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On certain classes of hyperbolic 3-manifolds

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Abstract. We study the topological structure and the homeomorphism problem for closed 3-manifolds $M(n, k)$ obtained by pairwise identifications of faces in the boundary of certain polyhedral 3-balls. We prove that they are (n/d)-fold cyclic coverings of the 3-sphere branched over certain hyperbolic links of $d + 1$ components, where $d = (n, k)$. Then we study the closed 3-manifolds obtained by Dehn surgeries on the components of these links.

1. Introduction

A well known theorem of Lickorish and Wallace [19] [38] states that any closed orientable 3-manifold can be obtained by Dehn surgeries on the components of an oriented link in the 3-sphere. If the link is hyperbolic [1] (i.e. its complement is a hyperbolic cusped 3-manifold), the Thurston–Jorgensen theory [43] of hyperbolic surgery implies that the resulting manifolds are hyperbolic for almost all surgery coefficients. Another result, due to Alexander (and successively improved by Hilden and Montesinos – see for example [38]) says that any closed 3-manifold can be represented as a branched covering of some link in the 3-sphere. Again as above, if the link is hyperbolic, the construction yields hyperbolic manifolds for branching indices sufficiently large. So, if we consider a (hyperbolic) link in the 3-sphere, we can construct many classes of (hyperbolic) closed orientable 3-manifolds by considering its branched coverings or by performing Dehn surgery along it. Moreover, these are very nice methods for representing closed orientable 3-manifolds by combinatorial tools, which can be easily visualized and manipulated. In the present paper we treat these two constructions for a hyperbolic link L_{d+1} of $d + 1$ components, which extends the Whitehead link ($d = 1$). In fact, L_{d+1} is formed by a chain of d unknotted circles (each of which linked with exactly two adjacent components with alternating crossings) plus an extra unknotted component which is the axis of d-symmetry of the chain. The orbifold, which is topologically the 3-sphere $\mathbb{S}^3$ and has the link L_{d+1} as singular set, is hyperbolic for almost all branching

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indices (see for example [1] and [8]). This means that the cyclic branched coverings of that orbifold are hyperbolic. Furthermore, the closed manifolds obtained by topological Dehn surgeries on the components of L_{d+1} are also hyperbolic for surgery coefficients sufficiently large. The strongly cyclic branched coverings of the Whitehead link were considered, and completely classified, in [13] and [44]. Here we construct the family of n-fold cyclic branched coverings $M(n, k)$ of the Whitehead link (where the branching indices of its components are n and n/d, $d = (n, k)$), and classify them, up to homeomorphism (or equivalently, up to isometry). The homeomorphism problem is solved in Section 2 by using a result which extends the main theorem of [44], and applying some techniques discussed in [34] and [37]. Moreover, we show that these coverings $M(n, k)$ can be topologically obtained as identification spaces of a family of polyhedra (the referee suggested to complete the study of the whole family of cyclic branched coverings of the Whitehead link; in fact, this can be done by using the results of Section 2 again, but for technical reasons we will treat the topic in the appendix). The description of closed 3-manifolds as polyhedral 3-balls, whose finitely many boundary faces are glued together in pairs, is another standard way to construct 3-manifolds (examples can be found in [13], [15], [34], [40], and [43]). If the polyhedra admit a geometric structure and the face identification is performed by means of geometric isometries, then the same geometric structure is inherited by the quotient manifold (compare again [13], [15], and [34]). Here we do not prove that the considered polyhedra are hyperbolic whenever the manifolds are, and deduce the geometric structure of our manifolds from that of their quotient orbifolds exploiting Dunbar's list [8]. In Section 3 we directly prove that the manifolds $M(n, k)$ are strongly cyclic coverings of the 3-sphere branched over L_{d+1}, where $d := (n, k)$. The result includes the main theorem of Helling, Kim, and Mennicke [13] when $d = 1$ (the Whitehead link). The method, we use in the proof, is a generalization to the orbifold case [32] [33] of a technique of cancelling handles on Heegaard diagrams, described in [12] for link complements. Section 4 deals with the combinatorial representation of closed 3-manifolds by a special class of edge-colored graphs, called *crystallizations* (see [3–6], [9–11], [20], [35], and [36]). This allows us to give an alternative proof of our result, obtaining also a geometric presentation of the fundamental group of $M(n, k)$ which arises from a (n/d)-symmetric Heegaard splitting of it (of genus $n-d$) in the sense of [2]. We remark that strongly cyclic branched coverings of links are widely studied in the literature; but however there are no simple geometric constructions of branched coverings which are not necessarily strongly cyclic. Thus we think that the class of closed 3-manifolds $M(n, k)$ is of a particular interest. In Section 5 we study the closed orientable 3-manifolds obtained by Dehn surgery along L_{d+1}. We obtain geometric presentations of the fundamental group of these manifolds, and discuss questions concerning their spines and certain RR-systems associated to them (see [7], [22], [29–31], and [42]). According to the Montesinos theorem [27], any manifold obtained by Dehn surgeries on a strongly invertible link can be presented as a 2-fold covering of the 3-sphere branched over some link. Moreover, there is an algorithm for constructing this branch set, as explained in [27]. We apply this algorithm to describe when our surgery manifolds are 2-fold branched coverings of $\mathbb{S}^3$. In particular, the result includes the topological classification of the closed 3-manifolds obtained by Dehn surgery on the Whitehead link, due to Mednykh and Vesnin [25]. Through the paper we work in the piecewise-linear (PL) category in the sense of [39] and [40]. The prefix PL will be omitted. For basic definitions and main results of the theory of (hyperbolic) 3-manifolds and 3-orbifolds we refer to [1], [8], [14], [16], [40], and [43]. All manifolds will be assumed to be connected and orientable. As references on knot theory and branched coverings see for example [2], [27], [28], [38], and [44].

