locally connected.*

Lemma 3.5. *Let $(G, \mathcal{P})$ be a geometrically finite convergence group on S^2 with connected Bowditch boundary $\partial(G, \mathcal{P}) = \Lambda$. The ordinary set $S^2 \setminus \Lambda$ is made of finitely many orbits, each with stabilizer which is a 2–orbifold group. When G is finitely generated, these are finite-type surface groups.*

Proof. We start by observing that the compact connected set Λ is also locally connected by Corollary 3.4. According to [33, Theorem VI.4.4], local connectivity of Λ assures that the components of the ordinary set $S^2 \setminus \Lambda$ form a null-sequence (Λ is an E-set) and that the boundary of each component is locally connected. Moreover, each component Ω is simply connected. This follows from the fact that every simple closed curve contained in the surface Ω separates the sphere into two disks, one containing the connected set $\Lambda \supset \partial\Omega$ and the other contained in Ω.

If $S^2 \setminus \Lambda$ has only finitely many components, then G contains a finite index subgroup that stabilizes each component, i.e., a relatively quasiconvex subgroup. Of course, in this case $S^2 \setminus \Lambda$ is made of finitely many orbits.

If $S^2 \setminus \Lambda$ has infinitely many components forming a null sequence, we may apply part 2 of Proposition 2.3 to conclude that $S^2 \setminus \Lambda$ is made of finitely many orbits and their boundaries are stabilized by relatively quasiconvex subgroups. We claim that the stabilizer H of a component Ω of $S^2 \setminus \Lambda$ is of finite index (at most 2) in the stabilizer of its boundary $\partial\Omega$. This shows that H is relatively quasiconvex. The claim follows by observing that the elements of the stabilizer of $\partial\Omega$ that do not leave Ω invariant must permute the components of $S^2 \setminus \Lambda$ that have the same boundary as Ω. If Ω is a Jordan domain, that is, if its closure is an embedded disk, then its boundary is a Jordan curve and either bounds one or two components of $S^2 \setminus \Lambda$. If Ω is not a Jordan domain, the boundary of Ω is not an embedded circle. However, since it is locally connected, the Carathéodory-Torhorst theorem applies: we can find a homeomorphism of the open disk onto Ω which extends continuously to the boundaries, $f : D^2 \to \bar{\Omega}$. Since $\partial\Omega$ is not a circle, such map cannot be injective. We can thus find a simple closed curve γ which is contained in the closure of Ω and meets $\partial\Omega$ in a single cut point. The curve γ is the image of a simple arc joining two points of the boundary of the closed disk which are mapped to the same point in $\partial\Omega$. Since every other component of $S^2 \setminus \Lambda$ must sit on either side of γ, another component cannot have the same boundary as Ω. We thus see that $H = Stab(\Omega)$ coincides with the stabilizer of $\partial\Omega$ in this case. Since Ω is an open disc and H acts properly discontinuously on Ω, H is a orbifold group by [19, Cor. 3.2].

When G is a finitely generated convergence group acting on S^2, the peripheral subgroups are finite-type orbifold subgroups by Proposition 3.2. We claim that the peripheral subgroups of H are finitely generated, hence H is finitely generated, since it is relatively quasiconvex, and hence hyperbolic relative to the induced peripheral subgroups. Any peripheral subgroup Q of H is $P \cap H$, where P is a peripheral subgroup of $(G, \mathcal{P})$. This is exactly the subgroup of P that takes Ω to itself. Then Ω/Q embeds in the finite-type orbifold $(S^2 \setminus \{p\})/P$, so is an embedded sub-orbifold of a finite-type orbifold, and hence of finite type. Since the peripheral subgroups are finitely generated, so is H.

□

Lemma 3.6. *Let $(G, \mathcal{P})$ be a geometrically finite, non elementary, convergence group on S^2 with limit set Λ. Let Ω be a simply connected component of $S^2 \setminus \Lambda$ and $h : \overline{\mathbb{D}} \to \overline{\Omega}$ the extension of the homeomorphism conjugating the action of the stabilizer H as above. Let $p \in \partial\Omega$ be a parabolic point with stabilizer P, and set $Q = h^{-1} \circ (P \cap H) \circ h$. Then the limit set Λ_Q is exactly the non-empty set $h^{-1}(\{p\})$.*

Proof. Let $K \subset S^2 \setminus \{p\}$ be a compact subset containing a fundamental domain for the action of P on $\Lambda \setminus \{p\}$. We may find $g \in P$ such that $g(\partial\Omega) \cap K \neq \emptyset$. Thus, $g(\Omega)$ contains in its closure the point p and at least one point of K. Such components form a finite set since Λ is an E-set (Def. 2.2). By considering a sequence of points in $\partial\Omega$ tending to p, we may pick an infinite sequence (g_n) in P that maps Ω to components whose closures intersect both $\{p\}$ and K. As there are only finitely many of them, we may assume that $g_n(\Omega) = V$ for a fixed component V, and all $n \geq 1$. Therefore, $(g_1^{-1} g_n)_n$ is an infinite collection of elements of $H \cap P$ which proves that Λ_Q is not empty.

Since $hQ \subset Ph$, it follows that $h^{-1}(\{p\})$ is Q-invariant and compact, hence it contains Λ_Q which by definition is the minimal compact invariant subset under the action of Q.

The equality will follow from the fact that p is a bounded parabolic point. We first rule out the case that $h^{-1}(\{p\})$ contains an interval. If this was the case, then it would be the whole circle by invariance so that we would have $H = P$; this contradicts that $\Lambda_P = \{p\}$ and $\Lambda_H = \partial\Omega$. Therefore, $h^{-1}(\{p\})$ is nowhere dense in $\mathbb{S}^1$.

Let $\Omega_1, \ldots \Omega_k$, be the P-translates of Ω whose closures intersect both $\{p\}$ and K and let us fix $g_1, \ldots g_k \in P$ such that $g_j(\Omega_j) = \Omega$. Set $L = h^{-1}(\cup_{1\le j\le k} g_j(K))$. This is a compact subset of $\overline{\mathbb{D}}$ disjoint from $h^{-1}(\{p\})$, hence from Λ_Q. Let $x \in h^{-1}(p)$. We want to prove that the action of Q is not equicontinuous at x. With that in mind, pick a point $y \in \mathbb{S}^1 \setminus h^{-1}(\{p\})$ arbitrarily close to x. Note that $h(y) \in \partial\Omega \setminus \{p\}$ and since K is a fundamental domain, we may find $g \in P$ and $j \in \{1, \ldots, k\}$ so that $g(h(y)) \in K \cap \partial\Omega_j$. It follows that $g_j g \in (H \cap P)$ so that we may find $q = h^{-1} g_j g h \in Q$ with $q(y) \in L$. This implies that $x \in \Lambda_Q$. Indeed, considering now a sequence (y_n) in $\mathbb{S}^1 \setminus h^{-1}(\{p\})$ tending to x, we obtain in this way a sequence (q_n) in Q such that $(q_n(x), q_n(y_n)) \in h^{-1}(\{p\}) \times L$: as L and $h^{-1}(\{p\})$ are disjoint compact subsets, $(q_n)_n$ cannot be equicontinuous at x. □

As already observed in the proof of Lemma 3.5, if h is not injective, i.e., if $h(x) = h(y)$ for some pair of points x, y in $\mathbb{S}^1$, then $h(x)$ is a cut point of Λ_G (we may build a Jordan arc in $\overline{\mathbb{D}}$, a crosscut, that maps under h to a separating Jordan curve), and so $h(x)$ is parabolic.

