Non-standard components of the character variety for a family of Montesinos knots

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October 29, 2012

Abstract

We show that for any integers $n > 3$ and $1 \leq d \leq n-3$, there exists Montesinos knots with n tangles whose $SL_2(\mathbb{C})$-character varieties have arbitrarily many irreducible components of dimension d. Moreover, these irreducible components can be chosen so that the trace of the meridian is non-constant.

AMS classification: Primary 57M25; Secondary 20C99; 57M50.

Keywords: Character varieties, Montesinos knots, bending, parabolic representations.

1 Introduction

The study of character varieties associated to representations of 3-manifold groups into $SL_2(\mathbb{C})$ has received great attention in recent years. Indeed, the understanding of the character variety of a manifold may give some insight on the structure of the manifold itself and notably on the existence of essential surfaces, by means of Culler-Shalen theory [7]. On the other hand, little is known about SL_2-character varieties over fields of positive characteristic. It follows from work of Gonzalez-Acuña and Montesinos [10], that the defining polynomial equations for an SL_2-character variety -which have coefficients in $\mathbb{Z}$- are the same over any field of characteristic different from 2, up to reduction mod p.

Standard results in algebraic geometry ensure that affine variety defined by polynomials with coefficients in $\mathbb{Z}$ has the same geometric properties, like the dimension and the number of irreducible components, when considered over $\mathbb{C}$ or over an algebraically closed field of characteristic p, for almost all primes p. This follows basically from the fact that the dimension of an affine variety (that is, the maximal dimension of its irreducible components) and its irreducible components can be computed algorithmically (see, for instance, [6, Chapter 9] for the dimension, and [6, page 209] for the decomposition into irreducible components).

In our situation, this says that for almost every prime $p > 2$ the SL_2-character variety for a manifold over the algebraic closure of the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ looks precisely like the SL_2-character variety over $\mathbb{C}$. It is, however, possible that for some exceptional prime $p \neq 2$ this is not the case, and the number of its irreducible components or their dimensions might change.

Our first motivation for wanting to establish the occurrence of this phenomenon comes from the fact that one may hope to find, in these special character varieties, "new" curves whose ideal points are associated to essential surfaces which cannot be detected in characteristic 0 (see [24] for examples of essential surfaces presenting this behaviour).

Although coming upon an actual example of surfaces detected only by curves in characteristic p appears to be extremely hard, work by Riley [23, 22] seems to provide evidence that this kind of phenomenon does happen, that is, there might be curves that appear only in certain characteristics. In his paper [23], Riley studied parabolic representations (i.e. where all meridians are sent to matrices with trace ± 2) in characteristic p for the group of a specific Montesinos knot with four tangles, and showed that the group admitted a one-parameter family of non-conjugate parabolic representations for each prime p.

The straightforward observation that in characteristic p parabolic elements have order p implies that the orbifold whose underlying topological space is the 3-sphere and whose singular set is the given Montesinos knot with order of ramification equal to p admits "several" representations in characteristic p, and possibly "more" than in characteristic 0. We shall give a precise meaning to this statement in Section 9.

The above observation motivated our study of the character variety of this and other Montesinos knots in characteristic 0. We are mainly interested in understanding the geometric reason behind the existence of Riley's representations, for its comprehension could lead to a proof of the existence of extra representations in other cases. This turns out to be related to the particular structure of the Montesinos knot considered by Riley, which allows one to perform what Riley calls (in a subsequent paper [22], again on parabolic representations of knots) the *commuting trick*. The commuting trick boils down to the elementary remark that crossings between two arcs whose associated generators commute in a given representation are nugatory and can be arbitrarily changed. Of course, Riley was not the first to exploit this basic fact as Riley himself remarks, cf. [18].

It is well-known that an analysis of the character variety of the knot can be carried out explicitly only for knots whose groups have a very limited number of generators, due to the computational complexity involved. It is thus helpful to find indirect methods to deduce properties (like the number of irreducible components, their dimensions, and their intersections) of the character variety. We will consider a family of Montesinos knots with at least 4 tangles that we shall call *Montesinos knots of Kinoshita-Terasaka type* (see Section 3 for a precise definition).

Using this elementary remark and *bending* (see Section 4), we are able to prove for this class of knots the existence of irreducible components of large dimension in their character variety which are *non-standard* in that they are different from the three *standard* ones: the distinguished curve containing the holonomy character, the abelian component, and the Teichmüller components whose points are associated to representations of the base of the Seifert fibration of the orbifold whose underlying topological space is the 3-sphere and whose singular set is the Montesinos knot with order of singularity equal to 2. These Teichmüller components have dimension at most $n-3$, where n is the number of rational tangles, and, since the meridian is mapped to a hyperbolic isometry of order two, the trace of the meridian is constant equal to 0 on them. Our first main result can thus be stated as follows:

Theorem 1. *Let K be a Montesinos knot of Kinoshita-Terasaka type with $n > 3$ tangles. Its character variety contains (at least) two irreducible components of dimension $\geq n-3$ which are not contained in the hyperplane defined by the condition that the trace of the meridian is equal to* 0.

This follows from two stronger and more precise statements (Theorems 21 and 36) whose proofs will be provided in Sections 5 and 7. In particular, one can establish the precise dimension of these components and say something more on their number (see Theorems 36 and 40).

Theorem 2. *For all integers $m > 0$ and $n > 3$ there is a Montesinos knot of Kinoshita-Terasaka type with n tangles whose character variety contains at least m irreducible components of dimension $n-3$.*

This improvement on Theorem 1 is a consequence of a result by Ohtsuki, Riley and Sakuma on the character varieties of 2-bridge knots which shows that the number of their components can be arbitrarily large [21].

The same methods allow to prove a generalisation of Theorem 2 in which the irreducible components of dimension $n-3$ are replaced by irreducible components of dimension d for any $0 < d < n-3$ (see again Theorem 36).

Although the non-standard components which are the object of Theorem 1 are obtained by bending, as already observed, the commuting trick is responsible for the existence of other non-standard components: this is for instance the case of the *r-components* detected by Mattman in the character variety of certain pretzel knots [19, Thm 1.6]. The geometric interpretation behind the existence of Mattman's non-standard components will be briefly discussed in Section 8.

The paper is organised as follows. In Section 2 we shall recall some basic facts about character varieties. The class of Montesinos knots we shall be dealing with will be introduced in Section 3: there we shall also see that the knots in this class are closely related to connected sums of 2-bridge knots. The main feature of connected sums of knots is that they admit several representations obtained from the representations of the single components by *bending*: this procedure will be described in Section 4. Section 5 and Section 6 will be devoted to the construction respectively of non-standard components of parabolic characters and non-standard components of non-parabolic characters, whose number can be arbitrarily large thus proving Theorems 1 and 2. The contents of Section 7 are more technical, and allow to establish the exact dimension of these non-standard components: the analysis of the parabolic components and the non-parabolic ones occupy a subsection each (Subsections 7.2 and 7.3). Finally, we shall discuss Mattman's non-standard components (Section 8) and comment on the character varieties of Montesinos knots of Kinoshita-Terasaka type over fields of positive characteristic (Section 9).

2 Character varieties

The variety of representations of a finitely presented group G is the set of representations of G in $SL_2(\mathbb{C})$:

$$R(G) = \hom(G, SL_2(\mathbb{C})).$$

Since G is finitely generated, $R(G)$ can be embedded in a product $SL_2(\mathbb{C}) \times \cdots \times SL_2(\mathbb{C})$ by mapping each representation to the image of a generating set. In this way G is an affine algebraic set, whose defining polynomials are induced by the relations of a presentation of G and whose coefficients are thus in $\mathbb{Z}$. By considering Tietze transformations, it is not hard to see that this structure is independent of the choice of presentation of G up to isomorphism, cf. [17][1]. Note that what stated above remains valid if $\mathbb{C}$ is replaced by any other field $\mathbb{K}$, that we shall assume to be algebraically closed for simplicity. In particular, the defining relations for $R(G)$ are the same over every field. We shall write $R(G)_{\mathbb{K}}$ whenever we wish to stress that we are considering representations in $SL_2(\mathbb{K})$. When the subscript $_{\mathbb{K}}$ is omitted, by convention $\mathbb{K} = \mathbb{C}$.

Given a representation $\rho \in R(G)$, its character is the map $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$, $\forall \gamma \in G$. The set of all characters is denoted by $X(G)$.

Given an element $\gamma \in G$, we define the map

$$\begin{array}{rcl} \tau_\gamma : X(G) & \to & \mathbb{C} \\ \chi & \mapsto & \chi(\gamma) \end{array} .$$

Proposition 3 ([7, 10])**.** *The set of characters $X(G)$ is an affine algebraic set defined over $\mathbb{Z}$, which embeds in $\mathbb{C}^N$ with coordinate functions $(\tau_{\gamma_1}, \dots, \tau_{\gamma_N})$ for some $\gamma_1, \dots, \gamma_N \in G$.*

The affine algebraic set $X(G)$ is called the *character variety* of G: it can be interpreted as the algebraic quotient of $R(G)$ by the conjugacy action of $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\mathcal{Z}(SL_2(\mathbb{C}))$.

Note that the set $\{\gamma_1, \dots, \gamma_N\}$ in the above proposition can be chosen to contain a generating set of G. For G the fundamental group of a knot exterior, we will then assume that it always contains a representative of the meridian.

A careful analysis of the arguments in [10] shows that Proposition 3 still holds if $\mathbb{C}$ is replaced by any algebraically closed field, provided that its characteristic is different from 2. Let $\mathbb{F}_p$ denote the field with p elements and $\overline{\mathbb{F}}_p$ its algebraic closure. We have:

Proposition 4 ([10])**.** *Let $p > 2$ be an odd prime number. The set of characters $X(G)_{\overline{\mathbb{F}}_p}$ associated to representations of G over the field $\overline{\mathbb{F}}_p$ is an algebraic set which embeds in ${\overline{\mathbb{F}}_p}^N$ with the same coordinate functions $(\tau_{\gamma_1}, \dots, \tau_{\gamma_N})$ seen in Proposition 3. Moreover, $X(G)_{\overline{\mathbb{F}}_p}$ is defined by the reductions mod p of the polynomials over $\mathbb{Z}$ which define $X(G)_{\mathbb{C}}$.*

A representation $\rho \in R(G)$ is called *irreducible* if no proper subspace of $\mathbb{C}^2$ is $\rho(G)$-invariant. The set of irreducible representations is Zariski open, and so is the set of irreducible characters [7]. We denote them by $R_{\mathrm{irr}}(G)$ and $X_{\mathrm{irr}}(G)$ respectively. The following lemma is proved by Culler and Shalen and González-Acuña and Montesinos in [7, 10] for $\mathbb{C}$.

Lemma 5 ([7, 10])**.** *The projection*

$$\begin{array}{rcl} R(G) & \to & X(G) \\ \rho & \mapsto & \chi_\rho \end{array}$$

[1] What stated here for finitely presented groups holds true for groups of finite type. The Hilbert basis theorem ensures the existence of a finite set of defining polynomials, whose coefficients, though, need not be in $\mathbb{Z}$.

is surjective. Moreover $R_{\text{irr}}(G) \to X_{\text{irr}}(G)$ is a local fibration with fibre the orbit by conjugacy.

The following is well-known for $\mathbb{C}$, but the same proof applies to $\overline{\mathbb{F}}_p$.

Lemma 6. *Let $\mathbb{K} = \mathbb{C}$ or $\overline{\mathbb{F}}_p$ for $p \neq 2$. For a free group on two generators $F_2 = \langle \gamma_1, \gamma_2 \mid \rangle$, $X(F_2)_{\mathbb{K}} \cong \mathbb{K}^3$ with coordinates $(\tau_{\gamma_1}, \tau_{\gamma_2}, \tau_{\gamma_1\gamma_2})$.*

For a compact manifold M, we use the notation $R(M) = R(\pi_1(M))$ and $X(M) = X(\pi_1(M))$. For a knot $K \subset S^3$, we write, $R(K) = R(S^3 \setminus \mathcal{N}(K))$ and $X(K) = X(S^3 \setminus \mathcal{N}(K))$, where $\mathcal{N}(K)$ denotes an open regular neighbourhood of K.

