A note on a paper by R. C. Alperin

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Abstract

We point out a mistake in the statement of Corollary 2 in R. C. Alperin's paper on Selberg's lemma.

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1 Introduction

The aim of this short note is to point out an incorrect statement in Alperin's paper "An elementary account of Selberg's lemma" which appeared in L'Enseignement Mathématique [Alp]. The main focus of the paper is to provide an accessible proof of Selberg's lemma, which affirms that any linear group defined over a field of characteristic zero contains a torsion free subgroup of finite index.

Selberg's lemma has several applications, both in algebra and geometry. Some straightforward consequences can also be found in Alperin's paper. Unfortunately, among these, part of the statement of Corollary 2 on page 272 is wrong. The corollary reads:

The torsion subgroups of a finitely generated linear group G are finite; moreover, in characteristic zero, these finite groups have bounded order.

Although the corollary is correct in characteristic zero, the assertion is false in positive characteristic. We believe that, in the proof, it is implicitely assumed that torsion subgroups of finitely generated linear groups are finitely generated. The rest of this note will be devoted to show that this is however not always the case.

2 Counterexamples to the statement

The examples in the following claim are not new and can be found in [Nic, Example 26.9, page 779].

Claim 1. *Let $n \geq 3$ and p a prime number. Let $\mathbb{F}_p[t]$ be the ring of polynomials in one variable over the field $\mathbb{F}_p$ with p elements. The special linear group $SL(n, \mathbb{F}_p[t])$ is finitely generated and contains the infinite torsion group $H = \{I_n + xE_{1\,n} \mid x \in \mathbb{F}_p[t]\}$, where $E_{1\,n}$ denotes the $n \times n$ matrix with the $(1, n)$ entry equal to 1 and all other entries equal to 0.*

Proof. The group $SL(n, \mathbb{F}_p[t])$ is clearly linear over the field $\mathbb{F}_p(t)$ of positive characteristic p. For $n \geq 3$, it was shown to be finitely generated by Behr in [Beh].

Since $(I_n + xE_{1\,n})(I_n + yE_{1\,n}) = I_n + (x + y)E_{1\,n}$, H is a subgroup of $SL(n, \mathbb{F}_p[t])$ isomorphic to the additive Abelian p-group $\mathbb{F}_p[t]$. This is an infinite dimensional vector space over the field $\mathbb{F}_p$, so it is a non finitely generated group of exponent p. This shows in particular that it is an infinite torsion group. $\square$

In positive characteristic there are thus torsion subgroups of finitely generated linear groups that are not finite, contradicting Corollary 2 of Alperin's paper. On the other hand, by Burnside's theorem (see [Alp, Corollary 3]) finitely generated torsion subgroups are indeed finite, regardless of the characteristic.

References

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