Given a parabolic point p with stabilizer P and a component Ω of the ordinary set which contains p in its boundary, we will say that p is *uniquely accessible from* Ω if the above map $h : \overline{\mathbb{D}} \to \overline{\Omega}$ is injective over p, i.e., $h^{-1}(\{p\})$ is a singleton. Likewise, we say that p is *doubly accessible from* Ω if $h^{-1}(\{p\})$ consists of two points. We expect that in general $h^{-1}(p)$ will be a Cantor set if p is not uniquely or doubly accessible.

Corollary 3.7. *Let p be a parabolic point with stabilizer P of a geometrically finite convergence group acting on S^2, $(G, \mathcal{P})$. Assume that the component of its Bowditch boundary containing p is not a singleton. Let Ω be any component such that $\partial\Omega$ contains p. If P is two-ended, then p is either uniquely or doubly accessible from Ω.*

Proof. Recall the notation of Lemma 3.6: Q is defined as $h^{-1} \circ (P \cap H) \circ h$, where h is the extension of the homeomorphism conjugating the action of the stabilizer H of Ω. The number of accesses to p from Ω are in bijection with the cardinality of Λ_Q. By Lemma 3.6, the limit set is non-empty so Q is infinite.

Since we assume that P is two-ended, this is also the case of Q. Hence there is a finite index cyclic subgroup in Q that is generated either by a loxodromic element, implying the point p is doubly accessible, or by a parabolic element, implying the point p is uniquely accessible. □

We note that the converse of Corollary 3.7 does not hold in full generality. Here is a counter-example: pick a convex-cocompact Kleinian group G that uniformizes a hyperbolic 3-manifold with totally geodesic boundary; consider one component F of its boundary and choose a compact π_1-injective proper subsurface S in F, with a non-Abelian free fundamental group P, such that each component of the complement of S has also non-Abelian free fundamental group. The pair $(G, \mathcal{P})$, where $\mathcal{P}$ consists of the conjugates of P, is a planar relatively hyperbolic group pair. To see this, P stabilises a component Ω_F of the ordinary set, hence the hyperbolic convex hull K of Λ_P in

Ω_F is connected and simply connected in Ω_F, and precisely invariant under P, i.e., if $g(K) \cap K \neq \emptyset$ for some $g \in G$, then $g \in P$. Therefore, as Λ_G is a Sierpiński carpet, $G(K)$ is a null sequence that satisfies the assumptions of Moore's Theorem 4.3: by collapsing each component of $G(K)$ we obtain a geometrically finite convergence group action on S^2 for which P is parabolic with fixed point p. Moreover, the parabolic point p is on the boundary of countably many components Ω such that $\mathrm{Stab}(\Omega) \cap P$ is cyclic but P is not, and p is uniquely accessible from each component.

Proposition 3.8. *Let p be a parabolic point with stabilizer P of a geometrically finite convergence group on S^2, $(G, \mathcal{P})$. We assume that the component of $\partial(G, \mathcal{P})$ containing p is not a singleton. Let Ω_p denote the union of the ordinary components which contain p on their boundary. The action of P on $S^2 \setminus (\{p\} \cup \Omega_p)$ is cocompact and the set of components of Ω_p forms finitely many orbits.*

In particular, if p is in the boundary of no ordinary component, then P acts cocompactly on $S^2 \setminus \{p\}$.

Proof. We may assume that Λ_G is connected according to Proposition 2.6. Let $K \subset S^2 \setminus \{p\}$ be a compact subset containing a fundamental domain for the action of P on $\Lambda_G \setminus \{p\}$.

Since Λ_G is an E-set, it follows that the closure of the union of ordinary components $\Omega \in \pi_0(\Omega_G \setminus \Omega_p)$ with $K \cap \partial\Omega \neq \emptyset$ is a compact subset L of $S^2 \setminus \{p\}$. Consider any component Ω disjoint from Ω_p, we may find $g \in P$ such that $g(\partial\Omega) \cap K \neq \emptyset$. It follows that the action is cocompact on $S^2 \setminus (\{p\} \cup \Omega_p)$.

We now consider components Ω which contain p on their boundary. As above, we may find $g \in P$ such that $g(\partial\Omega) \cap K \neq \emptyset$. Thus, $g(\Omega)$ contains in its closure the point p and at least one point of K. Such components form a finite set since Λ_G is an E-set. $\square$

We conclude with some general properties of the ordinary set, which are proved as for Kleinian groups. The next proposition was already known, but we were unable to find a formal proof in the literature.

Proposition 3.9. *Let G be a convergence group acting on S^2. Then the ordinary set has zero, one, two or infinitely many components.*

Proof. The conclusion is obvious if the limit set Λ_G is empty: in this case the ordinary set is connected and the action is elementary.

Let us first consider the case when $\Lambda_G \neq \emptyset$ is not connected. If all of its components are points, in particular if $\Lambda_G \neq \emptyset$ is finite and the action of G is elementary, then Ω_G is connected. Otherwise, there are infinitely many components of Λ_G which are non-trivial, so there are infinitely many components of the ordinary set by Lemma 2.7.

We may now assume that Λ_G is an infinite connected compact set. Let us assume furthermore that the ordinary set has at least two but finitely many components.

Considering a finite-index subgroup if necessary, one may assume that the group G fixes each component. Therefore, Λ_G is the boundary of each component of the ordinary set. This is the main point and follows from the fact that the boundary of each component is closed, contained in Λ_G, and G-invariant.

By density of loxodromic fixed points, the group G contains a loxodromic element g with fixed points a and b in Λ_G.

Consider a component Ω of the ordinary set and a point $x \in \Omega$. We may find a path c_0 in Ω that joins x to $g(x)$. The g-orbit $c_n = g^n(c_0)$, $n \in \mathbb{Z}$, defines a path which joins

a and b in Ω by the convergence property. Its image contains an arc c also joining a and b, i.e. a path without self-intersections.

Since, as remarked above, Λ_G is the boundary of every component, we can proceed similarly with a second component Ω' and denote by c' an arc in Ω' which joins a and b. Then $\{a,b\}\cup c\cup c'$ is a Jordan curve that separates Λ_G, for there are points of both Ω and Ω' on each side of the Jordan curve. If there were a third component in the complement of Λ_G it would sit on one side of this Jordan curve $\{a,b\}\cup c\cup c'$. Because of this, the boundary of this new component could not be Λ_G as it should. Therefore, if there are more than two components, there must be infinitely many. □

Corollary 3.10. *Let $(G,\mathcal{P})$ be a geometrically finite convergence group acting on S^2. If Ω_G is non empty and connected, then Λ_G is totally disconnected. If furthermore G is finitely generated, then G is covered by a Kleinian group. If Ω_G has exactly two components, then Λ_G is a circle and the action of G is either isomorphic to a Fuchsian group of finite coarea, or to a degree 2 extension of such a group.*

Proof. Let us assume that Ω_G is connected and let us assume for contradiction that Λ is a component of the limit set with at least two points. Then Λ is the limit set of its stabilizer H which is also hyperbolic relative to virtual surface groups, cf. Proposition 2.6. Since Λ does not separate the plane and does not contain an open disk, it is simply connected. It follows that Λ cannot contain a simple closed curve and, since it is locally connected by Corollary 3.4, it is a dendrite, which is impossible by Lemma 2.7. So Λ_G is totally disconnected. Now, when G is finitely generated, it follows from [21, Corollary 5.4] that G is covered by a Kleinian group.