Recall that by [2, Corollary 3.3] the fundamental group of a knot is generated by two meridians if and only if it is a 2-bridge knot (see Section 3).

Corollary 7. *Assume that $K \subset S^3$ is a 2-bridge knot, that is its fundamental group is generated by two meridians: $\pi_1(S^3 \setminus K) = \langle \mu_1, \mu_2 \mid r \rangle$. Then, for $\mathbb{K} = \mathbb{C}$ or $\overline{\mathbb{F}}_p$, $X(K)_{\mathbb{K}}$ is a plane curve with coordinates τ_{μ_1} and $\tau_{\mu_1\mu_2}$.*

This uses Lemma 6 and the fact that $\tau_{\mu_1} = \tau_{\mu_2}$, because μ_1 and μ_2 are conjugate. Moreover, by a theorem of Thurston [25] (see also [15]), each irreducible component has to be at least a curve.

Sometimes it will be convenient to work with $PSL_2(\mathbb{C})$ instead of $SL_2(\mathbb{C})$. In this case we use the notation $R(M, PSL_2(\mathbb{C}))$ for the representation variety while its quotient in invariant theory by conjugacy will be denoted by $X(M, PSL_2(\mathbb{C}))$ (cf. [4, 12] for an interpretation in terms of characters).

Proposition 8 ([9])**.** *Let $\mathcal{O}^2$ be a compact two dimensional orbifold with b cone points and c corners. If e denotes the Euler characteristic of the underlying surface $|\mathcal{O}^2|$, then*

$$\dim X(\mathcal{O}^2, PSL_2(\mathbb{C})) = -3e + 2b + c.$$

Here, $\dim X(\mathcal{O}^2, PSL_2(\mathbb{C}))$ means the maximal dimension of the irreducible components of $X(\mathcal{O}^2, PSL_2(\mathbb{C}))$.

3 Montesinos knots of Kinoshita-Terasaka type

The exposition in this section follows roughly the presentation in Zieschang's paper [26].

Recall that a *rational tangle* is any two-string tangle that can be obtained from the trivial tangle (i.e. two unknotting vertical arcs running parallel from the bottom to the top of a ball seen as a cube) by an isotopy of the ball which does not leave its boundary pointwise fixed. The general form of a rational tangle is shown in Figure 1 where the labels a_i', a_i'' and a_k denote the number of positive crossings, with the convention that a negative crossing counts for -1 positive crossings. It can be shown that the continued fraction $\frac{\beta}{\alpha} = \frac{1}{a_1 + \frac{1}{-a_2 + \dots}}$, where $a_i = a_i' + a_i''$ for $i = 1, \dots, k-1$, is an invariant of the isotopy class of the rational tangle, where isotopies in this case are required to leave the boundary pointwise fixed.

Rational tangles are closely related to 2-*bridge knots* and *links*. These are links obtained by gluing together two trivial tangles along their boundaries, or

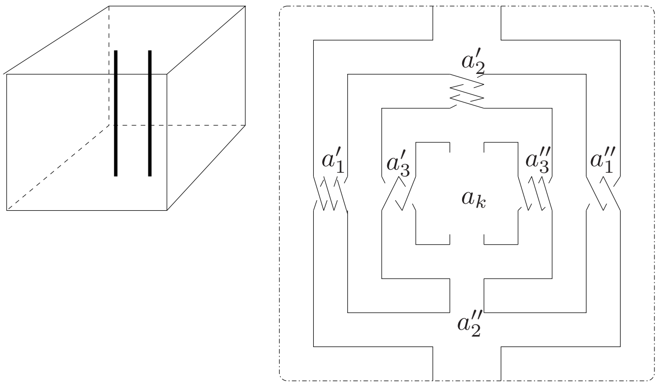

Figure 1: A trivial tangle, and a rational tangle in its standard form. An integer a'_i (respectively a''_i, a_k) represents $|a'_i|$ (respectively $|a''_i|$, $|a_k|$) crossings which are positive if the integer is positive and negative otherwise. Here $a'_1 = 3$, $a''_1 = 2$, $a'_2 = -3$, $a''_2 = 0$, $a'_3 = -2$ and $a''_3 = 3$.

equivalently by closing up a rational tangle by adding two arcs, one connecting the bottom ends of the tangle and one connecting its top ends (See Figure 2). We shall denote by $B(\frac{\beta}{\alpha})$ the 2-bridge link obtained by closing the rational tangle with invariant $\frac{\beta}{\alpha}$.

Montesinos links can be interpreted as a generalisation of 2-bridge links in which several tangles are stacked together one after the other in a circular pattern as shown in Figure 3, the 2-bridge link case corresponding to the situation where a unique tangle is used (see Figure 2). Note, though, that the tangle must be rotated by $\pi/2$ for the two constructions to be consistent; in particular the two continued fractions for the 2-bridge and the Montesinos presentations give rational numbers which are negative reciprocals of one another. It was proved by Bonahon (cf. [3]) that a Montesinos link with $n \geq 3$ tangles is completely determined by the ordered set of the n rational numbers $\frac{\beta_i}{\alpha_i} \in (0,1)$ associated to its n tangles up to cyclic permutation and reversal of order, together with the number $e_0 = e - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}$, where e is the number of crossings that appear outside the n tangles (see Figure 3). Note that these extra crossings can be absorbed in the rational tangles if we do not require their associated continued fractions to belong to $(0,1)$.

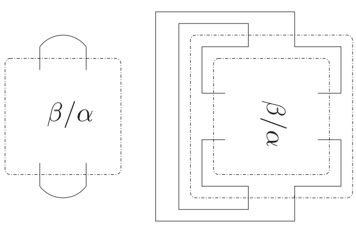

Figure 2: A 2-bridge knot and its Montesinos form; here $\frac{\beta}{\alpha}$ denotes the rational value of the continued fraction associated to the rational tangle.

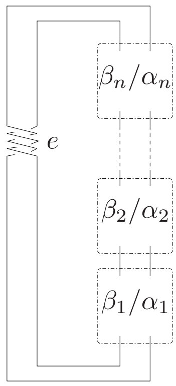

Figure 3: A Montesinos link with n rational tangles.

It is not hard to see that a Montesinos link is a knot if and only if either there is a unique even α_i, or there is no even α_i and the β_is and e satisfy some extra condition (see Boileau-Zimmermann [1, Fig. 4, page 570]). Note that when α_i is even, each arc of the i-th tangle enters and exits the ball on the same side. Since a Montesinos link is defined by its n tangles only up to a cyclic permutation, from now on we shall assume that if one of the α_i is even then α_n is even.

Definition 9. A Montesinos knot with n tangles will be called *of Kinoshita-Terasaka type*[2] if α_n is even.

According to the previous discussion, the α_is are all odd for $1 \leq i \leq n-1$. We can furthermore assume that $e = 0$, up to allowing $\frac{\beta_n}{\alpha_n}$ to be an arbitrary rational. For $n > 2$ we shall denote by $M(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_{n-1}}{\alpha_{n-1}}, \frac{\beta_n}{\alpha_n})$ the Montesinos knot of Kinoshita-Terasaka type obtained by stacking together n rational tangles of invariants $\frac{\beta_i}{\alpha_i}$, satisfying the aforementioned requirements.

Now let $K = M(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_{n-1}}{\alpha_{n-1}}, \frac{\beta_n}{\alpha_n})$ and let K' be the composite knot whose prime summands are the $n-1$ 2-bridge knots $B(\frac{\beta_i}{\alpha_i})$, $i = 1, \dots, n-1$, i.e. $K' = B(\frac{\beta_1}{\alpha_1})\sharp \dots \sharp B(\frac{\beta_{n-1}}{\alpha_{n-1}})$. Note that K' can be obtained from K by changing some crossings in the nth tangle.

For each of the two knots, consider the meridians μ_i and μ_i', $1 \leq i \leq n$, as shown in Figure 4: they generate the fundamental groups of (the exteriors of) K and K', and are in fact a redundant system of generators. This follows from the fact that the fundamental group of the (exterior) of a rational tangle is a free group of rank 2; in particular, the fundamental group of the ith tangle is generated by two meridians among μ_i, μ_i', μ_{i+1} and μ_{i+1}'. Using Wirtinger's method, one can deduce the following presentations for the fundamental groups of K and K':

$$\pi_1(K) = \langle \mu_1, \mu_1', \dots, \mu_n, \mu_n' \mid \mathcal{R}, w_1\mu_1 = \mu_1' w_1, w_n\mu_n = \mu_n' w_n \rangle,$$

and

$$\pi_1(K') = \langle \mu_1, \mu_1', \dots, \mu_n, \mu_n' \mid \mathcal{R}, \mu_1 = \mu_1', \mu_n = \mu_n' \rangle,$$

[2]The expression is already used by Riley in his paper: these knots can in fact be seen as a Kinoshita-Terasaka sum of a 2-component link, see [23] and also [16].

where $\mathcal{R}$ is a set of $2n-2$ relations expressing, for each $i = 1, \ldots, n-1$, two meridians among μ_i, μ'_i, μ_{i+1} and μ'_{i+1} as conjugates of the other two, and are obtained from the Wirtinger relations inside the ith tangle. Similarly w_1 and w_n are products of the elements μ_1, μ'_1, μ_n and μ'_n, and the last two relations are obtained from the Wirtinger relations in the nth tangle. Note that, for each i, the meridians μ_i and μ'_i cobound an annulus in the exterior of K'. Since for each i the annulus can be chosen to pass in between the two unknotted arcs on the left-hand side of the projection for K' in Figure 4, it is possible to choose a basepoint $*$ so that, for all i, $\mu_i = \mu'_i$ in $\pi_1(K', *)$, that is μ_i and μ'_i are not only conjugate but equal. Note, moreover, that the fundamental group of the composite knot K' can also be described as sum of the fundamental groups of its summands amalgamated over cyclic subgroups:

$$\pi_1(K') \cong \pi_1(B(\beta_1/\alpha_1)) *_{\mathbb{Z}} \cdots *_{\mathbb{Z}} \pi_1(B(\beta_{n-1}/\alpha_{n-1})),$$

where the amalgamating subgroups $\mathbb{Z}$ are generated by the meridians $\mu_i = \mu'_i$, $i = 2, \ldots, n-1$.

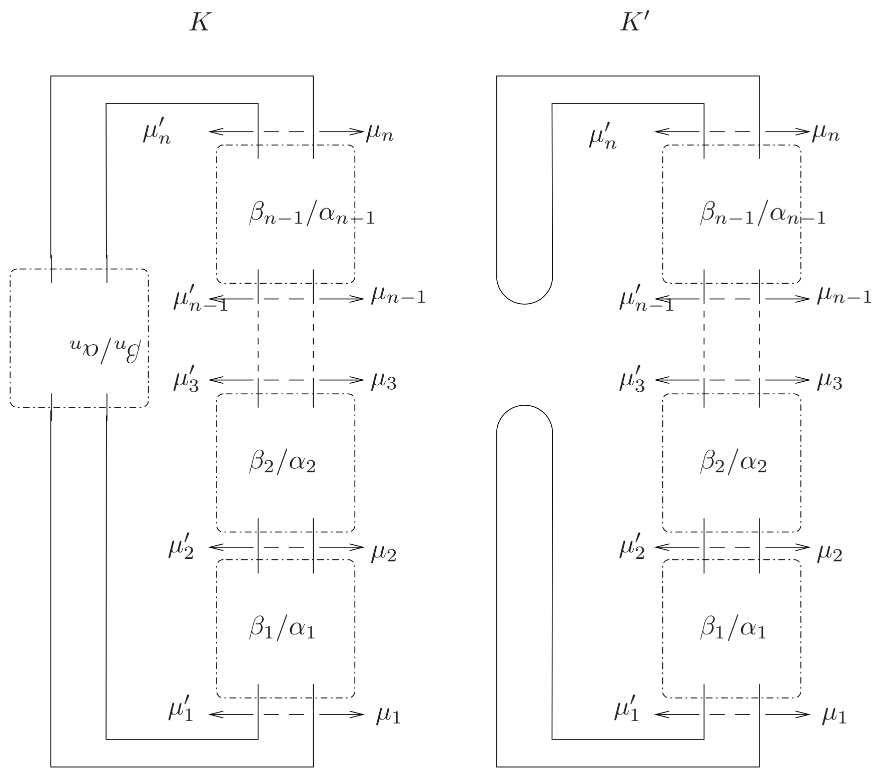

Figure 4: A Montesinos knot of Kinoshita-Terasaka type on the left and its associated composite knot on the right.