2. Branched coverings

A very nice method of constructing closed orientable 3-manifolds is given by the identification of oppositely oriented boundary faces of a polyhedral 3-cell $\mathcal{P}$. The only troublesome points in the resulting quotient space (in fact, a closed *pseudo-manifold*) are the 0-cells that arise from the vertices in the boundary of $\mathcal{P}$. They have regular neighborhoods that are cones over closed surfaces. The following criterion is well known [40]: the quotient space is a closed 3-manifold if and only if its Euler characteristic vanishes. Many authors have studied the connections between the face identification procedure and the representation of closed 3-manifolds as branched coverings of the 3-sphere (see for example [13], [15], [26], [32], and [34]).

In [13] Helling, Kim, and Mennicke classified, up to isometry, the strongly cyclic n-fold coverings of the 3-sphere branched over the Whitehead link. It turns out that these coverings can be obtained as identification spaces of a family of polyhedra $\mathcal{P}_n$ depending on a positive integer n. For any coprime positive integers n and k, they defined a pairwise glueing of faces in the boundary of the polyhedron $\mathcal{P}_n$ yielding a closed orientable 3-manifold $M(n,k)$. Then they proved that $M(n,k)$ is a cyclic n-fold covering of the 3-sphere branched over the Whitehead link. The polyhedra $\mathcal{P}_n$ are shown schematically in Fig. 2.1. The glueing pattern of index k is determined by the following pairing of faces (where indices are taken mod n)

$$a_1 a_2 \dots a_n \mapsto f_n f_1 \dots f_{n-1}$$
$$a_i b_i c_i d_i b_{i-1} \mapsto e_{i-k-1} f_{i-k} e_{i-k} d_{i-k} c_{i-k-1}$$

for any $i = 1, \dots, n$, and the boundary edges are identified respecting the order. According to notation in Fig. 2.1, the face X_i^+ (resp. Y^+) is glued to the face X_i^- (resp. Y^-) for all i's.

It is easily seen that a glueing can be defined for every integer $1 \le k \le n-1$. The resulting identification space, again denoted by $M(n,k)$, is a closed orientable 3-manifold since its Euler characteristic vanishes. Notice that the manifold $M(n,k)$ admits a rotation of order n, induced by the cylindrical n-symmetry of the polyhedron $\mathcal{P}_n$ which is preserved by the glueing. In Section 3, we shall give a direct proof of the fact that the manifolds $M(n,k)$ are n-fold coverings of the 3-sphere branched along the Whitehead link with orders of its two components n and n/d, respectively, where $d := (n,k)$. We shall also prove that $M(n,k)$ are (n/d)-fold cyclic coverings of $\mathbb{S}^3$ branched along a specified link with $d+1$ components. Since the Whitehead link is hyperbolic [43], applying Thurston's orbifold geometrization