Assume Ω_G has two components $\Omega_\pm$. Taking an index 2 subgroup if necessary, we may assume that both components are invariant under G so that $\overline{\Omega_+}\cap\overline{\Omega_-}=\Lambda_G$, the minimal G-invariant set. This implies that Λ_G is their common boundary and is connected by [33, Cor. VI.2.11], hence locally connected by Corollary 3.4, and that $\Omega_\pm$ are simply connected. Thus, by the Carathéodory-Torhorst theorem, there are continuous onto maps $\varphi_\pm:\overline{\mathbb{D}}\to\overline{\Omega_\pm}$ from the closed unit disk that restrict to homeomorphisms between their interiors. Since both images of the unit circle coincide, reasoning as in the proof of Lemma 3.5, we may conclude that Λ_G is a Jordan curve. Finally Lemma 3.5 enables us to conclude in this case. For more general results, see [23].

□

4 Blowing-up rank-one parabolic points

Definition 4.1. Let $(G,\mathcal{P})$ be a geometrically finite convergence group on S^2. We write $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$ where $\mathcal{P}_1$ consists of all stabilizers of rank 1 parabolic points.

Theorem 4.2. *Let $(G,\mathcal{P})$ be a geometrically finite convergence group on S^2, and $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$ as in Definition 4.1. Then $(G,\mathcal{P}_2)$ is a geometrically finite convergence group on S^2 and there is an equivariant degree 1 continuous map $\phi:S^2\to S^2$ mapping the Bowditch boundary of $(G,\mathcal{P}_2)$ onto that of $(G,\mathcal{P})$.*

Before giving the proof, we pause for some topological facts, starting with the following particular case of Moore's theorem [24].

Theorem 4.3 (Moore)**.** *Let $\mathcal{C}$ be a pairwise disjoint collection of compact and connected subsets of the sphere S^2 such that each $K\in\mathcal{C}$ is not a point. Assume that each* *element has a connected complement and the set $\mathcal{C}$ forms a null-sequence. Let $\sim$ be the equivalence relation generated by $x \sim y$ if there is some $K \in \mathcal{C}$ that contains $\{x, y\}$. Then $Z = S^2/\sim$ is a topological sphere when endowed with the quotient topology.*

We add some further properties that will be used in the proof of Theorem 4.2.

Proposition 4.4. *Under the assumptions of Theorem 4.3, set $Y = S^2 \setminus \cup_{K\in\mathcal{C}} K$. For any connected open subset U of S^2 such that $\partial U \subset Y$, the set $Y \cap U$ is arcwise connected. In particular Y is arcwise connected, and every point of Y admits a basis of neighborhoods such that the boundaries of these neighborhoods are disjoint from $S^2 \setminus Y = \cup_{K\in\mathcal{C}} K$.*

Proof. Denote by $\pi : S^2 \to Z$ the canonical projection and note that for all $y \in Y$, $\pi^{-1}(\pi(\{y\})) = \{y\}$, in particular the restriction of π to Y is injective. We first justify that if A is a compact arc or a Jordan curve in $\pi(Y)$, then so is $B = \pi^{-1}(A)$. To see this, note that B is compact and that $\pi : B \to A$ is bijective and continuous since $B \subset Y$.

Since the projection $\pi : S^2 \to Z$ maps Y to the complement of a countable set, we may find arcs joining any two points in $\pi(Y)$ and then lift them back to Y. This proves that Y is arcwise connected, as well as $U \cap Y$ for any connected open set with $\partial U \subset Y$, for in this case U is saturated and $\pi(U)$ is open. Similarly, if $x \in Y$, then we may construct a basis of disk-neighborhoods of $\pi(x)$ in Z with their boundaries contained in $\pi(Y)$. They lift as disk neighborhoods of x in S^2. Since $\mathcal{C}$ is a null sequence and x is disjoint from the collection $\mathcal{C}$, these disk-neighborhoods form a basis. □

Proof of Theorem 4.2. The proof goes as follows. We first define a set $\widehat{Y}$ that plays the role of a blown-up of S^2 over the parabolic fixed points coming from $\mathcal{P}_1$. This is a planar compact set bounded by Jordan curves. Then we prove that the action of the group G induces a geometrically finite convergence group action whose maximal parabolic subgroups are exactly those of $\mathcal{P}_2$, and we extend the action to the whole sphere.

Let $\mathbb{P}_1$ denote a set of representatives of each conjugacy class in $\mathcal{P}_1$.

Definition of the set $\widehat{Y}$.— Fix $P \in \mathbb{P}_1$ with parabolic point p and let us define two disjoint horoballs in Ω_G attached to p as follows. Since P is two-ended, P is either isomorphic to $\mathbb{Z}$ or to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ according to Proposition 3.2, and there is an element γ that acts as a translation on $S^2 \setminus \{p\}$ and that generates a subgroup of minimal index in P, cf. Lemma 3.1. Let us consider a chart that identifies $S^2 \setminus \{p\}$ with $\mathbb{C}$ and γ with the translation by 1. Note that the action of γ on $\Lambda_G \setminus \{p\}$ is cocompact since γ generates a finite index subgroup of P and that p is a bounded parabolic point. Therefore, we may enclose $\Lambda_G \setminus \{p\}$ into a horizontal open strip of bounded width. The complement of the strip in $\mathbb{C}$ is the union of two half-planes contained in Ω_G, each of which defines a closed horoball attached to p. Let H_P denote their union, and note that the fact that P is the stabilizer of p implies that one can choose the two half planes so that the stabilizer of H_P is exactly P. Set $\mathcal{C} = \cup_{P\in\mathbb{P}_1} GH_P$, the collection of all translates by G of the finite collection $\{H_P \mid P \in \mathbb{P}_1\}$.

Let us check that $\mathcal{C}$ forms a null sequence, by contradiction: we consider a sequence $(g_n)_{n\in\mathbb{N}}$ of G such that $\operatorname{diam} g_n(H_P) \geq \delta$ for some $\delta > 0$ and some fixed $P \in \mathbb{P}_1$ and associated point p. Up to taking a subsequence, by the convergence property, we may assume that (g_n) tends uniformly towards the constant map with image $b \in \Lambda_G$ on the compact subsets of $S^2 \setminus \{b'\}$, where $b, b' \in \Lambda_G$. We now remark that we must have

$b' = p$. Indeed, if that was not the case, the closure of H_P would be a compact set in the complement of b', intersecting Λ_G only in p. By the convergence property its images by the elements of the sequence should shrink to $\{b\}$, against the hypothesis that their diameter is bounded from below.