The following result is straightforward, in view of the above presentations:

Proposition 10. *Let H and H' denote the normal closures in $\pi_1(K)$ and $\pi_1(K')$ respectively of the six commutators* $[\mu_1, \mu'_1]$, $[\mu_1, \mu_n]$, $[\mu_1, \mu'_n]$, $[\mu'_1, \mu_n]$,

$[\mu_1', \mu_n']$, *and* $[\mu_n, \mu_n']$. *Then the groups* $\pi_1(K)/H$ *and* $\pi_1(K')/H'$ *are isomorphic. In particular, the representation variety of the group* $\Gamma = \pi_1(K)/H \cong \pi_1(K')/H'$ *is a subvariety of both representation varieties for* $\pi_1(K)$ *and* $\pi_1(K')$. *The analogue conclusion holds for the character varieties.*

In what follows we will be mainly concerned with Montesinos knots of Kinoshita-Terasaka type with at least 4 tangles (except in Section 8).

Remark 11. Let $\hat{K}$ be a Montesinos knot of Kinoshita-Terasaka type obtained from K by deleting some of its rational tangles $\frac{\beta_i}{\alpha_i}$ corresponding to indices $i \leq n-1$, (i.e. tangles with odd denominator) and let $\hat{K}'$ be its associated composite knot. It is easy to see that every representation of $\hat{K}'$ extends to a representation of K' whose restriction to the "missing" summands is the obvious abelian one. As a consequence, if $\hat{\Gamma}$ denotes the common quotient of $\pi_1(\hat{K})$ and $\pi_1(\hat{K}')$, as defined in this section, we have that $X(\hat{\Gamma}) \subset X(\Gamma)$.

4 Bending

Recall that $\pi_1(K')$ is the amalgamated product of $\pi_1(B(\frac{\beta_1}{\alpha_1}))$, ..., $\pi_1(B(\frac{\beta_{n-1}}{\alpha_{n-1}}))$ along the cyclic groups generated by $\mu_2, \dots \mu_{n-1}$.

Let $\rho \in R(K')$ be a non-trivial representation. Denote by $A_i < PSL_2(\mathbb{C})$ the projection of the centraliser of $\rho(\mu_i)$ (and $\rho(\mu_i')$) in $SL_2(\mathbb{C})$, for $i = 2, \dots, n-1$. By hypothesis $\rho(\mu_i)$ is non-trivial. For the centraliser A_i we have:

- If $\rho(\mu_i)$ is parabolic, then $A_i \cong \mathbb{C}$ is a parabolic group that stabilises the same point of $\partial_\infty \mathbf{H}^3 \cong \mathbb{P}^1(\mathbb{C}) \cong \hat{\mathbb{C}}$ as $\rho(\mu_i)$.
- If $\rho(\mu_i)$ is hyperbolic or elliptic (i.e. $trace(\rho(\mu_i)) \neq \pm 2$), then $A_i \cong \mathbb{C}^*$ is the group that preserves the same oriented geodesic as $\rho(\mu_i)$.

Denote by ρ_i the restriction of the representation $\rho \in R(K')$ to the vertex group $\pi_1(B(\frac{\beta_i}{\alpha_i}))$ for each $i = 1, \dots, n-1$. If $a = (a_2, \dots, a_{n-1}) \in A_2 \times \cdots \times A_{n-1}$ where $A_i \subset PSL_2(\mathbb{C})$ is the projection of the centraliser in $SL_2(\mathbb{C})$ of $\rho_{i-1}(\mu_i) = \rho_i(\mu_i)$, then $a\rho$ is the representation defined as

$$a\rho{=}\rho_1 * {}^{a_2}\rho_2 * {}^{a_2 a_3}\rho_3 * \cdots * {}^{a_2 a_3 \dots a_{n-1}}\rho_{n-1},$$

where ${}^x\rho$ is the representation obtained by conjugating ρ by $x \in PSL_2(\mathbb{C})$.

By [14, Lemma 5.6] we have:

Lemma 12 (Johnson and Millson [14])**.** *Suppose that* ρ_i *is an irreducible representation for each* $i = 1, \dots, n-1$. *Then there is a neighbourhood* U *of the identity in* $A_2 \times \cdots \times A_{n-1}$ *such that for* $a, b \in U$ $a\rho$ *is conjugate to* $b\rho$ *if and only if* $a = b$.

Consider the map whose components are the restriction of characters to each $\pi_1(B(\frac{\beta_i}{\alpha_i}))$:

$$\pi : X(K') \to X(B(\tfrac{\beta_1}{\alpha_1})) \times \cdots \times X(B(\tfrac{\beta_{n-1}}{\alpha_{n-1}})).$$

Corollary 13. *Let* $(\chi_1, \dots, \chi_{n-1}) \in X(B(\frac{\beta_1}{\alpha_1})) \times \cdots \times X(B(\frac{\beta_{n-1}}{\alpha_{n-1}}))$. *If non-empty, the fibre* $\pi^{-1}(\chi_1, \dots, \chi_{n-1})$ *has dimension* $\geq s-1$, *where* s *is the number of irreducible characters among* $\chi_1, \dots, \chi_{n-1}$.

Proof. Assume first that all the χ_i are irreducible, i.e. $s = n-1$. Then the dimension of the fibre is $\geq n-2$ by Lemmas 12 and 5. For an arbitrary $s \geq 1$, we just use the argument of Remark 11. □

Note that if all χ_i are irreducible, then the inequality in Corollary 13 is an equality by Lemma 5. We now want to give also an upper bound on the dimension of the fibre in the general case. For that we need to understand the reducible characters. We begin with the case where all χ_i are parabolic, and start with a remark:

Remark 14. A parabolic representation of $\pi_1(B(\frac{\beta}{\alpha}))$ that is reducible is also abelian and, up to conjugacy, it maps $\gamma \in \pi_1(B(\frac{\beta}{\alpha}))$ to

$$\pm \begin{pmatrix} 1 & h(\gamma) \\ 0 & 1 \end{pmatrix}$$

for some homomorphism $h : \pi_1(B(\frac{\beta}{\alpha})) \to \mathbb{C}$. Its character is the trivial one, though the representation may be non-trivial.

Since $\pi_1(K')$ is normally generated by any meridian, we are only interested in the case where the previous h is nontrivial.

Thus when a character χ_i is parabolic and reducible, then χ_i is the character of an abelian representation ρ_i for which $\rho_i(\mu_i) = \rho_i(\mu_{i+1})$. It follows that the global representation $\rho \in R(K')$ is in fact a representation of some $\hat{K}'$ in which the i-th tangle is omitted (see Remark 11). Therefore we have:

Corollary 15. *Let* $(\chi_1, \ldots, \chi_{n-1}) \in X(B(\frac{\beta_1}{\alpha_1})) \times \cdots \times X(B(\frac{\beta_{n-1}}{\alpha_{n-1}}))$ *be parabolic (i.e.* χ_i *of the meridian is* ± 2*). If non-empty, the fibre* $\pi^{-1}(\chi_1, \ldots, \chi_{n-1})$ *has dimension precisely* $s-1$*, where* s *is the number of irreducible characters among* $\chi_1, \ldots, \chi_{n-1}$*.*

For non-parabolic characters that are reducible, we have to distinguish between those that are characters of only abelian representations and those that are also characters of (reducible) non-abelian ones.

The following lemma is due to [5] and [8], cf [13].

Lemma 16. *Let* $\chi \in X(B(\frac{\beta}{\alpha}))$ *be a reducible character. Then the following are equivalent:*

(i) χ *is the character of a non-abelian representation (besides abelian ones).*

(ii) χ *belongs to the Zariski closure of an irreducible component of* $X(B(\frac{\beta}{\alpha}))$ *that contains irreducible characters.*

(iii) $\chi(\mu) = \theta + 1/\theta$*, where* θ^2 *is a root of the Alexander polynomial of the knot and* μ *a meridian.*

Definition 17. A character in $X(B(\frac{\beta}{\alpha}))$ is called *generic reducible* if it is reducible and does not satisfy the assertions of Lemma 16.

By Lemma 16, since ± 1 is not a root of the Alexander polynomial, when $\chi(\mu) = 0$ or ± 2, if χ is reducible then it is generic reducible (i.e. it is not the character of a reducible non-abelian representation). Hence as in Corollary 15, we have:

Corollary 18. *Let* $(\chi_1, \dots, \chi_{n-1}) \in X(B(\frac{\beta_1}{\alpha_1})) \times \cdots \times X(B(\frac{\beta_{n-1}}{\alpha_{n-1}}))$*. Assume that if* χ_i *is reducible then it is generic reducible (i.e. it does not satisfy the assertions of Lemma 16). If non-empty, the fibre* $\pi^{-1}(\chi_1, \dots, \chi_{n-1})$ *has dimension precisely* $s-1$*, where* s *is the number of irreducible characters among* $\chi_1, \dots, \chi_{n-1}$.

5 Parabolic representations

In this section we shall define parabolic representations for the Montesinos knot of Kinoshita-Terasaka type $M(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_{n-1}}{\alpha_{n-1}}, \frac{\beta_n}{\alpha_n})$ which are induced by parabolic representations of the $n-1$ 2-bridge knots $B(\frac{\beta_i}{\alpha_i})$, $i = 1, \dots, n-1$.

Definition 19. We shall denote by $X_{\text{par}}(K)$ the subvariety of the character variety of a knot K consisting of characters associated to *parabolic representations*, i.e. those in which the meridian μ of K is mapped to a parabolic matrix:

$$X_{\text{par}}(K) = \{\chi \in X(K) \mid \chi(\mu) = \pm 2\}.$$

Note that $X_{\text{par}}(K)$ is never empty for it always contains at least the trivial character (associated to the trivial representation).

Let now K denote again the Montesinos knot of Kinoshita-Terasaka type $M(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_{n-1}}{\alpha_{n-1}}, \frac{\beta_n}{\alpha_n})$ and K' its associated composite knot as defined in Section 3.

For each $i = 1, \dots, n-1$, let $\rho_i : \pi_1(B(\frac{\beta_i}{\alpha_i})) \longrightarrow SL_2(\mathbb{C})$ be an irreducible parabolic representation of the 2-bridge knot $B(\frac{\beta_i}{\alpha_i})$. The existence of such a ρ_i can be seen as follows: $\pi_1(B(\frac{\beta_i}{\alpha_i}))$ admits an irreducible representation in $PSL_2(\mathbb{C})$ which corresponds to the holonomy representation of $B(\frac{\beta_i}{\alpha_i})$ if the 2-bridge knot is hyperbolic or to the holonomy representation of the base of its fibration if $B(\frac{\beta_i}{\alpha_i})$ is a torus knot. It then suffices to lift this irreducible representation to $SL_2(\mathbb{C})$ by choosing the same trace sign for each generator. This is consistent because all generators are conjugate and because of the very nature of the Wirtinger's relations.