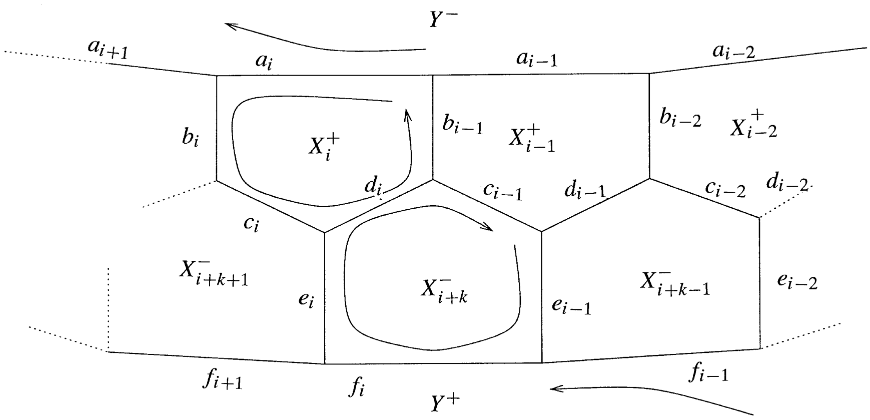


Fig. 2.1. Polyhedral schemata of $M(n, k)$

theorem we can conclude that our manifolds are hyperbolic for all $n \geq 3$ and $d < n/2$. If $d = n/2$, then the manifolds are Seifert fibered. This can be directly seen as follows. We shall prove that $M(n, n/2)$ is the 2-fold branched covering of a chain with $n/2$ components plus its symmetry axis. Let us consider the 2-fold covering of the 3-sphere branched along the symmetry axis of the chain. The preimage of the chain is again a chain with n rings. Such a chain is a Montesinos link, so its 2-fold cyclic branched covering is a Seifert fibered space (this is just a $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$-fold branched covering of the initial link). To conclude, we observe that this fibered space is a 2-fold non-branched covering of $M(n, n/2)$. By checking Dunbar's list of non-hyperbolic 3-orbifolds [8], we can see that $M(4, 2)$ is in fact a Nil-manifold (compare also with Theorem 4.5). Here we exploit these facts to classify the homeomorphism type of the manifolds $M(n, k)$, and generalize a result on cyclic branched coverings of hyperbolic links, due to Zimmermann [44]. First of all, we analyze the fundamental group of $M(n, k)$. Let us denote, for semplicity, $G(n, k) := \pi_1(M(n, k))$. We can obtain a finite presentation for $G(n, k)$ just looking at the Heegaard diagram of $M(n, k)$, as described in Section 3. Indeed, we have:

$$G(n,k) = \langle x_1, \ldots, x_n, y : \ \{x_i x_{i-1}^{-1} x_{i+k}^{-1} x_{i+k-1} y^{-1} = 1\}_{i=1,\ldots,n}, \ \{x_i x_{i-k} x_{i-2k} \ldots x_{i+k} = 1\}_{i=1,\ldots,d}\rangle,$$

where indices are taken mod n. Thanks to this presentation, we have patiently computed by standard methods the first integral homology group of our manifolds, i.e. $H(n, k) := H_1(M(n, k)) = G(n, k)/G(n, k)^{(1)}$.

We state the result.

Theorem 2.1. *Let $m := n/d$, where $d = (n,k)$. Then the group $H(n,k)$ is isomorphic to*

$$m \equiv 0 (\text{mod } 6) \begin{cases} \mathbb{Z}_{m/6} \oplus \mathbb{Z}_{m/2} \oplus \mathbb{Z}_m^{d-1} \oplus \mathbb{Z}_{12m} & \text{if } (d,6) = 1 \\ \mathbb{Z}_{m/6} \oplus \mathbb{Z}_{m/2} \oplus \mathbb{Z}_m^{d-2} \oplus \mathbb{Z}_{2m} \oplus \mathbb{Z}_{6m} & \text{if } d \equiv \pm 2 (\text{mod } 6) \\ \mathbb{Z}_{m/2} \oplus \mathbb{Z}_{m/2} \oplus \mathbb{Z}_m^{d-2} \oplus \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2m} & \text{if } d \equiv 0 (\text{mod } 3) \end{cases}$$

$$m \equiv \pm 2 (mod 6) \begin{cases} \mathbb{Z}_{m/2} \oplus \mathbb{Z}_{m/2} \oplus \mathbb{Z}_m^{d-1} \oplus \mathbb{Z}_{4m} & \text{if } d \equiv 1 (\text{mod } 2) \\ \mathbb{Z}_{m/2} \oplus \mathbb{Z}_{m/2} \oplus \mathbb{Z}_m^{d-2} \oplus \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2m} & \text{if } d \equiv 0 (\text{mod } 2) \end{cases}$$

$$m \equiv 3 (mod\ 6) \begin{cases} \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m^{d-1} \oplus \mathbb{Z}_{4m} & \text{if } d \equiv \pm 1 (\text{mod } 3) \\ \mathbb{Z}_m^{d+2} & \text{if } d \equiv 0 (\text{mod } 3) \end{cases}$$

$$(m,6) = 1 \quad \mathbb{Z}_m^{d+2}.$$

From the homology, it follows that $M(n,k)$ and $M(n',k')$ are homeomorphic only if $m = m'$ and $d = d'$ or, equivalently, only if $n = n'$ and $d = d'$. The case when $d = 1$ has already been discussed in [13] and [44], where these manifolds were completely classified, up to isometry. Now we want to deal with the remaining cases, by extending Theorem 1 of [44]. We start by noting that $M(n,k)$ and $M(n,-k)$ are homeomorphic as their fundamental groups are isomorphic. This can be easily seen as in [13] since the Whitehead link orbifold with two distinct branching indices still admits the symmetries preserving its singular components. So the claim follows by Mostow's rigidity theorem because of the hyperbolicity of our manifolds.