Pick $c \in \Lambda_G \setminus (\{b\} \cup Gp)$. Since $c \neq b$, it follows that $(g_n^{-1}(c))$ tends to p. As p is a bounded parabolic point and $c \notin Gp$, up to passing to a subsequence, we may find a sequence (h_n) in P such that $(h_n(g_n^{-1}(c))$ tends to a point $a \in \Lambda_G \setminus \{p\}$. Pick a neighborhood V of a that is disjoint from H_P. It follows that $(g_n \circ h_n^{-1})_n$ tends uniformly to the constant map b on $S^2 \setminus V$. Since $h_n(H_P) = H_P$, it follows that we have uniform convergence of $(g_n|_{H_P})_n$ to the constant map b, contradicting our assumptions.

Choose the horoballs small enough so that the collection is pairwise disjoint in Ω_G. This is possible since the action of G is properly discontinuous on Ω_G and $\mathbb{P}_1$ is finite.

Set $Y = S^2 \setminus \cup_{K \in \mathcal{C}} K$ and observe that we are under the assumptions of Proposition 4.4. In particular, Y is arcwise connected.

It will be convenient to endow S^2 with a distance d_S compatible with its topology. We define, on Y, $d_Y(x, y) = \inf \mathrm{diam}_S L$ where L runs over all continua of Y which contain $\{x, y\}$. This defines a metric. Let us denote by $\widehat{Y}$ its completion.

Properties of the set $\widehat{Y}$.— We claim that $\widehat{Y}$ is a planar, locally connected and arcwise connected compact set with open disks as complementary components. To see this, we define a notion of *regular neighborhoods* for points in $\overline{Y} \subset S^2$.

- By Proposition 4.4, every point in $y \in Y$ admits a basis of neighborhoods in S^2 whose boundaries are disjoint from the elements of $\mathcal{C}$. We call such neighborhoods regular of type (Y) for y.

  Let $K \in \mathcal{C}$ be the element associated to a rank 1 parabolic point p. Note that K is a union of two closed disks attached at p and that $K \setminus \{p\}$ is contained in Ω_G, so isolated from the other components.

- It follows from the fact that the components in $\mathcal{C}$ are disjoint in Ω_G that any point $x \in \partial K \setminus \{p\}$ admits a basis of neighborhoods which are discs in $\overline{Y}$ bounded by the union of an arc in ∂K and an arc in $\Omega_G \setminus K$. We call such neighborhoods regular of type (K) for x.

- For the point p, we may consider a basis of Jordan disks that is regular for the collection $\mathcal{C} \setminus \{K\}$ and that intersects K in exactly two arcs, one in each horoball. We call such neighborhoods regular of type (P) for p.

By construction, if V is a regular neighborhood of type (Y) or (K), then $Y \cap V$ is arcwise connected, whereas if V is of type (P), then $Y \cap V$ has exactly two arcwise connected components.

Recall that d_S is the metric on S^2 used to define d_Y above so that $d_S \leq d_Y$. Thus every Cauchy sequence for d_Y is a Cauchy sequence for d_S. This ensures the existence of a canonical continuous map $\pi : \widehat{Y} \to (\overline{Y}, d_S)$.

Let $(x_n)_n$ be a Cauchy sequence in (Y, d_S) with limit $x \in \overline{Y}$. If x is not a rank-one parabolic point, then it admits a basis of regular neighborhoods in $\overline{Y}$ of types (Y) or (K) that intersect Y in an arcwise connected set, so that $(x_n)_n$ is also a Cauchy sequence in (Y, d_Y) that defines a unique limit point in $\widehat{Y}$. If x is a rank-one parabolic point with stabilizer P, then $\overline{Y} \setminus \{x\}$ has two ends associated to the arcwise connected components of its regular neighborhoods of type (P). It follows that $\pi^{-1}(\{x\})$ has exactly two preimages that correspond to each end. Thus, π is also surjective and a point has two preimages if it is a parabolic point of rank one and one preimage otherwise. Moreover, we may define regular neighborhoods $\widehat{V}$ for points in $\widehat{Y}$ that will be connected by lifting regular neighborhoods of types (Y) and (K) and half neighborhoods of type (P).

All these observations enable us to conclude that $\widehat{Y}$ is arcwise connected, locally connected, compact, with no local cut points and that each component of $\widehat{Y} \setminus Y$ is a Jordan curve.

It remains to check that $\widehat{Y}$ is planar. Claytor's theorem [7] asserts that a continuum without local cut points is embeddable in the sphere if and only if it contains neither a copy of the complete graph on five vertices K_5 nor of the complete bipartite graph with six vertices $K_{3,3}$.

Let us consider a finite connected graph L and an embedding $j : L \hookrightarrow \widehat{Y}$. We will modify the embedding j so that $\pi \circ j$ is also injective, implying that L cannot be one of the forbidden graphs.

Let $T \subset L$ denote the closure of the set of points $z \in L$ for which we may find $w \neq z$ in L such that $(\pi \circ j)(z) = (\pi \circ j)(w)$. Note that T is a compact subset of L. If T is empty, then there is nothing to be done. Let us assume it is not empty. Let $z \in T$. If z belongs to an edge, we consider an open interval neighborhood $J_z \subset L$ contained in the same edge; if z is a vertex, then we consider a star-shaped open neighborhood J_z contained in the union of the edges incident to z. Since $L \setminus J_z$ is compact, we have $d_Y(j(z), j(\overline{L \setminus J_z})) > 0$ so that we may find a regular neighborhood $\widehat{V}_z \subset \widehat{Y}$ of $j(z)$ such that $j^{-1}(\widehat{V}_z) \subset J_z$; we let $V_z \subset \overline{Y}$ be the corresponding regular neighborhood of $(\pi \circ j)(z)$.

We now extract a finite subcover of $(\pi \circ j)(T)$ given by the above regular neighborhoods that we order $V_1, \ldots, V_n$. Each V_k comes with a point $z_k \in L$, a neighborhood $J_k \subset L$ and a regular neighborhood $\widehat{V}_k$ of $j(z_k)$. We modify the embedding j inductively on the neighborhoods V_k. Let us fix $1 \leq k \leq n$, and let us assume that $\pi \circ j$ is injective on $L \setminus (\cup_{k \leq i \leq n} J_{z_i})$. We note that $j(L) \cap \widehat{V}_k \subset j(J_k)$ and $\widehat{V_k} \cap Y$ is homeomorphic to the complement of a countable subset of a Jordan domain. Therefore, we may modify $j|_{J_k}$ so that its image in $\widehat{V}_k$ is contained in Y. As $\pi|_Y$ is injective, the map $\pi \circ j$ is now injective on $L \setminus (\cup_{k < i \leq n} J_{z_i})$.

In conclusion, given any embedding of a finite graph L in $\widehat{Y}$, there is an embedding of L in $\overline{Y}$, hence in S^2. As the latter space is planar, we may conclude that L is not isomorphic to K_5 nor $K_{3,3}$, and so $\widehat{Y}$ is planar.