Since all non-diagonal parabolic matrices belong to two conjugacy classes according to the sign of their trace, up to conjugacy, we can assume that $\rho_{i-1}(\mu_i) = \rho_i(\mu_i)$ for all $i = 2, \dots, n-1$. One can thus define a representation

$$\rho = \rho_1 * \cdots * \rho_{n-1} : \pi_1(K') \longrightarrow SL_2(\mathbb{C}).$$

We remark that in fact one can define several different representations in this way, just by conjugating ρ_i by an element in the centraliser of $\rho_{i-1}(\mu_i) = \rho_i(\mu_i)$ as discussed in the previous section.

We now want to show that one can find representations of this kind which factor through representations of Γ, where Γ is the common quotient of $\pi_1(K)$ and $\pi_1(K')$ defined in Section 3. Recall from the previous section that if $a = (a_2, \dots, a_{n-1}) \in A_2 \times \cdots \times A_{n-1}$ where A_i is the projection in $PSL_2(\mathbb{C})$ of the centraliser of $\rho_{i-1}(\mu_i) = \rho_i(\mu_i)$, then $a\rho$ is the representation defined as $\rho_1 * {}^{a_2}\rho_2 * {}^{a_2 a_3}\rho_3 * \cdots * {}^{a_2 a_3 \dots a_{n-1}}\rho_{n-1}$, where ${}^{x}\rho$ is the representation obtained by conjugating ρ by x. We have

Lemma 20. *For each ρ, there is an $a \in A_2 \times \cdots \times A_{n-1}$, such that $a\rho(\mu_1)$ and $a\rho(\mu_n)$ commute.*

Proof. Recall that the subgroup $\pi_1(B(\frac{\beta_i}{\alpha_i}))$ of $\pi_1(K')$ is generated by μ_i and μ_{i+1}, for $i = 1, \ldots, n-1$. This implies that that $\rho_i(\mu_i)$ and $\rho_i(\mu_{i+1})$ cannot belong to the same reducible subgroup of $SL_2(\mathbb{C})$ for ρ_i is irreducible. If we consider the natural action of $SL_2(\mathbb{C})$ on $\mathbb{CP}^1 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, this is equivalent to saying that the fixed points of $\rho_i(\mu_i)$ and $\rho_i(\mu_{i+1})$ are different. Note also that the centraliser A_i of $\rho_i(\mu_i)$ acts transitively on $\hat{\mathbb{C}} \setminus Fix(\rho_i(\mu_i))$.

We shall start by proving the lemma for $n = 4$. Without loss of generality, we may assume that the fixed point of $\rho_2(\mu_2)$ is 0 and that of $\rho_2(\mu_3)$ is ∞. We can choose an element $a_2 \in A_2$ and an element $a_3 \in A_3$ such that the fixed points of ${}^{a_2}\rho_1(\mu_1)$ and that of ${}^{a_3}\rho_3(\mu_4)$ are both equal to, say, 1. This is possible because the fixed point of $\rho_1(\mu_1)$ is in $\hat{\mathbb{C}} \setminus \{0\}$ and that of $\rho_3(\mu_4)$ is in $\hat{\mathbb{C}} \setminus \{\infty\}$. In fact, one can choose the common fixed point for ${}^{a_2}\rho_1(\mu_1)$ and ${}^{a_3}\rho_3(\mu_4)$ to be any point in $\mathbb{C} \setminus \{0\}$. It is now evident that $\rho_1(\mu_1)$ and $\rho_3(\mu_4)$ commute, because they are parabolic elements fixing the same point. The desired representation is then ${}^{a_2}\rho_1 * \rho_2 * {}^{a_3}\rho_3$, which is conjugate to $a\rho$ with $a = (a_2^{-1}, a_3)$.

Assume now that $n > 4$. The same argument applies using μ_{n-1} instead of μ_3. Note that, for the argument to work, there is no need for the fixed points of $\rho_2(\mu_2)$ and of $\rho_{n-2}(\mu_{n-1})$ to be distinct. In fact, in the case when they coincide, one only needs to conjugate ρ_{n-1} (and not ρ_1). □

We already knew that the intersection $X_{\text{par}}(K') \cap X(\Gamma)$ is not empty, for it contains the trivial character. The previous lemma shows moreover that this intersection contains an irreducible character. The bending procedure seen in Section 4 assures that this irreducible character is contained in an irreducible component Y' of $X_{\text{par}}(K')$ of dimension at least $n-2$. It is now easy to see that the subvariety $X_{\text{par}}(K') \cap X(\Gamma)$ is obtained by intersecting $X_{\text{par}}(K')$ with the hypersurface defined by the equation $\chi([\rho(\mu_1), \rho(\mu_n)]) = 2$. It is indeed elementary to see that the commutator of two parabolic elements is trivial if and only if its trace is equal to 2. As a consequence the dimension of $Y' \cap X(\Gamma)$ is at least $n-3$ since $Y' \cap X(\Gamma)$ is non-empty.

The above considerations together with the fact, seen in Section 3, that $X(\Gamma) \subset X(K)$ give the following:

Theorem 21. *Let K be a Montesinos knot of Kinoshita-Terasaka type with n tangles, $n > 3$. The subvariety $X_{\text{par}}(K)$ contains a parabolic component of dimension at least $n-3$.*

In addition, the number of such components can be arbitrarily large.

For the last assertion, we use that the parabolic component of a 2-bridge knot consists of finitely many points, but its cardinality can be arbitrarily large by [21].

Observe that the irreducible components described in the above theorem correspond to the components studied by Riley for a specific Montesinos knot of Kinoshita-Terasaka type with 4 tangles over fields of positive characteristic in [23]. More precisely, Riley only considered the $\mathbb{F}_p$-rational points corresponding to homomorphisms of the knot group to the finite group $SL_2(\mathbb{F}_p)$. Observe also that, although an easy upper bound on the dimension of $X_{\text{par}}(K)$ can be given in terms of the number of generators of $\pi_1(K)$ or, equivalently, in terms of the number of rational tangles of K, at this point we are unable to establish the precise dimension of $X_{\mathrm{par}}(K)$. This requires some extra considerations and will be achieved in Theorem 40.

Finally note that the characters that we have constructed belong to components of $X_{\mathrm{par}}(\Gamma) = X_{\mathrm{par}}(K) \cap X(\Gamma)$, which a priori can be contained in some larger components of $X_{\mathrm{par}}(K)$. We will see later that the components of $X_{\mathrm{par}}(K) \cap X(\Gamma)$ are in fact components of $X_{\mathrm{par}}(K)$ (see Lemma 38).

Remark 14 allows us to construct parabolic representations of $\pi_1(K')$ that are irreducible on some of the $\pi_1(B(\frac{\beta_i}{\alpha_i}))$ and abelian on the others.

Remark 22. Reasoning as in Remark 11, one can see that it is possible to construct other parabolic components of $X_{\mathrm{par}}(\Gamma)$ of smaller dimension by choosing some of the ρ_is to be abelian: If ℓ representations among the $n-1$ are abelian, with $0 \leq \ell \leq n-3$, the resulting components of $X_{\mathrm{par}}(\Gamma)$ have dimension $\geq n-\ell-3$. Note that if $\ell = n-1$ we obtain a point corresponding to the abelian parabolic character. On the other hand, the case $\ell = n-2$ is impossible, because the images of the two meridians in an irreducible representation cannot commute.

6 The non-parabolic case

We turn now to consider the case of non-parabolic representations of a Montesinos knot of Kinoshita-Terasaka type $K = M(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_{n-1}}{\alpha_{n-1}}, \frac{\beta_n}{\alpha_n})$ arising from representations of the 2-bridge knots $B(\frac{\beta_i}{\alpha_i})$, $i = 1, \ldots, n-1$. The construction will be similar to the one seen in the previous section. We start by summarising some properties of representations for 2-bridge knots.

Proposition 23. *The character variety of a 2-bridge knot $X(B(\frac{\beta}{\alpha}))$ is a union of plane curves $\mathcal{C}_0 \cup \cdots \cup \mathcal{C}_r \subset \mathbb{C}^2$, $r \geq 1$. Moreover, the map*

$$\tau_\mu : \mathcal{C}_j \to \mathbb{C},$$

where μ denotes a meridian, is proper.

The reducible characters form a component $\mathcal{C}_0 = X_{\mathrm{red}}(B(\frac{\beta}{\alpha}))$ such that $\tau_\mu : \mathcal{C}_0 \to \mathbb{C}$ is an isomorphism.

Proof. The components are plane curves by Corollary 7. Properness of τ_μ means that whenever a sequence $\chi_n \in X(B(\frac{\beta}{\alpha}))$ goes to infinity, then $\tau_\mu(\chi_n) = \chi_n(\mu) \to \infty$: this is a consequence of the fact that μ is not a boundary slope (cf. [7, 11]). For the second assertion, just notice that each reducible character is also the character of an abelian representation, and the abelianisation of a knot group is $\mathbb{Z}$, generated by the representative of μ. □

It follows from Lemma 16 that for $j > 0$ the component $\mathcal{C}_j$ contains only a finite number of reducible characters, because the Alexander polynomial has a finite number of zeros. Thus for almost every value of $\tau \in \mathbb{C} \setminus \{\pm 2\}$ and for all $1 \leq i \leq n-1$ there is an irreducible representation ρ_i of $\pi_1(B(\frac{\beta_i}{\alpha_i}))$ such that $\chi_{\rho_i}(\mu_i) = \tau$. Since any two matrices of $SL_2(\mathbb{C})$ having the same trace $\tau \neq \pm 2$ are conjugate, it follows easily that the ρ_is can be matched together to give a representation of the composite knot K'. It follows at once that for each choice of irreducible 1-dimensional components $Z_1 \subset X(B(\frac{\beta_1}{\alpha_1}))$,...,

$Z_{n-1} \subset X(B(\frac{\beta_{n-1}}{\alpha_{n-1}}))$, each containing irreducible characters, one can construct an irreducible component $\mathcal{C}$ of $X(K')$. The bending argument of Corollary 13 shows that the dimension of $\mathcal{C}$ is at least $n-1$, the extra dimension with respect to the parabolic case coming from the fact that τ is a free parameter.

We now want to show that $\mathcal{C} \cap X(\Gamma)$ is non-empty. The argument will follow the same lines of Lemma 20. However, since the elements $\rho(\mu_1)$ and $\rho(\mu_n)$ are not parabolic, they commute if and only if they have the same axis (cf. Section 4). Thus this time we need to keep track of two points in $\hat{\mathbb{C}}$. As in the parabolic case, one of these two points can be moved in an arbitrary way, however the position of the second point will be determined by the position of the first, because cross-ratios are preserved by the $SL_2(\mathbb{C})$-action on $\hat{\mathbb{C}}$. The following elementary observations will be useful.

Lemma 24.
- *For each $\lambda \in \mathbb{C} \setminus \{0,1\}$, the subgroup of $SL_2(\mathbb{C})$ which fixes pointwise a and b in $\hat{\mathbb{C}}$ acts simply transitively on the pairs of distinct points p, q of $\hat{\mathbb{C}} \setminus \{a,b\}$ such that the cross-ratio $[a,b,p,q] = \lambda$.*
- *Let $a,b,c,d \in \hat{\mathbb{C}}$ be four pairwise different points. For all $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0,1\}$, there are two points $x \neq y \in \hat{\mathbb{C}} \setminus \{a,b,c,d\}$ such that $[a,b,x,y] = \lambda_1$ and $[c,d,x,y] = \lambda_2$.*

Proof. The first part of the lemma follows from the fact that the subgroup of $SL_2(\mathbb{C})$ which fixes pointwise two points of $\hat{\mathbb{C}}$ acts simply transitively on the remaining points and the fact that once a, b and p are fixed there is a unique q such that $[a,b,p,q] = \lambda$.