Let us introduce some notation. Consider an oriented link with $r+1$ components $L := L_0 \cup \cdots \cup L_r$ in the right-hand oriented 3-sphere $\mathbb{S}^3$. Let $\mathcal{O}_{n_0,n_1,\dots,n_r}(L)$ denote the 3-orbifold which is topologically the 3-sphere, and with singular set the link L, whose i-th component L_i has branching index $n_i > 1$, for any $i = 0, \dots, r$. Assume that – up to a reordering of indices– n_i divides $n := n_0$ for all i's, and that the inequalities $n_0 \geq n_1 \geq \cdots \geq n_r$ hold. If $\mathbf{m}_i$ denotes an oriented meridian of L_i, for any $i = 0, \dots, r$, then the orbifold fundamental group $\pi_1(\mathcal{O}_{n_0,\dots,n_r}(L))$ is isomorphic to the factor group $\pi_1(\mathbb{S}^3 \backslash L)/\langle \mathbf{m}_0^{n_0}, \dots, \mathbf{m}_r^{n_r} \rangle$. The abelianized group $\pi_1^{\text{ab}}(\mathcal{O}_{n_0,\dots,n_r}(L))$ is isomorphic to the direct product $\mathbb{Z}_{n_0} \times \cdots \times \mathbb{Z}_{n_r}$, and it is generated by the images of the $r+1$ meridians. If d_i is such that $n = n_i d_i$, then choose k_i satisfying $(n, k_i) = d_i$ for any $i \geq 1$, and set $k_0 := 1$. Let us define the homomorphism

$$\psi_{n,k_1,\dots,k_r} : \pi_1(\mathcal{O}_{n_0,\dots,n_r}(L)) \longrightarrow \pi_1^{\text{ab}}(\mathcal{O}_{n_0,\dots,n_r}(L)) \longrightarrow \mathbb{Z}_n$$

which sends $\mathbf{m}_i$ to k_i for any i. Let $M(n, k_1, \dots, k_r)$ be the closed orientable 3-manifold corresponding to the kernel of $\psi_{n,k_1,\dots,k_r}$: of course, it is an n-fold cyclic branched covering of the orbifold $\mathcal{O}_{n_0,\dots,n_r}(L)$. Assume that L is hyperbolic so that $\mathcal{O}_{n_0,\dots,n_r}(L)$ is a hyperbolic orbifold for sufficiently large n_i. This immediately implies that $M(n, k_1, \dots, k_r)$ is a hyperbolic manifold, and all covering transformations are isometries. Note that the isometry groups of $M(n, k_1, \dots, k_r)$ and $\mathcal{O}_{n_0,\dots,n_r}(L)$ are finite.

Theorem 2.2. *With the above notation, let $k_i, k_i' \neq 0$ satisfy $(n, k_i) = (n, k_i') = d_i$ for any $i \geq 1$. Suppose there exists a prime p such that, for some positive* *integer σ, p^σ does not divide the order of the symmetry group of the orbifold $\mathcal{O} := \mathcal{O}_{n_0,\ldots,n_r}(L)$, and moreover it does no divide n_i for all $n_i \neq n$. Then, if the hyperbolic 3-manifolds $M := M(n, k_1, \ldots, k_r)$ and $M' := M(n, k'_1, \ldots, k'_r)$ are isometric (or equivalently, homeomorphic) there exist isometries $\varphi : M \longrightarrow M'$ and $\phi : \mathcal{O} \longrightarrow \mathcal{O}$ such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M' \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow[\phi]{} & \mathcal{O}. \end{array}$$

This means that φ conjugates the two cyclic groups of covering transformations of the orbifold acting on the considered manifolds.

In particular, if $r = 1$, then the theorem becomes: in the above assumptions (with $k_1 =: k$ and $k'_1 =: k'$), we have that:

a) *If M and M' are isometric, then one of the following condition holds:*
   (i) $k \equiv k' \pmod{n}$*;*
   (ii) $k \equiv -k' \pmod{n}$ *and there exists an isometry of $\mathcal{O}$ fixing L_0 and L_1 and reversing the orientation of exactly one of L_0 and L_1;*
   (iii) $kk' \equiv 1 \pmod{n}$ *and there exists an isometry of $\mathcal{O}$ exchanging L_0 and L_1 whose square preserves orientations of both L_0 and L_1;*
   (iv) $kk' \equiv -1 \pmod{n}$ *and there exists an isometry of $\mathcal{O}$ exchanging L_0 and L_1 whose square reverses the orientation of both L_0 and L_1.*

b) *Conversely, if one of the conditions i), ..., iv) holds, then the hyperbolic closed 3-manifolds M and M' are isometric.*

Remark that for the simple case when $r = 1$ we can just check all possible actions on $\mathbb{Z}_n$ induced by isometries ϕ of $\mathcal{O}$. Obviously, the number of cases to consider increases with the number of components involved in the initial link.

We also observe that conditions iii) and iv) can not happen if $d := d_1 \neq 1$: there can not exist isometries exchanging the two components of L since they have different branching indices, and $kk' \not\equiv \pm 1$ in this case.