Extension of the action of G to $\widehat{Y}$.— Let us now consider the action of G on $\widehat{Y}$: regular neighborhoods enable us to conclude that the action on (Y, d_Y) extends continuously to $\widehat{Y}$.

Let us check that the action remains a geometrically finite convergence action. For this, we pick a sequence of distinct elements (g_n). We may as well assume that there are two points a and b in $\overline{Y}$ such that the sequence (g_n) of homeomorphisms of the sphere tends uniformly to the constant map a on the compact subsets of $S^2 \setminus \{b\}$. When both a and b are distinct from the rank 1 parabolic points, then this property lifts to $\widehat{Y}$.

Let us assume that a is a rank 1 parabolic point and write $\{x, x'\} = \pi^{-1}(a)$. Let us consider a regular neighborhood of type (P). It defines two disjoint connected neighborhoods W and W' in $\widehat{Y}$ of x and x' respectively. We may pick a point $z \in \widehat{Y}$ and assume that $(g_n(z))$ tends to x for instance, implying that $g_n(z) \in W$ for all n large enough. Note that we may exhaust $\widehat{Y} \setminus \pi^{-1}(\{b\})$ by connected compact subsets. Since $\pi^{-1}(a)$ is discrete, for any connected compact subset $K \subset \overline{Y} \setminus \{b\}$ containing

$\pi(z)$, the convergence property implies that $g_n(K)$ has to be contained in $\pi(W)$ for n large enough. This implies that (g_n) tends to the constant x in $\widehat{Y} \setminus \pi^{-1}(b)$. If b is not parabolic, then we are done.

On the other hand, if $\pi^{-1}(b) = \{y, y'\}$, then the same reasoning for $(g_n^{-1})_n$ shows that we may also assume that all compact subsets disjoint from $\{x, x'\}$ tend to y under (g_n^{-1}). Let $V \subset \widehat{Y}$ be a disk-neighborhood of y disjoint from $W \cup W'$ and $K = \widehat{Y} \setminus (W \cup W')$. Note that, for any n large enough, $g_n^{-1}(K)$ is contained in V so that the connected set $\widehat{Y} \setminus V$ is covered by the two disjoint open sets $g_n^{-1}(W)$ and $g_n^{-1}(W')$. The connectedness of $\widehat{Y} \setminus V$ implies that $g_n^{-1}(W') \subset V$ since (g_n) pushes points into $W \subset (\widehat{Y} \setminus V)$. Therefore, we have uniform convergence of (g_n^{-1}) on W' to the constant map y. By symmetry, we get uniform convergence of (g_n) to the constant map x on compact subsets disjoint from y. This shows that G has also a convergence action on $\widehat{Y}$.

Let us note that since the action of G on $\Lambda_G \cap Y$ is invariant and minimal, its closure $\widehat{\Lambda}$ in $\widehat{Y}$ will be a minimal invariant subset, hence the limit set of this new action.

We may check that the action on it is geometrically finite with maximal parabolic subgroups in $\mathcal{P}_2$. Since $\widehat{Y}$ is planar, we may now consider it as a subset of S^2 and extend the action to the whole sphere using [15, Thm. 5.8]. $\square$

5 Topological Schottky sets

Definition 5.1. A *topological Schottky set* $\mathcal{S}$ is a proper compact subspace of S^2 defined by the following topological properties enjoyed by Schottky sets.

(S1) the set of components $\{D_i\}_{i \in I}$ of $S^2 \setminus \mathcal{S}$ is countable and not empty;

(S2) for each i, $\bar{D}_i = D_i \cup \partial D_i$ is a closed disc; that is D_i is a Jordan domain.

(S3) for each pair $i \neq j \in I$ $\bar{D}_i$ and $\bar{D}_j$ meet in at most one point.

(S4) for each triple of distinct indices $i, j, k \in I$, $\bar{D}_i \cap \bar{D}_j \cap \bar{D}_k = \emptyset$,

(S5) for every open cover $\mathcal{U}$ of S^2 and for all but finitely many $i \in I$ there is a $U_i \in \mathcal{U}$ such that $D_i \subset U_i$.

Remark 5.2. If the 2-sphere is endowed with a metric, the purely topological condition (S5) is equivalent to asking that $\mathcal{S}$ is an E-set (Def. 2.2). This is an easy consequence of the Lebesgue number lemma.

The most well-known topological Schottky sets are the Sierpiński carpet and the Apollonian Gasket. These both occur as the limit sets of geometrically finite Kleinian groups, [20], [17]. Hence they are also the (Bowditch) boundaries of relatively hyperbolic groups. Observe that in contrast to the definition of a Schottky set, the cardinality of I is not required to be at least 3. However, if $|I| \leq 2$ then $\mathcal{S}$ has non empty interior and cannot be the boundary of a relatively hyperbolic group.

Proposition 5.3. *A topological Schottky set is connected, locally connected, hence arcwise connected, with no cut points and no cut pairs.*

Lemma 5.4. *Let $\mathcal{S}$ be a topological Schottky set and Ω a non-empty open connected subset of S^2 such that, for each $i \in I$, $\partial D_i \cap \Omega$ is connected. The set $X = \mathcal{S} \cap \Omega$ is connected.*

Note that the proof of this lemma does not use the E-set condition (S5).

Proof. Let us consider two open subsets of Ω, R and B, for red and blue, such that $X \subset R \cup B$, $X \cap R$ and $X \cap B$ are not empty, but $X \cap R \cap B = \emptyset$. We may assume that each component of R and B intersects X, by removing any components that do not intersect X (note that the sphere is locally connected so every component of an open subset is itself open).

We will increase these sets (by adding in disks associated to the two components) into two open and disjoint subsets that cover Ω: this will prove that one of them has to be empty, hence that X is connected. With this in view, we split the set of components $\{D_i\}_{i\in I}$ into three sets $I = I_0 \sqcup I_R \sqcup I_B$.

Let $i \in I$, and let us write $C_i = \partial D_i$. If $C_i \cap X = \emptyset$, since Ω is a connected set intersecting $\mathcal{S}$ and C_i is a Jordan curve, then $\Omega \cap D_i = \emptyset$ and we let i belong to I_0. If not, $(C_i \cap X)$ is connected by assumption, and covered by R and B. Hence, $R \cap (C_i \cap X) = \emptyset$ or $B \cap (C_i \cap X) = \emptyset$. In the former case, $D_i \cap R = \emptyset$ as each component of R intersects X, but not C_i, so we let i belong to I_B; in the latter, we let i belong to I_R. Thus $i \in I_R$ if and only if $(C_i \cap X) \subset R$ and $i \in I_B$ if and only if $(C_i \cap X) \subset B$.