For the second part, without loss of generality, we may assume that $a = 0$, $b = \infty$ and $c = 1$. We must find two points x and y such that $\lambda_1 = [0,\infty,x,y] = \frac{x}{y}$ and $\lambda_2 = [1,d,x,y] = \frac{(1-x)(d-y)}{(d-x)(1-y)}$. From the first condition we get $x = y\lambda_1$. Replacing in the second we get a polynomial equation of degree 2 in the unknown y:

$$\lambda_1(\lambda_2 - 1)y^2 + y(1 + d\lambda_1 - d\lambda_2 - \lambda_1\lambda_2) + d(\lambda_2 - 1) = 0. \qquad (1)$$

This equation always admits a solution in $\mathbb{C}$ since $\lambda_1(\lambda_2 - 1) \neq 0$ by hypothesis. We still need to verify that the solution is admissible. Note that $y = 0$ cannot be a solution for $d(\lambda_2 - 1) \neq 0$, $y = 1$ cannot be a solution for $d(\lambda_1 - 1) \neq 1$, and $y = d$ cannot be a solution for $d(1-d)(1-\lambda_1)\lambda_2 \neq 0$. Similarly one sees that x cannot be equal to 0, 1 or d. □

The fact that Equation 1 has two solutions may be understood in terms of symmetries as follows. Consider the rotation r of angle π in hyperbolic space $\mathbf{H}^3$, that permutes a and b, as well as c and d. Thus if $\overline{ab}$ and $\overline{cd} \subset \mathbf{H}^3$ denote the hyperbolic geodesics with respective end-points a and b, and c and d, then r is the π-rotation around the geodesic perpendicular to $\overline{ab}$ and $\overline{cd}$. Moreover r induces an involution on each of these geodesics that reverses the orientation.

It is easy to check that if (x,y) satisfies the second assertion of Lemma 24, then so does $(r(y), r(x))$, and those are all solutions. Namely, in the the notation of the proof ($a = 0$, $b = \infty$, $c = 1$ and $d = d$) r is the Möbius transformation of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z \mapsto d/z$ for all $z \in \hat{\mathbb{C}}$. Therefore $r(y) = d/y$ and $r(x) = d/x = \frac{d}{\lambda_1 y}$. Notice that the product of the solutions of Equation 1 is precisely $y\, r(x) = d/\lambda_1$.

Remark 25. There are precisely two ordered pairs of points satisfying the second assertion of Lemma 24, (x, y) and $(r(y), r(x))$, where r is the hyperbolic rotation of order two that satisfies $r(a) = b$ and $r(c) = d$.

With the same notation as in the previous section we have:

Lemma 26. *For each ρ there is an $a \in A_2 \times \cdots \times A_{n-1}$, such that $a\rho(\mu_1)$ and $a\rho(\mu_n)$ commute.*

Proof. Note that as in the parabolic case the two elements $\rho_i(\mu_i)$ and $\rho_i(\mu_{i+1})$ acting on $\hat{\mathbb{C}}$ have no fixed point in common, for the representation ρ_i is irreducible by hypothesis. Assume that $n = 4$. Let a and $b \in \hat{\mathbb{C}}$ be the fixed points of $\rho_2(\mu_2)$, and c and $d \in \hat{\mathbb{C}}$ be the fixed points of $\rho_2(\mu_3)$. Let $p, q \in \hat{\mathbb{C}}$ and $r, s \in \hat{\mathbb{C}}$ be the fixed points of $\rho_1(\mu_1)$ and $\rho_3(\mu_4)$ respectively. We define $\lambda_1 = [a, b, p, q]$ and $\lambda_2 = [c, d, r, s]$. The previous lemma tells that there is an element in the centraliser of $\rho_2(\mu_2)$ and one in the centraliser of $\rho_2(\mu_3)$ that conjugate $\rho_1(\mu_1)$ and $\rho_3(\mu_4)$ respectively to elements with the same fixed points. As a consequence, one can find a representation $a\rho$ such that $a\rho(\mu_1)$ and $a\rho(\mu_n)$ commute.

If $n > 4$, consider the elements $\rho_2(\mu_2) = \rho_1(\mu_2)$ and $\rho_{n-2}(\mu_{n-1}) = \rho_{n-1}(\mu_{n-1})$: if they have no common fixed point it suffices to apply verbatim the argument seen for $n = 4$. Otherwise, one can start by conjugating ρ_{n-2} and ρ_{n-1} by an element in the centraliser of $\rho_{n-2}(\mu_{n-2})$ to make sure that this is indeed the case. □

Remark 27. Two hyperpolic isometries which are conjugate and have the same axis are either equal or inverses of one another. By appropriately choosing the order of the end-points of the axis of $\rho_1(\mu_1)$ and $\rho_{n-1}(\mu_n)$ we can ensure that $a\rho(\mu_1)$ and $a\rho(\mu_n)$ satisfy either one of the situations above. We will always assume we have made the choice that $a\rho(\mu_1) = a\rho(\mu_n)$.

Proposition 28. *Let K be a Montesinos knot of Kinoshita-Terasaka type with n tangles, $n > 3$. The subvariety $X(\Gamma)$ of $X(K)$ contains components of dimension $\geq n - 3$ on which the trace of the meridian is non-constant.*

In addition, the number of such components can be arbitrarily large.

Proof. We know that $X(K')$ contains irreducible components $\mathcal{C}$ of dimension $n - 1$ on which the trace of the meridian is non-constant. Lemma 26 ensures that the intersection $\mathcal{C} \cap X(\Gamma)$ is non-empty. Since we need to impose two conditions to the points of the components of $\mathcal{C}$ for them to belong to $X(\Gamma)$ we see that we obtain in $X(\Gamma) \subset X(K)$ components of dimension at least $n - 3$. Note that the construction shows that even on these components the trace of the meridian is not constant.

For the last assertion, we use again that the number of irreducible 1-dimensional components of a 2-bridge knot containing irreducible characters can be arbitrarily large, according to [21]. □

For later use, we discuss the space of solutions in Lemma 26. We start with the case $n = 4$. Assume that $\rho = \rho_1 * \rho_2 * \rho_3 \in R(\Gamma)$ is a representation so that ρ_1, ρ_2 and ρ_3 are irreducible. In particular $a = (a_2, a_3) = (Id, Id)$ is a solution. To find further solutions, let r_i denote the π-rotation of $\mathbf{H}^3$ around the geodesic perpendicular to the axes of both $\rho_i(\mu_i)$ and $\rho_i(\mu_{i+1})$ (here $\rho_3(\mu_4) = \rho_1(\mu_1)$). Then

$$^{r_1}\rho_1 *^{r_2} \rho_2 *^{r_3} \rho_3$$

is also a representation of $R(\Gamma)$, since r_i conjugates $\rho_i(\mu_i)$ and $\rho_i(\mu_{i+1})$ to their inverses. As $r_i^2 = Id$, $^{r_1}\rho_1 *^{r_2} \rho_2 *^{r_3} \rho_3$ is conjugate to

$$^{r_2 r_1}\rho_1 * \rho_2 *^{r_2 r_3} \rho_3.$$

This representation corresponds to the second solution of Equation 1 and is obtained as explained in Remark 25. Indeed both r_1 and r_3 permute the endpoints of the axis of $\rho_1(\mu_1) = \rho_3(\mu_4)$, while r_2 is the rotation r in Remark 25. Since this representation is also conjugate to

$$\rho_1 *^{r_1 r_2} \rho_2 *^{r_1 r_2 r_2 r_3} \rho_3,$$

we have that $(a_2, a_3) = (r_1 r_2, r_2 r_3)$ is another solution different from the trivial one. It follows moreover from the discussion in Remark 25 that these are all solutions.

For larger n, there are more indeterminacies, but by the same argument we get the following lemma:

Lemma 29. *Assume that $a\rho$ satisfies Lemma 26 and the $a_2, \dots, a_{n-3}$ are chosen generically, so that the group generated by $a\rho(\mu_1)$ and $a\rho(\mu_{n-2})$ is irreducible. Let r, r' and r'' be hyperbolic rotations of order two, so that the axis of r is perpendicular to the axes of $a\rho(\mu_1)$ and $a\rho(\mu_{n-2})$, the axis of r' to the axes of $a\rho(\mu_{n-2})$ and $a\rho(\mu_{n-1})$, and similarly for r'' and $a\rho(\mu_{n-1})$ and $a\rho(\mu_n) = a\rho(\mu_1)$. Once the $a_2, \dots, a_{n-3}$ are fixed, the only other solution for the parameters a_{n-2} and a_{n-1} is $a'_{n-2} = r\, r' a_{n-2}$ and $a'_{n-1} = r' r'' a_{n-1}$.*

7 Bounding dimensions from above

We have seen that it is relatively easy to establish lower bounds on the dimension of the non-standard components we constructed in the previous sections. Determining their exact dimension, which turns out to coincide with the lower bound, requires a finer analysis which will be carried out in this section. In Subsection 7.1 we give a sufficient condition to guarantee convergence of characters in $X(\Gamma)$, once we know that the restrictions to the 2-bridge factors converge. Subsection 7.2 deals with the non-parabolic case, and Subsection 7.3 with the parabolic one.

7.1 Convergence of characters and displacement function

The goal of this subsection is to prove Proposition 33 about convergence of characters. Proposition 33 admits an elementary proof when the limiting characters χ_i^∞ are non parabolic. This follows from Lemma 24 and the continuity of the only two solutions of Equation 1. Since Equation 1 does not apply to the parabolic case, we need a different argument when the limit is parabolic; in fact we are going to give an argument that holds in general. For this purpose, first we need to recall the definition of the displacement function of an isometry in hyperbolic space and its main properties.

Definition 30. Let $h \in \mathrm{Isom}(\mathbf{H}^3)$ be an isometry. Its *displacement function* is

$$\begin{array}{rcl} d_h : \mathbf{H}^3 & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d_h(x) = d(x, h(x)) \end{array} .$$

Lemma 31. *(i) For every isometry $h \in \mathrm{Isom}(\mathbf{H}^3)$, d_h is convex.*

(ii) For every $h \in \mathrm{Isom}(\mathbf{H}^3)$ and $x, y \in \mathbf{H}^3$,

$$|d_h(x) - d_h(y)| \leq 2d(x, y).$$

(iii) Let $(h_k)_{k\in\mathbb{N}} \subset \mathrm{Isom}(\mathbf{H}^3)$ be a sequence of isometries. If h_k converges then $(d_{h_k}(x))_{k\in\mathbb{N}}$ is bounded for every $x \in \mathbf{H}^3$.

(iv) Let $(h_k)_{k\in\mathbb{N}} \subset \mathrm{Isom}(\mathbf{H}^3)$ be a sequence of isometries. If there is $x \in \mathbf{H}^3$ so that $(d_{h_k}(x))_{k\in\mathbb{N}}$ is bounded, then $(h_k)_{k\in\mathbb{N}}$ has a convergent subsequence.