Proof. Note that the proof of the theorem goes through in the same way as in [44], once we rearrange Lemma 1 of [44] to our situation. First we need some notation. Let n, k_i and k'_i, $i = 0, \ldots, r$, be fixed. Denote by H and H' the two groups of covering transformations of $\mathcal{O}$ acting on M and M', respectively, both isomorphic to $\mathbb{Z}_n$. Denote by $\tilde{L}_i$, $i = 0, \ldots, r$, and by $\tilde{L}$ the preimages in M of L_i and L, respectively – the same can be done for M' –. It is easy to see that $\tilde{L}(n) := \tilde{L}_0 \cup \cdots \cup \tilde{L}_t$, where $t := \max\{i : n_i = n\}$, is the fixed point set for the generator h of H while $\tilde{L}$ is fixed by $h^{n/d}$, where $d = \text{g.c.d.}(n_1, \ldots, n_r)$. Notice that we can choose h to act locally as a rotation of $2\pi/n$ around $\tilde{L}(n)$. We prove that if there exists an isometry $\varphi : M \longrightarrow M'$, then it can be chosen so that $\varphi H \varphi^{-1} = H$. Suppose, on the contrary, that this is not true. Then the two groups H and $\bar{H} := \varphi H' \varphi^{-1}$ are not conjugated in the isometry group of M. The element h^{n/p^σ} can not be conjugated to the element $\bar{h}^{n/p^\sigma}$, where $\bar{h}$ is defined analogously to h. Otherwise, the element conjugating them would conjugate $\bar{H}$ to H. In fact, all elements of H not contained in $\cup_{i>t}\langle h^{d_i}\rangle$ have the same rotational axis as the generator h, and conjugation by an element λ maps rotations around $\tilde{L}(n)$ to rotations with the same angle around $\lambda^{-1}(\tilde{L}(n))$. In other terms, we see that any isometry, normalizing an element of H not contained in $\cup_{i>t}\langle h^{d_i}\rangle$, must normalize the whole cyclic group. Thus, one deduces that the p-Sylow subgroups of H are not p-Sylow subgroups of the group of isometries of M, and that the group $\langle h^{n/p^\sigma}\rangle$ is properly contained in its normalizer with respect to the p-Sylow subgroup of the group of isometries of M (compare also [43: 1.5, p. 88]). Then there would exist an isometry of M projecting to an isometry of the orbifold of order p, which is against our assumptions. □

We now apply this result to our class of hyperbolic manifolds. In particular, we are interested in the case when d is greater than 1. We also treat the particular situation in which $r = 1$ and L is the Whitehead link, as in [44].

The following classifies the homeomorphism type of $M(n,k)$.

Theorem 2.3. *Let* $(n,k) = (n,k') \neq 1$. *Then the manifolds* $M(n,k)$ *and* $M(n,k')$ *are homeomorphic if and only if* $k \equiv \pm k' \pmod n$. *The homeomorphism between* $M(n,k)$ *and* $M(n,-k)$ *is induced by the rotation of* $\mathcal{P}_n$ *exchanging* Y^+ *and* Y^-. *More precisely, its action on the edges is defined as follows:*

$$\begin{aligned} a_i &\mapsto f_{-i}, & d_i &\mapsto d_{-i},\\ b_i &\mapsto e_{-i-1}, & e_i &\mapsto b_{-i-1},\\ c_i &\mapsto c_{-i-1}, & f_i &\mapsto a_{-i}. \end{aligned}$$

3. Orbifolds and polyhedral schemata

In this section we study the quotient spaces of the manifolds $M(n,k)$ by the action of the cyclic group of rotations induced by the cylindrical symmetry of the polyhedra $\mathcal{P}_n$. We prove that $M(n,k)$ are obtained as strongly cyclic (n/d)-fold coverings of the 3-sphere, where $d = (n,k)$. The corresponding branch sets are completely described. Moreover, we obtain the following commutative diagram of branched coverings:

$$\begin{array}{ccc} M(n,k) & \xrightarrow{\;n/d\;} & \mathcal{O}_{n/d}(L_{d+1}) \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ \mathcal{O}(M(n/d, k \pmod{n/d})) & \xrightarrow[\;n/d\;]{} & \mathcal{O}_{n,n/d}(L_2) \end{array}$$