We let

$$R' = R \cup (\cup_{i\in I_R} D_i \cap \Omega) \quad \text{and} \quad B' = B \cup (\cup_{i\in I_B} D_i \cap \Omega)\,.$$

We obtain in this way a cover of Ω by two disjoint open sets, so that one of them has to be empty. Therefore, one of R or B has to be empty as well, establishing the connectedness of X. □

Proof of Proposition 5.3. To show that $\mathcal{S}$ is connected and with no cut points, we apply Lemma 5.4 twice: with $\Omega = S^2$ first and then with $\Omega = S^2 \setminus \{x\}$, for any $x \in \mathcal{S}$. As each boundary component of $\mathcal{S}$ is a closed simple curve, it cannot be disconnected by removing at most one point and it follows that $\mathcal{S}$ and $\mathcal{S} \setminus \{x\}$ are both connected.

Now let $x, y \in \mathcal{S}$ be two points and consider $\Omega = S^2 \setminus \{x, y\}$. If no boundary component of $\mathcal{S}$ contains both x and y, the previous argument applies and we see that x, y cannot form a cut pair. We can thus assume that there is an $i \in I$ such that $x, y \in C_i$. Let γ be a properly embedded arc in $\bar{D}_i$ connecting x to y. $D_i \setminus \gamma$ is the union of two open disks, D and D', each adjacent to precisely one connected component of $C_i \setminus \{x, y\}$. We can now repeat the same strategy used in the proof of Lemma 5.4 with $\Omega = S^2 \setminus \gamma$ to conclude that $\mathcal{S} \setminus \{x, y\}$ must be connected. This shows that $\mathcal{S}$ has no cut pairs.

As $\mathcal{S}$ is an E-set, we deduce from [33, Theorem VI.4.4] that it is also locally connected, hence arcwise connected [33, Theorem II.5.1]. □

Our first key result is that the boundaries of the D_i are topologically distinguished, generalizing the case of a Sierpiński carpet.

Proposition 5.5. *Let $\mathcal{S}$ be a topological Schottky set with $\mathcal{S} \simeq S^2 \setminus \cup(D_i)$, where each D_i is open. Then the non-separating embedded circles of $\mathcal{S}$ are exactly the $C_i = \bar{D}_i \cap \mathcal{S}$.*

Proof. Let C be an embedded circle in $\mathcal{S} \subset S^2$. By the Jordan curve theorem, the complement of C consists of two open discs O and O'. Assume that C is contained in $\mathcal{S}$. By construction C is a C_i if and only if either O or O' coincides with D_i. If this is not the case, both O and O' contain points of $\mathcal{S}$ and C separates $\mathcal{S}$.

We want to show that if $C = C_i$ then C does not separate $\mathcal{S}$. Let us consider $\Omega = S^2 \setminus \bar{D}_i$. Condition (S3) ensures that Lemma 5.4 applies to prove the connectedness of $\mathcal{S} \setminus C_i$.

□

Corollary 5.6. *Any homeomorphism $h : \mathcal{S}_1 \to \mathcal{S}_2$ between two topological Schottky sets is the restriction of a self-homeomorphism $H : S^2 \to S^2$ of the sphere.*

This implies that we may define a topological Schottky set as an abstract compact subset homeomorphic to that of an embedded topological Schottky set as above

Proof. By Proposition 5.5, h maps boundary components $\{C_i^1\}$ to boundary components $\{C_i^2\}$. As these components are Jordan curves, one may extend $h : C_i^1 \to C_i^2$ as a homeomorphism $H_i : D_i^1 \to D_i^2$ for each $i \in I$. Since topological Schottky sets are E-sets, these local homeomorphisms induce a global homeomorphism $H : S^2 \to S^2$. □

Proposition 5.7. *Let $(G, \mathcal{P})$ be a relatively hyperbolic pair. If its Bowditch boundary is homeomorphic to a topological Schottky set, then $(G, \mathcal{P})$ is a geometrically finite convergence group on S^2.*

Proof. We may assume that G acts as a convergence group action on a topological Schottky $\mathcal{S} \subset S^2$. Proposition 5.5 implies that G preserves the collection of boundary circles. Therefore, we may apply [15, Thm. 5.8] and extend in this way the action as a global convergence of the sphere. □

Corollary 5.8. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group pair with Bowditch boundary a topological Schottky set. The set $\cup_{i \neq j \in I}(\bar{D}_i \cap \bar{D}_j)$ corresponds to the set of parabolic points whose stabilizers are 2-ended.*

Proof. Let p be a parabolic point. Let Ω_p denote the union of components of the ordinary set that contain p on their boundaries: according to the definition of a topological Schottky set, Ω_p is either empty, or has one or two components. By Proposition 3.8, the action on $S^2 \setminus (\{p\} \cup \Omega_p)$ is cocompact.

If $\Omega_p = \emptyset$, then $(S^2 \setminus \{p\})/\mathrm{Stab}(p)$ is a compact surface orbifold. If Ω_p has a single component, then $\mathrm{Stab}(p)$ is cyclic since it preserves $\partial\Omega_p$, but this prevents the quotient to be compact on its complement as the action of the cyclic group is generated by a translation by Lemma 3.1. Therefore, if Ω_p is non-empty, then it is the union of two discs. Conversely, if two boundary components intersect, then Proposition 2.8 implies that the intersection of their stabilisers is a parabolic point p. Up to index 2, $\mathrm{Stab}(p)$ fixes each component, hence is a rank 1 parabolic point. □

6 Incidence graphs for topological Schottky sets

We recall Definition 5.1. A *topological Schottky set* $\mathcal{S}$ is a connected, locally connected, 1-dimensional subset of the sphere such that the complement is a union of pairwise disjoint Jordan domains. The closure of each component of the complement is homeomorphic to a disc $\bar{D}_i$. The intersection $\bar{D}_i \cap \bar{D}_j$ is at most one point and a point of $\mathcal{S}$ belongs to at most two $\bar{(}D_i)$. A topological Schottky set has no cut points nor cut pairs.

In this situation, we can draw more conclusions from the above construction in Section 4.

Definition 6.1. We define the *incidence graph* $\Gamma(\mathcal{S})$ *of the topological Schottky set* $\mathcal{S}$. Let Γ be the bipartite graph with vertex set the union of vertices $\{v_i\}_{i \in I}$, associated to the components $\{D_i\}_{i \in I}$ or, equivalently by Proposition 5.5, to the embedded non separating circles in $\mathcal{S}$, and vertices v_p, associated to intersections $\bar{D}_i \cap \bar{D}_j$, such that there is a non oriented edge between v_i and v_p if and only if $p \in \partial D_i$. Since we are working with a topological Schottky set, Γ embeds into S^2. To see this we pick for each component D_i a base-point $v_i \in D_i$ and join v_i to each $p \in \partial D_i \cap \partial D_j$ with an arc in $\bar{D}_i$.

If $(G, \mathcal{P})$ is a relatively hyperbolic group pair whose boundary is a topological Schottky set, we will often denote this graph by $\Gamma(G)$ or $\Gamma(G, \mathcal{P})$. As observed in Lemma 6.3, each edge corresponds to a rank-1 parabolic point. Also, we may ignore the vertices corresponding to the rank-1 parabolic points since this will not change the topology of the graph.

The following is a consequence of Proposition 5.5.