Proof. Assertion (i) is consequence of convexity of the distance function in hyperbolic space. Assertion (ii) is a straightforward application of the triangle inequality, and (iii) follows from continuity. Finally, (iv) follows from the fact that

$$\begin{array}{ccccc} O(3) & \to & \mathrm{Isom}(\mathbf{H}^3) & \to & \mathbf{H}^3 \\ & & h & \mapsto & h(x) \end{array}$$

is a fibre bundle with compact fibre. □

The following corollary is based on Assertion (iv):

Corollary 32. *Let $G = \langle g_1, \ldots, g_s \mid (r_j)_{j\in J}\rangle$ be a finitely generated group and $(\rho^k)_{k\in\mathbb{N}} \in R(G)$ a sequence of representations. If there exists $x \in \mathbf{H}^3$ such that $(\sum_{i=1}^{s} d_{\rho^k(g_i)}(x))_{k\in\mathbb{N}}$ is uniformly bounded, then $(\rho^k)_{k\in\mathbb{N}}$ has a convergent subsequence.*

Proposition 33. *For $i = 1, \ldots, n-1$, let $(\chi_i^k)_{k\in\mathbb{N}}$ be a sequence in $X(B(\beta_i/\alpha_i))$ converging to an irreducible character χ_i^∞. Assume that $\chi_1^k(\mu) = \cdots = \chi_{n-1}^k(\mu) \neq \pm 2$. Then there exist $\chi^k \in X(\Gamma)$ such that χ^k restricted to $\pi_1(B(\beta_i/\alpha_i))$ equals χ_i^k and $(\chi^k)_{k\in\mathbb{N}}$ converges up to a subsequence.*

Notice that even if the χ_i^k are characters of representations ρ_i^k such that the sequences $(\rho_i^k)_{k\in\mathbb{N}}$ converge, the conjugating matrices in the amalgam between ρ_i^k and ρ_{i+1}^k could go to infinity.

Proof. Assume first that $n = 4$. Let $\rho_i^k \in R(B(\beta_i/\alpha_i))$ be a representation with character χ_i^k, for $i = 1, 2, 3$ and $k \in \mathbb{N}$. Since χ_i^∞ is irreducible, there exists $\rho_i^\infty \in R(B(\beta_i/\alpha_i))$, unique up to conjugacy, with character χ_i^∞. In particular, after conjugacy, we can assume that the sequence $(\rho_i^k)_{k\in\mathbb{N}}$ converges to ρ_i^∞, for $i = 1, 2, 3$.

Reasoning as in Lemma 26, we see that we can find isometries $h_k, g_k \in \mathrm{Isom}^+(\mathbf{H}^3)$ for each $k \in \mathbb{N}$ such that the following three conditions are fulfilled:

$$\begin{array}{rcl} h_k^{-1}\rho_1^k(\mu_2)h_k & = & \rho_2^k(\mu_2) \\ g_k^{-1}\rho_3^k(\mu_3)g_k & = & \rho_2^k(\mu_3) \\ h_k^{-1}\rho_1^k(\mu_1)h_k & = & g_k^{-1}\rho_3^k(\mu_4)g_k. \end{array}$$

As a consequence, for each $k \in \mathbb{N}$ we are able to construct a representation ρ^k, with character χ^k. We want to exploit Lemma 31 to prove that there is a $(\rho'^k)_{k\in\mathbb{N}}$ which converges up to a subsequence, where for each k ρ'^k is conjugate to ρ^k. In particular, $(\chi^k)_{k\in\mathbb{N}}$ converges up to a subsequence.

We fix a point $x \in \mathbf{H}^3$ and we look at displacement functions at $h_k^{-1}(x)$, x and $g_k^{-1}(x)$. According to Lemma 31(iii) we have that the sequences

$$\begin{aligned}
&\Big(d_{\rho^k(\mu_1)}\big(h_k^{-1}(x)\big) = d_{\rho_1^k(\mu_1)}(x)\Big)_{k\in\mathbb{N}}\\
&\Big(d_{\rho^k(\mu_2)}\big(h_k^{-1}(x)\big) = d_{\rho_1^k(\mu_2)}(x)\Big)_{k\in\mathbb{N}}\\
&\Big(d_{\rho^k(\mu_2)}(x) = d_{\rho_2^k(\mu_2)}(x)\Big)_{k\in\mathbb{N}}\\
&\Big(d_{\rho^k(\mu_3)}(x) = d_{\rho_2^k(\mu_3)}(x)\Big)_{k\in\mathbb{N}}\\
&\Big(d_{\rho^k(\mu_3)}\big(g_k^{-1}(x)\big) = d_{\rho_3^k(\mu_3)}(x)\Big)_{k\in\mathbb{N}}\\
&\Big(d_{\rho^k(\mu_4)}\big(g_k^{-1}(x)\big) = d_{\rho_3^k(\mu_4)}(x)\Big)_{k\in\mathbb{N}}
\end{aligned}$$

are bounded by some constant $C > 0$.

We are looking for a sequence $(y_k)_{k\in\mathbb{N}} \subset \mathbf{H}^3$ such that $d_{\rho^k(\mu_i)}(y_k)$ is bounded above independently of k, for $i = 1, 2, 3$. For each $k \in \mathbb{N}$, we consider the hyperbolic triangle with vertices x, $h_k^{-1}(x)$ and $g_k^{-1}(x)$. Thinness of hyperbolic triangles says that there is a point y_k whose distance to each edge of this triangle is less than $\log(2+\sqrt{3})$. Let y_k be such point. To prove the upper bound for $d_{\rho^k(\mu_1)} = d_{\rho^k(\mu_4)}$ on y_k, we notice that $d_{\rho^k(\mu_1)}$ is $\leq C$ on the segment between $h_k^{-1}(x)$ and $g_k^{-1}(x)$, by convexity. Thus, using Lemma 31(ii),

$$d_{\rho^k(\mu_1)}(y_k) \leq C + 2\log(2+\sqrt{3}) = C'.$$

By a similar argument we bound $d_{\rho^k(\mu_2)}$ (using the segment between $h_k^{-1}(x)$ and x) and $d_{\rho^k(\mu_3)}$ (using the segment between x and $g_k^{-1}(x)$). Once we have that $d_{\rho^k(\mu_i)}(y_k) \leq C'$, for each $k \in \mathbb{N}$ let ρ'^k be the conjugate of ρ^k by an isometry that maps y_k to a fixed point y_0, so that $d_{\rho'^k(\mu_i)}(y_0) = d_{\rho^k(\mu_i)}(y_k) \leq C'$. By Corollary 32, $(\rho'^k)_{k\in\mathbb{N}}$ has a convergent subsequence.

For $n > 4$ we proceed by induction on n. By the induction hypothesis, there is a convergent sequence of representations $(\phi^k)_{k\in\mathbb{N}}$ of the subgroup generated by the meridians $\mu_1, \ldots, \mu_{n-1}$ such that the restriction to $\langle \mu_i, \mu_{i+1}\rangle$ is conjugate to ρ_i^k and $\phi^k(\mu_1) = \phi^k(\mu_{n-1})$. We only need its restriction φ^k to $\langle \mu_2, \ldots, \mu_{n-1}\rangle$. Notice that the irreducibility of ρ_1^k implies that $\varphi^k(\mu_2)$ and $\varphi^k(\mu_{n-1}) = \phi^k(\mu_1)$ generate an irreducible representation. Now we can repeat the argument for $n = 4$ applied to ρ_1^k, φ^k and ρ_{n-1}^k. □

7.2 The non-parabolic case

To bound dimensions of the components of $X(K)$ constructed in the previous section, we will use their intersection with the Teichmüller part (the set of the characters of the basis of the Seifert fibred orbifold). Namely, in Lemma 34 we will prove that the components intersect the hyperplane H_μ defined by the condition that the trace of the meridian is 0, and in Lemma 35 we will bound the dimension of the intersection with the Teichmüller part.

Lemma 34. *Let $Z \subset X(\Gamma)$ be an irreducible component as in Proposition 28. If μ denotes a meridian, then $Z \cap H_\mu \neq \emptyset$.*

Proof. By construction, the restriction $Z \to X(B(\frac{\beta_i}{\alpha_i}))$ is non-constant, for each $i = 1, \dots, n-1$. In particular the Zariski closure of the image of Z in $X(B(\frac{\beta_i}{\alpha_i}))$ is a curve $Z_i \subset X(B(\frac{\beta_i}{\alpha_i}))$. Consider the algebraic set

$$\{(\chi_1, \dots, \chi_{n-1}) \in Z_1 \times \cdots \times Z_{n-1} \mid \chi_1(\mu_1) = \cdots = \chi_{n-1}(\mu_{n-1})\}$$

and the projection induced by taking restrictions:

$$\pi : Z \to \{(\chi_1, \dots, \chi_{n-1}) \in Z_1 \times \cdots \times Z_{n-1} \mid \chi_1(\mu_1) = \cdots = \chi_{n-1}(\mu_{n-1})\}.$$

By Proposition 23, the closure of the image $\overline{\pi(Z)}$ is a curve and contains a point $(\chi_1, \dots, \chi_{n-1}) \in \overline{\pi(Z)}$ with $\chi_i(\mu) = 0$ for every meridian μ. Since -1 is not a root of the Alexander polynomial, χ_i is irreducible (a matrix with determinant 1 and trace 0 has eigenvalues $\pm i$, hence irreducibility comes from Lemma 16).

We take $n-1$ sequences of characters $(\chi_i^k)_{k\in\mathbb{N}} \subset X(B(\frac{\beta_i}{\alpha_i}))$ so that $\chi_i^k \to \chi_i$ as $k \to \infty$ and $(\chi_{\rho_1}^k, \dots, \chi_{\rho_{n-1}}^k) \in \pi(Z)$. Since χ_i is irreducible, we may apply Proposition 33 to conclude that there is a convergent sequence of characters $(\chi^k)_{k\in\mathbb{N}} \subset X(\Gamma)$, $\chi^k \to \chi$, so that $\pi(\chi^k) = (\chi_1^k, \dots, \chi_{n-1}^k)$. In particular $\pi(\chi) = (\chi_1, \dots, \chi_{n-1})$. We want to show that χ^k may be chosen to belong to the irreducible component Z. Notice that the χ^k are defined using the construction of Lemma 26, $\chi^k = \chi_{\rho^k}$ where ρ^k is the amalgam of ${\rho_1}^k, \dots, {\rho_{n-1}}^k$ with conjugating matrices $a_2, \dots, a_{n-1}$. The $a_2, \dots, a_{n-3}$ can be perturbed in an open (hence Zariski dense) subset of $A_2 \times \ldots \times A_{n-3}$, and once those are chosen then a_{n-2} and a_{n-1} are subject to a compatibility condition. We have seen in Lemma 29 that there are two solutions for (a_{n-2}, a_{n-1}) related by rotations around axes perpendicular to the axes of the meridians. Notice that this construction gives at most two irreducible components for the fibres of $\pi^{-1}(\chi_1^k, \dots, \chi_{n-1}^k)$. We can assume that the sequence (χ^k) is contained in one of these two components. If this component is Z we are done, else using the construction we just recalled we can find a new sequence contained in Z and which converges by continuity (Lemma 29). Therefore $\chi \in Z \cap H_\mu$. □

In the previous proof we used that the Zariski closure of the image of $Z \subset X(\Gamma)$ in $X(B(\frac{\beta_i}{\alpha_i}))$ is a curve $Z_i \subset X(B(\frac{\beta_i}{\alpha_i}))$. Let ℓ be the number of Z_is that consist of abelian representations. Reasoning as in Remark 22, we can prove that $\ell \neq n-2$. In addition it seems unlikely that $\ell = n-3$ occurs for any choice of 2-bridge knots. This case will indeed occur sometimes, for example for the sum of two copies of the same knot, or combining this with surjections between 2-bridge knot groups (see Section 8). When $\ell = n-1$ then all representations of Z are abelian.