where the labels of the maps indicate the degree of the covering. Exploiting a method described in [33] and [32], which generalizes a result of Grunewald and Hirsch [12], we prove that $M(n,k)$ are cyclic branched (n/d)-coverings of a link L_{d+1} with $d+1$ components in the 3-sphere. Such a link consists of a chain with d rings with alternating crossings plus an extra unknotted component which is the axis of d-symmetry of the chain (depicted in Fig. 5.1). This last component is precisely the image of the axis of the rotation of order n of $M(n,k)$ induced by the cylindrical symmetry of $\mathcal{P}_n$. This immediately implies that the quotient orbifold $\mathcal{O}_{n,n/d}(L_2)$ is topologically the 3-sphere branched again over the Whitehead link whose components have orders n (the image of the symmetry axis) and n/d (the images of the rings of the chain which derive from certain edges of $\mathcal{P}_n$), respectively. Indeed, this is exactly what one expects by letting $d = 1$. Before proving this fact, we briefly describe what the orbifolds $\mathcal{O}(M(n/d, k \pmod{n/d}))$ are. First of all, notice that they are obtained by identifying a d-th "slice" of $\mathcal{P}_n$. Topologically this is again a polyhedron of our family once we glue the two sides identified by the d-rotational symmetry. More precisely, it is $\mathcal{P}_{n/d}$. In this case, however, the symmetry axis is singular. Moreover, note that the identification of $\mathcal{P}_n$, yielding $M(n,k)$, induces an identification of $\mathcal{P}_{n/d}$. Thus it is easy to see that, topologically, the quotient is $M(n/d, k \pmod{n/d})$. The singular set consists of the image of the symmetry axis, as we have already observed, with branching index d and, if $\text{g.c.d.}(n/d, d) = \text{g.c.d.}(n/d, k) = \text{g.c.d.}(n/d, k \pmod{n/d}) =: s \neq 1$, then also certain edges of the polyhedron have singular images: there are exactly s singular components of this type, all of them with branching index s.

Theorem 3.1. *The closed connected orientable* 3*-manifolds* $M(n,k)$ *are strongly cyclic* (n/d)*-fold coverings of the* 3*-sphere* $\mathbb{S}^3$ *branched over a link* L_{d+1} *with* $d+1$ *components, where* $d = (n,k)$*. This link is formed by a chain of* d *unknotted circles, each of which is linked with exactly two adjacent components of the chain with alternating crossings, plus an extra unknotted component which is the axis of* d*-symmetry of the chain. Furthermore,* $M(n,k)$ *are cyclic branched* n*-fold coverings of the Whitehead link in the* 3*-sphere, where the branching indices of its components are* n *and* n/d*, respectively.*

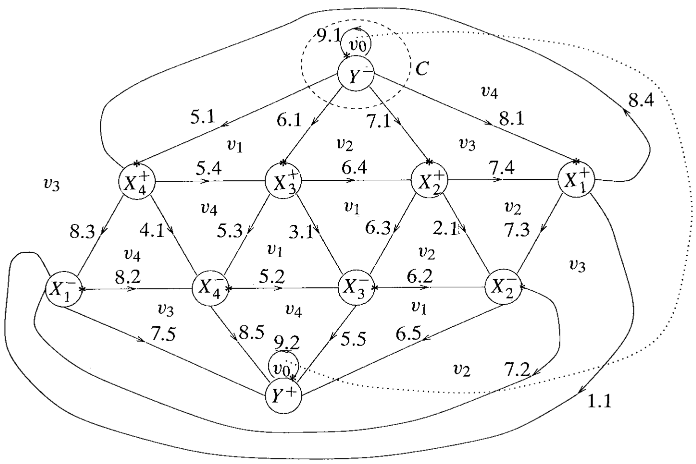
Proof. In order to prove the theorem, it suffices to illustrate a particular case. Indeed, the symmetry of $\mathcal{P}_n$ guarantees that the construction works in all cases. So we are going to determine the quotient $M(8,4)/\mathbb{Z}_2 = \mathcal{O}_2(L_4)$. As above, we remark that the orbifold $\mathcal{O}_{n/d}(L_{d+1}) = M(n,k)/\mathbb{Z}_{n/d}$ (resp. $\mathcal{O}_{n,n/d}(L_2)$) is simply the identification space of the polyhedron $\mathcal{P}_d$ (resp. $\mathcal{P}_1$) of identification index 0. The singular set is the image in the quotient of the axis of symmetry of the rotation and of the edges with label d_i. Notice that these edges are identified by the glueing in d groups of n/d edges and for a symmetry matter the dihedral angle along these edges must be $2d\pi/n$. The polyhedron $\mathcal{P}_d$ defines in a natural way a decomposition of $\mathcal{O}_{n/d}(L_{d+1})$ into handles. In fact, the 3-handles are the neighborhoods in $\mathcal{O}_{n/d}(L_{d+1})$ of the images of the vertices of $\mathcal{P}_d$, the 2-handles are the neighborhoods of the images of its edges, and the 1-handles are the neighborhoods of the images of its faces. There is a unique 0-handle obtained after cutting off all other handles: it is the image of a 3-ball inside the polyhedron. All the information concerning the glueing pattern can be stored in a planar graph consisting of 2-discs joined by certain nonintersecting arcs with suitable labels attached to them. Actually, this graph represents a Heegaard diagram for the identification space. If we delete one vertex of $\mathcal{P}_d$, then we can flatten out the boundary of the polyhedron onto a plane while the interior of the polyhedron itself lies in the semispace "behind" the plane. If we remove from the closed semispace the preimages of the 3- and 2-handles of our decomposition, then the boundary of $\mathcal{P}_d$, that remains, is a collection of 2-discs on the plane, one disc for each face of $\mathcal{P}_d$. Mark a point along the boundary of the disc for every edge surrounding that face, and label identified faces with the same letter. The glueing of the faces is completely defined once we choose for each pair of faces a base point (i.e. two edges that are glued together when the two faces are identified) and two opposite orientations: one positive (counterclockwise) and one negative (clockwise). Now we join with an arc two points if they represent the same edge in $\mathcal{P}_d$. In this way, any point is connected to exactly one other point. This tell us how to glue back the 2-handles we removed.