Lemma 6.2. *If $\partial(G, \mathcal{P})$ has Bowditch boundary a topological Schottky set, then G acts on $\Gamma(G)$.*

The following was established in the proof of Corollary 5.8

Lemma 6.3. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group pair with Bowditch boundary a topological Schottky set. The intersection of two ∂D_i, which corresponds to an edge in the incidence graph, is a parabolic point with a 2-ended stabiliser. All other parabolic points have stabilisers isomorphic to a compact surface orbifold group.*

Corollary 6.4. *Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ be as in Definition 4.1. The components of the ordinary set of $\partial(G, \mathcal{P}_2)$ are in bijection with the components of Γ. Each cycle in Γ separates the Bowditch boundary of $(G, \mathcal{P}_2)$.*

Proof. We will use the same notation introduced in Section 4 for the proof of Theorem 4.2: Y is the complement of the union of pairs K of closed horoballs attached to each rank-1 parabolic point p and $\widehat{Y}$ its completion after blowing-up, so that there is a natural quotient map $\pi : \widehat{Y} \longrightarrow \bar{Y}$. Let $\Gamma_T = \Gamma \cap Y$ and let us consider its closure Γ'_T in $\widehat{Y}$. This graph is disjoint from the limit set, and each edge is cut into two pieces by a Jordan domain $D = \pi^{-1}(K)$, $K \in \mathcal{C}$. We may then connect both sides of the edge in D to reconstruct a graph Γ' isomorphic to Γ which will now be disjoint from the limit set.

Let us observe that this edge separates in $\overline{D}$ the preimages of the parabolic point, so that any cycle in Γ' separates the limit set. By construction each connected component of the new ordinary set contains a component of Γ' (which might be reduced to a single point). To see there is at most one, we may proceed by contradiction as follows: if two components of Γ' belonged to the same component of $\widehat{\Omega}_G$, we could consider a curve joining them in Y: a contradiction. □

Theorem A. *Let $\mathcal{S}$ be a topological Schottky set with $\mathcal{S} = \partial(G, \mathcal{P})$. Then the incidence graph $\Gamma(\mathcal{S})$ has 1, 2 or infinitely many components. Their stabilizers are virtual surface groups.*

Proof. According to Proposition 5.5, boundary components do not separate a topological Schottky set so the group G maps boundary components to themselves. Therefore, [15, Thm. 5.8] enables us to extend the action onto the whole sphere. The parabolic points are either surface groups that are not accessible from any components or rank 1 parabolic points, which correspond to two intersecting disks, as observed in Lemma 6.3. According to Corollary 6.4, the components of the graph are thus in bijection with those of the blown-up ordinary set. Since the action is geometrically finite, there are 1, 2 or infinitely many components as seen in Proposition 3.9. By Proposition 2.3 we deduce that the stabilizers of components are relatively quasiconvex subgroups. In addition they have no parabolics. Since these stabilizers stabilize disks, they are virtually closed surface groups. □

7 One component in the incidence graph

Here we prove Theorem B:

Theorem B. *Let $\mathcal{S}$ be a topological Schottky set with $\mathcal{S} = \partial(G, \mathcal{P})$.*

When the incidence graph $\Gamma(\mathcal{S})$ has one component, then G is virtually a free product of a free group F_n of rank $n \geq 0$ and some finite index subgroups of groups in $\mathcal{P}$. Moreover, if G is finitely generated, its action is faithful and orientation preserving, then G is covered by a geometrically finite Kleinian group K.

Recall from Therorem 4.2 that if $(G, \mathcal{P})$ is a relatively hyperbolic group pair and $\mathcal{P}'$ is the set of non 2-ended subgroups of $\mathcal{P}$, then $(G, \mathcal{P}')$ is a relatively hyperbolic group pair and the Bowditch boundary $\partial(G, \mathcal{P}')$ is obtained from $\partial(G, \mathcal{P})$ by unpinching the parabolic points of $\partial(G, \mathcal{P})$ with two-ended stabilizers. Furthermore in our situation (in fact whenever $\partial(G, \mathcal{P})$ is planar) the unpinched boundary $\partial(G, \mathcal{P}')$ is also planar.

There are three cases to consider for relatively hyperbolic group pairs with Schottky set boundary. The first is when the incidence graph has one component.

Theorem 7.1. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group pair such that $\partial(G, \mathcal{P})$ is a Schottky set with connected incidence graph. Let $(G, \mathcal{P}')$ be the relatively hyperbolic group pair where $\mathcal{P}'$ consists of the subgroups in $\mathcal{P}$ that are not two-ended. Then $\partial(G, \mathcal{P}')$ is a Cantor set.*

Proof. We will prove that if $\partial(G, \mathcal{P}')$ has a non-trivial component, it is a dendrite. However, this is impossible according to Lemma 2.7. The theorem will then follow.

Take a component L of $\partial(G, \mathcal{P})$. Suppose that L contains at least two points x and y.

- L is a connected, locally connected compact metrizable space. The component is connected by definition. A component L is itself the boundary of a relatively hyperbolic subgroup pair: the subgroup stabilizing L along with the peripheral subgroups whose fixed points belong to L [4]. Thus L is compact and a metric space. Furthermore, the set of peripheral subgroups is a subset of $\mathcal{P}'$, each of whose elements is a closed surface group. Therefore by Bowditch [3] the boundary L is locally connected.
- The component L contains no simple closed curve. Any simple closed curve bounds two discs in S^2 which are either contained in L or not. At least one must be contained in L as the complementary region would correspond to an additional component of the incidence graph, which is connected. If the simple closed curve bounds a disk in L, then the boundary has non-empty interior but then it must be all of S^2, since it is the boundary of a relatively hyperbolic group (limit points of loxodromic elements are dense).

Thus, L should be a drendrite, so we may now conclude that there are no non-trivial components.

□

Proof of Theorem B. Let $(G, \mathcal{P}')$ be the relatively hyperbolic group pair where $\mathcal{P}'$ consists of the subgroups in $\mathcal{P}$ that are not two-ended. According to Theorem 7.1, the Bowditch boundary of the group pair $(G, \mathcal{P}')$ is a Cantor set, hence its ordinary set on S^2 is connected. We are now in a position to apply Theorem 2.11 and conclude that, in this case, the group G is virtually a free product of infinite cyclic groups and finite index subgroups of peripheral groups, which are virtual surface groups. It follows from Corollary 3.10 that G is covered by a Kleinian group if it is finitely generated. The conclusion follows.

□

Remark 7.2. In the same circle of ideas, Otal proves that if $(\mathbb{F}, \mathcal{P})$ is a free relatively hyperbolic group pair such that its Bowditch boundary is a topological Schottky set, then there exists a handlebody with fundamental group $\mathbb{F}$ and disjoint homotopy classes of simple curves on its boundary that represent the peripheral structure $\mathcal{P}$ [27].

8 More components in the incidence graph

In the previous section, under the hypothesis that $\partial(G, \mathcal{P})$ is a topological Schottky set with connected incidence graph, we determined the structure of the group G. Since the incidence graph has 1, 2, or infinitely many components, we now analyze what happens in the latter two cases.