Lemma 35. *Let Z be a component of $X(\Gamma)$ contained in a component V of $X(K)$ as in Proposition 28. Let ℓ be the number of Z_i that consist of abelian characters, as above. If $\ell \leq n-4$, then $\dim V \leq n-3-\ell$. If $\ell = n-1$ or $\ell = n-3$, then $\dim V \leq 1$. Moreover $V = Z$.*

Proof. Embed $X(K)$ in $\mathbb{C}^N$ with coordinates some trace functions, according to Proposition 3. One of these coordinates is chosen to be the trace τ_μ of the meridian μ. By Lemma 34, Z and V intersect the hyperplane H_μ defined by trace of the meridian equal to zero; therefore, since H_μ has codimension 1 in the ambient space,

$$\dim(V \cap H_\mu) \geq \dim V - 1.$$

Any matrix in $SL_2(\mathbb{C})$ with zero trace has order two in $PSL_2(\mathbb{C})$, i.e. it is a rotation of angle π in hyperbolic space, hence every representation contained in $V \cap H_\mu$ factors through a representation of $\pi_1(\mathcal{O}_2)$ into $PSL_2(\mathbb{C})$, where $\mathcal{O}_2$ is the three-dimensional orbifold with underlying space S^3, with branching locus K and ramification index 2. Since K is a Montesinos knot, $\mathcal{O}_2$ is Seifert fibred, with basis a Coxeter 2-orbifold P^2 on a polygon with n vertices (one for each rational tangle). The representations of $Z \cap H_\mu$ are irreducible by Lemma 16. Hence the representations corresponding to points of a Zariski open nonempty subset of $V \cap H_\mu$ (containing $Z \cap H_\mu$) are also irreducible and thus they map the fibre to $\pm$ the identity. It follows that each component of $V \cap H_\mu$ that meets Z admits a finite-to-one map onto a subvariety W of $X(P^2, PSL_2(\mathbb{C}))$, and

$$\dim V - 1 \leq \dim(V \cap H_\mu) = \dim W.$$

Assume first that $\ell = 0$. We claim that $\dim W \leq n-4$ for the components of $X(P^2, PSL_2(\mathbb{C}))$ that contain characters induced by characters of $Z \cap H_\mu$. The corners of P^2 correspond to the tangles of K, and the stabiliser of each of these corners is a dihedral group. In particular, the nth corner is a dihedral group of order $2\alpha_n$. The stabilisers of the adjacent edges are order two groups, generated by reflections of the plane, that in $PSL_2(\mathbb{C})$ are mapped to rotations. Thus, the meridians of the arcs adjacent to the nth tangle are mapped to rotations whose axes form an angle which is an integer multiple of π/α_n. In particular this angle is constant on the irreducible component W of $X(P^2, PSL_2(\mathbb{C}))$.

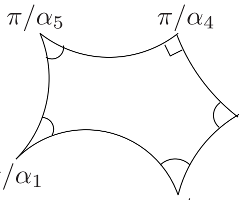

Figure 5: A Coxeter orbifold P^2 for $n = 5$

Now chose W a component of $X(P^2, PSL_2(\mathbb{C}))$ that contains characters induced by characters in $Z \cap H_\mu$. For any representation $\rho \in R(P^2, PSL_2(\mathbb{C}))$ coming from $Z \subset X(\Gamma)$, since the axes in $\mathbf{H}^3$ of $\rho(\mu_1)$ and $\rho(\mu_n)$ are assumed to be the same for $\chi_\rho \in X(\Gamma)$, the axes of the rotations that stabilise the edges adjacent to the nth vertex of P^2 coincide and so the generators of the stabilisers are mapped to the same element. Therefore the dihedral stabiliser of the nth vertex is mapped to a group of order two. This holds true for the whole component W, because this dihedral group is finite. Thus the characters of W factor through characters of P', the Coxeter orbifold obtained by forgetting the last vertex of P^2, and $\dim W \leq \dim X(P', PSL_2(\mathbb{C}))$. Since P' has $n-1$ vertices, $\dim X(P', PSL_2(\mathbb{C})) = n - 4$, by Proposition 8. As $\dim W \leq n-4$, $\dim Z \leq \dim V \leq \dim W + 1 = n - 3$, and we are done when $\ell = 0$.

Assume next that $\ell \leq n-4$. Then one can apply the same argument to the ℓ vertices corresponding to the 2-bridge factors whose representations are abelian, and therefore we can still remove ℓ vertices to P', to get the desired estimate of the dimension. The critical case occurs when $\ell = n-3$, the resulting P' would just be a segment, and in this case the dimension is still zero. Finally, in the abelian case $\ell = n-1$, the component is a curve, for the abelianisation of a knot is cyclic. □

Using Lemma 35 and Proposition 28, we can prove the following theorem. Notice that in Proposition 28 we found a lower bound for the dimension assuming that the restriction to every 2-bridge factor contained irreducible representations, but a similar argument applies to bound the dimension when there are some abelian ones (see Remark 11 and Corollary 18). In addition, we use [21] to find arbitrarily many components.

Theorem 36. *Let K be a Montesinos knot of Kinoshita-Terasaka type with n tangles, $n > 3$. Then $X(K)$ contains components of dimension d, for $d = 1, \dots, n-3$ on which the trace of the meridian is non-constant, and which are entirely contained in $X(\Gamma)$. Moreover there exist knots K for which the number of such components is arbitrarily large.*

7.3 More on intersections and the parabolic case

This proposition describes how the different components we have constructed meet each other.

Proposition 37. *Let $K = M(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_{n-1}}{\alpha_{n-1}}, \frac{\beta_n}{\alpha_n})$. For all $i = 1, \dots, n-1$, let ρ_i be a parabolic representation of the 2-bridge knot $B(\frac{\beta_i}{\alpha_i})$ with character χ_i. For each i, let Z_i be an irreducible component of $X(B(\frac{\beta_i}{\alpha_i}))$ containing χ_i. Denote by Z the irreducible component of $X(K)$ contained in $X(\Gamma)$ and constructed from the Z_is as in Proposition 28. Let Y the parabolic component constructed from the ρ_is as in Theorem 21. One has $Z \cap Y \neq \emptyset$.*

Proof. The proof of this lemma is similar to the proof of Lemma 34, because 1 is not a root of the Alexander polynomial of any knot. However here we have to use that Y contains characters of representations ρ satisfying $\rho(\mu_1) = \rho(\mu_n)$ and not only that $\rho(\mu_1)$ and $\rho(\mu_n)$ commute. We suppose first that $n = 4$. Let ρ be a representation with character $\chi_\rho \in Y$. Let ρ_1, ρ_2 and ρ_3 be the restrictions of ρ. We have that, up to conjugacy, the parabolic transformation $\rho_1(\mu_2) = \rho_2(\mu_2)$ fixes $\infty \in \hat{\mathbb{C}}$, $\rho_2(\mu_3) = \rho_3(\mu_3)$ fixes 0, and $\rho_1(\mu_1)$ and $\rho_3(\mu_4)$ fix 1. This can be achieved thanks to Lemma 20. We want to conjugate ρ_1 and ρ_3 by matrices $g, h \in PSL_2(\mathbb{C})$ so that $g\rho_1(\mu_1)g^{-1} = h\rho_3(\mu_4)h^{-1}$ (i.e. we want them to be equal, not only commuting), and so that $g\rho_1(\mu_2)g^{-1} = \rho_1(\mu_2)$ and $h\rho_3(\mu_3)h^{-1} = \rho_3(\mu_3)$. Thus we have to choose

$$g = \pm \begin{pmatrix} 1 & x-1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h = \pm \begin{pmatrix} 1 & 0 \\ y-1 & 1 \end{pmatrix},$$

for $x, y \in \mathbb{C}$. Since $\rho_1(\mu_1)$ and $\rho_3(\mu_4)$ are parabolic matrices that fix 1,

$$\rho_1(\mu_1) = \begin{pmatrix} 1+a & -a \\ a & 1-a \end{pmatrix} \quad \text{and} \quad \rho_3(\mu_4) = \begin{pmatrix} 1+b & -b \\ b & 1-b \end{pmatrix}$$

where $a, b \in \mathbb{C} \setminus \{0\}$. A straightforward computation shows that the equation

$$g\rho_1(\mu_1)g^{-1} = h\rho_3(\mu_4)h^{-1}$$

has solutions:

$$x = \pm\sqrt{b/a} \text{ and } y = 1/x = \pm\sqrt{a/b}.$$

The resulting matrices are:

$$g\rho_1(\mu_1)g^{-1} = h\rho_3(\mu_4)h^{-1} = \begin{pmatrix} 1 \pm \sqrt{ab} & -b \\ a & 1 \mp \sqrt{ba} \end{pmatrix}.$$

Notice that changing the sign of the square root corresponds to conjugating:

$$\begin{pmatrix} 1 - \sqrt{ab} & -b \\ a & 1 + \sqrt{ba} \end{pmatrix} = R \begin{pmatrix} 1 + \sqrt{ab} & -b \\ a & 1 - \sqrt{ba} \end{pmatrix}^{-1} R^{-1}$$

where

$$R = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

is the matrix of a rotation around the axis with end-points 0 and ∞, which are precisely the points in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ fixed by $\rho_1(\mu_2)$ and $\rho_3(\mu_3)$. This relation by a rotation can be seen as the limit of the rotations that appear for the non-parabolic representations in Remark 25 and Lemma 29. This guarantees that the parabolic representations are the limit of non-parabolic representations in Z provided by Proposition 33.

Similarly, for $n > 4$, we apply the openness argument to the $n-4$ conjugating matrices $a_2, \ldots, a_{n-2}$ in $A_2, \ldots, A_{n-2}$: a_i can be chosen in an open (Zariski dense) subset of A_i, for $i = 1, \cdots, n-2$. Then we apply the previous argument to a_{n-2} and a_{n-1}. □

Lemma 38. *Let $Y \subset X_{\text{par}}(K)$ be an irreducible component such that $Y \cap X(\Gamma) \neq \emptyset$. Assume there is a character $\chi_{\rho_0} \in Y \cap X(\Gamma)$ such that for three meridians μ_i, μ_j and μ_k the points of $\hat{\mathbb{C}}$ fixed by $\rho_0(\mu_i)$, $\rho_0(\mu_j)$ and $\rho_0(\mu_k)$ are all different. Then $Y \subset X(\Gamma)$.*

Proof. Let $\rho_0 \in R_{\text{par}}(K)$ be a parabolic representation with character $\chi_0 \in Y \cap X(\Gamma)$ satisfying the hypothesis of the lemma. Seeking a contradiction, we assume that $Y \cap X(\Gamma)$ is not equal to Y. Then, by the curve selection lemma [20, Lemma 3.1], there exist an $\varepsilon > 0$ and a deformation $\rho_s \in R_{\text{par}}(K)$ of ρ_0, analytic in $s \in [0, \varepsilon)$, such that for all $s \in (0, \varepsilon)$ $\chi_{\rho_s} \in Y \setminus (Y \cap X(\Gamma))$. Using the notation of Figure 4, we have the following relations

$$\mu_1^{-1}\mu_1' = \mu_2^{-1}\mu_2' = \cdots = \mu_n^{-1}\mu_n'.$$

Let $l \in \{1, \ldots, n\}$ and $f = \mu_l^{-1}\mu_l'$: We have that $\rho_0(f)$ is the identity. We claim that $\rho_s(f)$ is also trivial for every $s \in [0, \varepsilon)$. Otherwise, by analyticity there is $0 < \varepsilon' < \varepsilon$ such that $\rho_s(f)$ is nontrivial for every $s \in (0, \varepsilon')$. Then, by Claim 39 below and by chosing $s > 0$ small enough, one of the fixed points of $\rho_s(f)$ in $\hat{\mathbb{C}}$ would be arbitrarily close to the points fixed by $\rho_0(\mu_l) = \rho_0(\mu_l')$, for each $l = 1, \ldots, n$. But since we assume that the points fixed by $\rho_0(\mu_i)$, $\rho_0(\mu_j)$ and $\rho_0(\mu_k)$ are different, and since $\rho_s(f)$ has at most two fixed points in $\hat{\mathbb{C}}$, $\rho_s(f)$

must be trivial. We deduce that $\rho_s(\mu_{n-1}) = \rho_s(\mu'_{n-1})$ and $\rho_s(\mu_n) = \rho_s(\mu'_n)$. In particular the restriction of ρ_s determines a representation

$$\varphi_s : \pi_1(B(\tfrac{\alpha_n}{\beta_n})) \to SL_2(\mathbb{C}).$$

Since, for each $s \in [0, \varepsilon)$, φ_s is parabolic and it is a deformation of a parabolic abelian representation φ_0 of $B(\frac{\alpha_n}{\beta_n})$, φ_s is still abelian for each $s \in [0, \varepsilon)$, by Lemma 16, and therefore $\rho_s \in R_{\text{par}}(\Gamma)$ and $\chi_{\rho_s} \in X_{\text{par}}(\Gamma)$. Hence we get a contradiction that proves the lemma, assuming Claim 39. $\square$