Choose now an arc and an orientation on it. Denote it by 1.1. The endpoint of this arc is glued onto another point by the identification of the faces. Denote by 1.2 the oriented arc starting at this latter point. Go on like that until you return to the arc 1.1. If there are arcs left unlabelled, pick up one of them and call it 2.1 and repeat the operation till all arcs are labelled. This is a way to recover the cycle relations along an edge (in the language of Poincaré's theorem), i.e. a way to detemine which edges are identified by the glueing. From the graph we can also recover a presentation for the fundamental group of the space (not the orbifold fundamental group, however). This presentation has a generator for each 1-handle. According to notation in Fig. 2.1, we have generators x_i, for any $i = 1, \dots, n$, and y associated to the 1-handles X_i and Y, respectively. To compute the relations, it is enough to see how the 2-handles are attached, i.e. to consider the cycle relations we described above. For instance, consider the sequence of arcs $1.1, \dots, 1.t$ and write the sequence of faces (generators with exponents ± 1) entered by the arcs of the sequence with the given order: this yields one relation. The same procedure for all sequences gives a complete set of relations.

We define now the *complexity* of the graph as a pair (g, m), where g is the number of 1-handles of the manifold (i.e. half of the number of the faces) and m is the number of arcs of the graph (i.e. the number of edges of $\mathcal{P}_d$); complexities are ordered lexicographically. We describe two ways to diminish the complexity (g, m) of the graph:

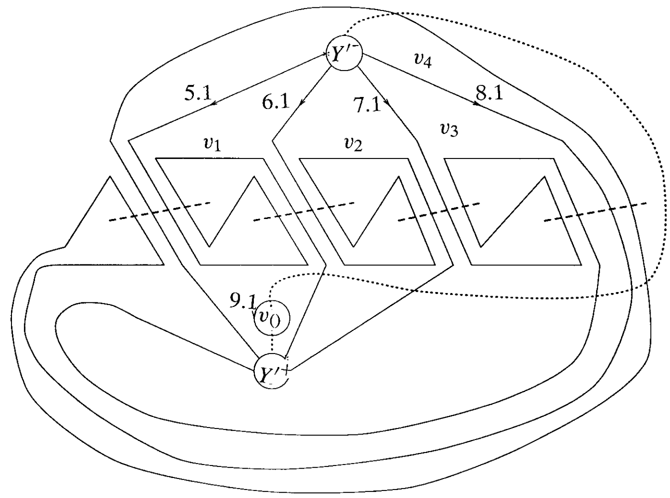
1) First we examine how to alter the graph (without changing the identification space) in such a way that the new graph has complexity (g, m'), where $m' < m$. Suppose that it is possible to draw in the plane a Jordan curve with the following properties:
   i) the curve intersects the graph only along its arcs and transversally;
   ii) the curve separates a pair of identified discs such that the number of points along their boundaries is greater than the number of intersections of the curve with the graph.

   All we need to do now is to dig a ball along the curve and glue it back along the pair of disks determined in ii).
2) If we want to diminish g instead, then we have to cancel a 1-handle with a 2-handle. To do this we need to check that the sequence of arcs defining the glueing diagram for the 2-handle intersects the two discs determing the 1-handle in exactly one point. In this case, we must see how the remaining arcs of the graph are deformed.