Theorem C. *Let $\mathcal{S}$ be a topological Schottky set with $\mathcal{S} = \partial(G, \mathcal{P})$. When the incidence graph $\Gamma(\mathcal{S})$ has exactly 2 components G is virtually a closed surface group.*

Proof. We recall that the rank 1 parabolic points in $\mathcal{P}$ correspond to the edges of the incidence graph by Lemma 6.3 and the very definition of the incidence graph.

Then, we unpinch the rank-1 parabolic points as in Theorem 4.2. This results in a different geometrically finite action of the group G. For every parabolic point removed, the two components of the domain of discontinuity that corresponded to the endpoints of the edge are contained in the same component. So when there are no more rank-1 parabolic points, there are two components of the domain of discontinuity. Then by Corollary 3.10, G is virtually Fuchsian with limit set S^1.

Since we already removed all of the rank-1 parabolic points, G is virtually a closed surface group and $P_2 = \emptyset$.

□

When the incidence graph has infinitely many components, the topology of the blown-up limit set can be extremely varied so there is no hope of getting a meaningful description of the underlying group. Indeed, the next theorem shows in particular that the limit set of any finitely generated Kleinian group with infinitely many components in its regular set and no two-ended parabolic subgroups is isomorphic to the boundary of some $(G, \mathcal{P}_2)$ obtained by blowing up all the rank-one parabolics of a relatively hyperbolic group $(G, \mathcal{P}_1 \cup \mathcal{P}_2)$ where $\partial(G, \mathcal{P}_1 \cup \mathcal{P}_2)$ is a topological Schottky set.

Theorem D. *Let K be a geometrically finite Kleinian group with non-empty domain of discontinuity. Then there is a peripheral structure $\mathcal{P}_{K'}$ on a finite index subgroup K' of K, such that $(K', \mathcal{P}_{K'})$ is a relatively hyperbolic group pair and $\partial(K', \mathcal{P}_{K'})$ is a topological Schottky set. Moreover, $\mathcal{P}_{K'}$ contains the natural peripheral structure of the Kleinian group $K' \subset K$.*

Proof. We choose K' to be a torsion-free finite-index subgroup of K contained in $PSL(2,\mathbb{C})$. Below we will define a peripheral structure $\mathcal{P}_{K'}$ of K' that will contain all the parabolic subgroups of $K' < PSL(2,\mathbb{C})$ but will in general be larger.

In this situation, there is an irreducible and orientable manifold with boundary $M_{K'}$ obtained as the quotient of the 1-neighborhood of the convex hull of $\Lambda_{K'}$, the limit set of K', by the action of K'. There is at least one geometrically finite end, as the group is geometrically finite and its limit set is not all of S^2. This manifold comes equipped with a natural pared structure, given by the parabolic structure on K'. This realizes the boundary of $M_{K'}$ as a union of connected surfaces with boundary, which corresponds to the rank-1 cusps in the hyperbolic structure. We will add curves to the peripheral structure so that the resulting pared manifold contains no essential annuli or disks, and thus admits a hyperbolic structure with totally geodesic boundary [25, Theorem B' page 70].

We will first consider the case when these surfaces are incompressible. Now, in this situation, $M_{K'}$ admits a JSJ-decomposition along a finite family of pairwise disjoint and non parallel incompressible annuli A_i into "geometric pieces" (see [32] for a description): I-bundles over surfaces (Seifert fibered pieces) and anannular manifolds with boundary (hyperbolic pieces). By taking a further cover if necessary, that is by taking a further finite-index subgroup, we assume no twisted I-bundle appears in the decomposition. Note that a piece can have different structures. For instance, a solid torus can be seen as a circle bundle over a disk, an interval times an annulus, as well as a twisted I-bundle over a Möbius band. We only require each piece to admit some product structure.

The characteristic submanifold $C_{K'}$ in $M_{K'}$ consists of all the surface-times-interval components together with small neighborhoods of the JSJ annuli A_i, which are solid tori T_i. Note that if $C_{K'}$ is empty, $M_{K'}$ with its natural pared structure admits a hyperbolic metric with totally geodesic boundary so that $\partial(K', \mathcal{P}_{K'})$ is a Sierpiński carpet and hence a topological Schottky set.

Otherwise, we observe that the boundary of each solid torus T_i is partitioned into four annuli: two of them contained in $\partial M_{K'}$ and two others properly embedded in $M_{K'}$ and parallel to A_i. For each T_i, we mark two points on each of the four circles that delimit the four annuli in its boundary. We then connect these pairs of points with two arcs in $\partial T_i \cap \partial M_{K'}$ running from one circle to the other.

Remark that $\partial C_{K'} \setminus \partial M_{K'}$ consists of properly embedded annuli contained in the boundary of some tori T_i. The rest of the boundary $\partial C_{K'}$ in $\partial M_{K'}$ is a union of subsurfaces, possibly with boundary or cusps.

For each complementary piece of $\partial M_{K'} \setminus \partial C_{K'}$, we connect all the marked points on its boundary components with an embedded collection of essential (pairwise non-parallel) arcs. Next, if some component of $\partial M_{K'} \setminus \cup_i T_i$ is an annulus, (for instance, if a piece is a solid torus) we connect the pair of points on one boundary component directly with the pair of points on the other boundary component. Each remaining component of $C_{K'} \setminus \cup_i T_i$ is a surface times an interval $S \times I$. In this case again we first connect the marked points on the boundary circles along an embedded collection of essential arcs in $S \times I \cap \partial M_{K'}$ (as was done in the complementary components). Then we take a pair of pants decomposition of each remaining component after cutting along these arcs. The pair of pants decomposition for the pieces of $S \times \{0\}$ should be different from the decomposition for $S \times \{1\}$, in particular, the curves of the pants decomposition for $S \times \{0\}$ should be transverse to curves going through $S \times \{1\}$. Since there are two arcs meeting at each marked point, the union of these arcs and curves is a collection of curves so that any essential annulus in $\partial C_{K'}$ is transverse to some curve in this collection. Since these curves are essential and non-parallel, we can make this collection peripheral. The resulting pared manifold with this peripheral structure will admit a Kleinian representation where the quotient of the 1-neighborhood of the convex hull of the limit set is a hyperbolic manifold with totally geodesic boundary. Therefore its limit set can be realized as a Schottky set.

Assume now that $\partial M_{K'}$ is compressible. In this case, the limit set of K' is not connected. We can choose a finite family $\mathcal{D}$ of properly embedded pairwise disjoint essential disks such that (the closure of) each component of the complement of the disks, $M_{K'} \setminus \cup_{D\in\mathcal{D}} D$, has incompressible boundary and the family $\mathcal{D}$ is minimal with respect to this property. As we did with the JSJ-annuli in the previous case, for each disk D we remove small cylindrical neighborhood C_D and mark two points on each of the circles delimiting the two disks on the boundary of C_D. We then connect the two pairs of points by two arcs in the annulus contained in ∂C_D. For each component N of $M_{K'} \setminus \cup_{C_D\in\mathcal{D}} D$ let us denote C_N the characteristic submanifold of N. Note that we can assume that the annuli of the JSJ-decomposition of N are disjoint from the disks of the family $\mathcal{D}$. We can now repeat the previous argument keeping in mind that this time we need to connect also the marked points on the boundary of the disks.

□

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