Claim 39. *Let $\rho_s \in R_{\text{par}}(K)$ be a deformation of ρ_0 as in the proof of Lemma 38, analytic in $s \in (-\varepsilon, \varepsilon)$. Suppose that $\rho_s(\mu_i^{-1}\mu'_i)$ is nontrivial for $s \neq 0$. Then at least one of the fixed points of $\rho_s(\mu_i^{-1}\mu'_i)$ in $\hat{\mathbb{C}}$ converges to the fixed point of $\rho_0(\mu_i) = \rho_0(\mu'_i)$ as $s \to 0$.*

Proof. We may assume that

$$\rho_0(\mu_i) = \rho_0(\mu'_i) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In addition, since $\rho_s(\mu_i)$ is parabolic, its fixed point in $\hat{\mathbb{C}}$ changes analytically and, after conjugating by matrices that map it to ∞, we may assume that this fixed point is constant. Furthermore, after conjugating by diagonal matrices that depend analytically on s, we may assume that $\rho_s(\mu_i) = \rho_0(\mu_i)$ remains constant for $s \in (-\varepsilon, \varepsilon)$. Since $\rho_s(\mu'_i)$ is parabolic, we may write

$$\rho_s(\mu'_i) = \begin{pmatrix} 1 + a(s) & 1 + b(s) \\ c(s) & 1 - a(s) \end{pmatrix},$$

where a, b and c are analytic functions in s satisfying $a(0) = b(0) = c(0) = 0$ and $a^2 + (1+b)c = 0$. Then, a straightforward computation gives that the points of $\hat{\mathbb{C}}$ fixed by

$$\rho_s(\mu_i^{-1}\mu'_i) = \begin{pmatrix} 1 + a(s) - c(s) & b(s) - a(s) \\ c(s) & 1 - a(s) \end{pmatrix}$$

are:

$$\{z \in \hat{\mathbb{C}} \mid cz^2 + (c - 2a)z + a - b = 0\}.$$

Using $c = -a^2/(1+b)$, the sum of the two solutions of this quadratic equation is

$$\frac{2a(s) - c(s)}{c(s)} = \frac{-2(1 + b(s))}{a(s)} - 1,$$

which converges to infinity as $s \to 0$. Thus, for s sufficiently small, at least one of the solutions is arbitrarily close to ∞, the point fixed by $\rho_0(\mu_i)$. $\square$

We can now prove:

Theorem 40. *Let Y be a component of $X_{\text{par}}(K)$ constructed in Theorem 21. Then* $\dim Y = n - 3$.

Proof. By Lemma 38, we may assume that Y is a component of $X_{par}(\Gamma)$. According to Proposition 37 there is a component Z of $X(K)$ which intersects Y and on which the trace of the meridian is non-constant. We have that $Z \cap Y$ is contained in the intersection of Z with the hyperplanes defined by the condition that the trace of the meridian is equal to ± 2. Using the fact that $\dim Z \leq n-3$ (see Theorem 36) and that the trace of the meridian is non-constant on Z, we deduce that $\dim(Z \cap Y) \leq n-4$. On the other hand, $Z \cap Y$ is obtained from Y by imposing just one condition: indeed, this is the condition required for two parabolic matrices which commute to be the same. It follows that $n-4 \geq \dim(Z \cap Y) \geq \dim Y - 1$, and $\dim Y \leq n-3$. The last statement follows from the fact that $\dim Y \geq n-3$.

The same dimensional bound can be obtained directly by observing that $\dim X_{par}(\Gamma) < \dim X_{par}(K')$, because for a generic character $\chi_\rho \in X_{par}(K')$, $\rho(\mu_1)$ and $\rho(\mu_n)$ do not commute (the fixed points in $\hat{\mathbb{C}}$ are different) and in addition, by Corollary 15, the dimension of the component of $X_{par}(K')$ containing Y is $\leq n-2$. □

Of course, a similar result holds for the parabolic components of smaller dimension described in Remark 22, using the same argument, Remark 11 and Corollary 15.

8 Other non-standard components

In the previous sections we relied on bending to be able to construct new non-standard components. The commuting trick, however, can be used to construct other non-standard components which are not obtained by bending. Consider for instance a Montesinos knot of Kinoshita-Terasaka type with 3 rational tangles $K = M(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3})$. In this case, the construction of Section 5 can be carried out, but only gives a finite number of parabolic representation up to conjugacy. On the other hand, the argument of Section 6 does not apply anymore.

It was however shown by Mattman that some of these knots admit non-standard components on which the trace of the meridian is not constant. We start with a simple observation.

Remark 41. Assume that for $i = 1, 2$, there is an epimorphism $\psi_i : \pi_1(B(\frac{\beta_i}{\alpha_i})) \longrightarrow G$ such that $\psi_1(\mu_2) = \psi_2(\mu_2)$, and such that $\psi_1(\mu_1)$, $\psi_2(\mu_3)$ commute. Then each representation ρ of G into $SL_2(\mathbb{C})$ induces a representation of K in which the images of μ_1 and μ_3 commute. The induced representation is obtained by "doubling" ρ.

Of course, one can adapt the reasoning in the above remark to the case of Montesinos knots of Kinoshita-Terasaka type with more than three rational tangles, or more generally to other knots obtained as *Kinoshita-Terasaka sums*, that is to knots obtained by stacking together three tangles in the same circular pattern seen for Montesinos knots but where the first two tangles are not necessarily rational and the third one is a rational tangle of invariant $\frac{\beta_3}{\alpha_3} = \frac{1}{2a}$ for some integer a (see [16]).

Note that the hypothesis of Remark 41 is trivially satisfied when $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = \frac{\beta}{\alpha}$ by taking $G = \pi_1(B(\frac{\beta}{\alpha}))$. Indeed, with the notation introduced in Section 3 for rational tangles, one can show that one can choose a continued fraction expansion for $\frac{\beta}{\alpha}$ in which $a_i' = a_i''$ for all i.

Mattman considered the case where $\frac{\beta_1}{\alpha_1} = \frac{1}{3}$, and $\frac{\beta_2}{\alpha_2} = \frac{1}{m}$, and found non-standard components in the case where m is a multiple of 3. This follows from the fact that there is a π_1-surjective map of degree $\frac{m}{3}$ from the $(2, m)$-torus knot onto the trefoil knot if 3 divides m (see Figure 6). As a consequence, the character variety of each of these pretzel knots contains the character variety of the trefoil knot as non-standard component.

It is worthwhile to point out that in general the non-standard components obtained in this way have small dimension with respect to the non-standard components obtained by bending.

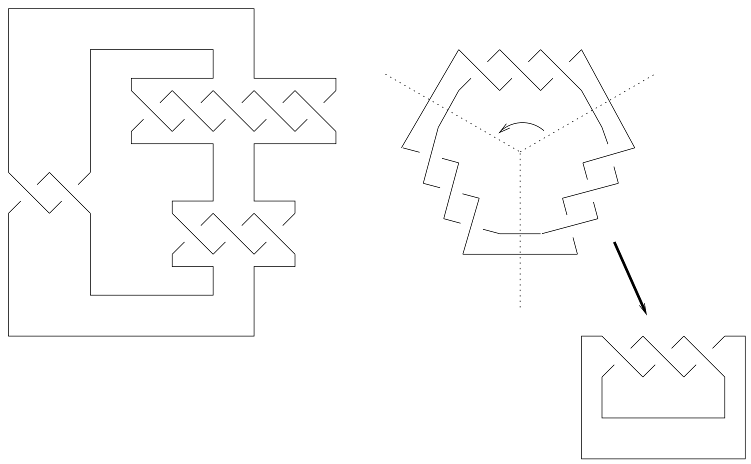

Figure 6: One of the $(2, 3, k)$ pretzel knots considered by Mattman, and a π_1-surjective branched covering from the $(2, 9)$-torus knot onto the trefoil knot.

9 Representations over fields of positive characteristic

For an odd prime p, define

$$\Gamma_p = \Gamma / \langle \mu^p \rangle$$

where μ denotes a meridian as usual.

Lemma 42. *For almost all primes p,* $\dim X(\Gamma_p) \leq n - 4$.

Proof. By construction,

$$X(\Gamma_p) = X(\Gamma) \cap \{\tau_\mu = 2\cos(\tfrac{k\pi}{p}) \mid k = 1, \ldots, \tfrac{p-i}{2}\}.$$

Hence, for almost all p, $X(\Gamma) \cap \{\tau_\mu = 2\cos(\frac{k\pi}{p}) \mid k = 1, \ldots, \frac{p-i}{2}\}$ is contained in the union of some irreducible components $Z_1, \ldots, Z_r$ of $X(\Gamma)$, for which τ_μ is non-constant. Lemma 34 and Theorem 36 apply to Z_i and $\dim Z_i \leq n - 3$. □

Given p as in the previous lemma, for almost every odd prime q, $\dim X(\Gamma_p)_{\overline{\mathbb{F}}_q} \leq n-4$. Thus the following results tells that $X(\Gamma_p)$ ramifies at p:

Proposition 43. *For almost all primes p, $\dim X(\Gamma_p)_{\overline{\mathbb{F}}_p} \geq n-3$. In particular $X(\Gamma_p)$ ramifies at p.*

Proof. Since $\overline{\mathbb{F}}_p$ has characteristic p, then a representation of Γ in $SL_2(\overline{\mathbb{F}}_p)$ factors through Γ_p iff μ is mapped to a parabolic element. Thus $X(\Gamma_p)_{\overline{\mathbb{F}}_p} = X_{par}(\Gamma)_{\overline{\mathbb{F}}_p}$. Moreover, for almost all p, $X_{par}(\Gamma)_{\overline{\mathbb{F}}_p}$ has the same dimension as $X_{par}(\Gamma)$, that is $\geq n-3$. □

Let $\mathcal{O}_p$ denote the orbifold with underlying space S^3, singular locus K and ramification of order p, an odd prime. Recall that the orbifold fundamental group of $\mathcal{O}_p$ is $\pi_1(S^3 \setminus \mathcal{N}(K))/\langle \mu^p \rangle$. The results above show that the subvariety $X(\Gamma_p)_{\mathbb{K}} \subseteq X(\mathcal{O}_p)_{\mathbb{K}}$ has a larger dimension for $\mathbb{K} = \overline{\mathbb{F}}_p$ than for $\mathbb{K} = \mathbb{C}$.

The extra ideal points of $X(\Gamma_p)_{\overline{\mathbb{F}}_p} \subseteq X(\mathcal{O}_p)_{\overline{\mathbb{F}}_p}$ give rise to essential 2-suborbifolds of $\mathcal{O}_p$ which meet K. They correspond to properly embedded essential surfaces in the exterior of K whose boundary components are meridians. Typically these surfaces are Conway spheres.

It would be interesting to understand whether these essential 2-suborbifolds of $\mathcal{O}_p$ can be associated to ideal points of curves in $X(\mathcal{O}_p)_{\mathbb{K}}$ for an arbitrary $\mathbb{K}$.

Acknowledgements The authors are indebted to Ariane Mézard for suggesting the book of Cox, Little and O'Shea [6] as a reference in arithmetic geometry and for valuable discussion. The second author is partially supported by the Spanish Micinn through grant MTM2009-0759 and by the Catalan AGAUR through grant SGR2009-1207. He received the prize "ICREA Acadèmia" for excellence in research, funded by the Generalitat de Catalunya. He thanks the Université de Provence for support and hospitality.

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