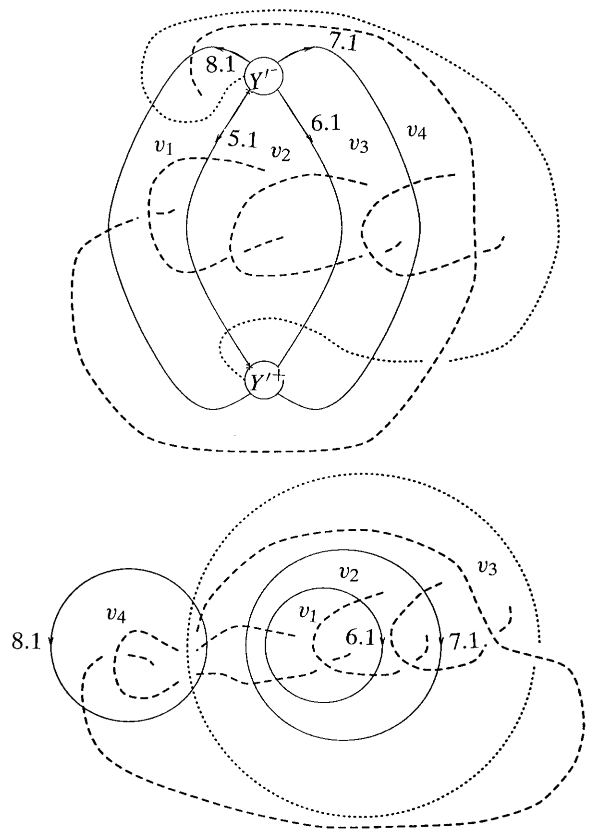
Base points:	$*$	1-handles:	X_1, X_2, X_3, X_4, Y
Singular axis:	$\cdots\cdots$	2-handles:	1 2 3 4 5 6 7 8 9
Jordan curve:	C	3-handles:	v_0 v_1 v_2 v_3 v_4

Fig. 3.1. The planar graph for $M(4,0)$. Eliminate 1-handles X_i with 2-handles i, $i = 1, \dots, 4$. Decrease complexity by digging a ball along C. Glue 1-handle Y and create 1-handle Y'



– – – – Singular cores of 2-handles
Cancel 2-handle 9 with 3 handle v_0.
Deform the graph

Fig. 3.2.



Top: Cancel 1-handle Y' with 2-handle 5
bottom: Cancel 2-handles 6 7 8 with 3-handles v_1 v_2 v_3 respectively
Glue back 3 handle v_4

Fig. 3.3.

In Figs. 3.1, 3.2, 3.3 and 3.4 we show how these simplifications apply to our case. Observe that the axis of symmetry is a line inside the semispace starting and ending in two points of the plane belonging to the boundary of the neighborhood of two vertices, while the edges are the cores of the 1-handles we attach at the end. The axis is represented by a dotted line in our figures. It determines the singular set together with the core of the 1-handles corresponding to the edges with label d_i-cycle relations number 1, 2, 3 and 4- again represented by (different) dotted lines. Remark that the axis must be deformed accordingly to the simplifications of the graph we have described. □

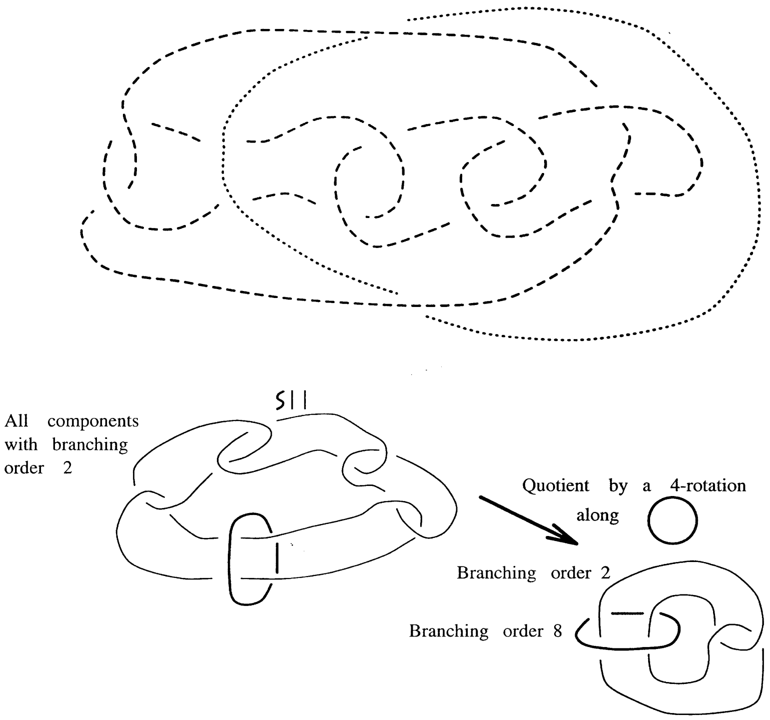


Fig. 3.4.

4. Colored graphs

In this section we give an alternative proof of Theorem 3.1 by using the representation theory of closed triangulated n-manifolds via colored graphs (as general references see [3], [4], [6], [11], and [20]). As a consequence, we obtain a geometric presentation of the fundamental group of $M(n, k)$ induced by a (n/d)-symmetric Heegaard splitting of it (of genus $n - d$, $d = (n, k)$). By [5] this presentation also corresponds to a spine of $M(n, k)$. Then we prove further results concerning the topological and geometric structures of our manifolds. We first recall some basic definitions and results on colored graphs and pseudocomplexes. We always work in the piecewise linear category [39] [40] without an explicit mention of it. An *n-pseudocomplex* is an n-dimensional ball complex K whose h-balls, considered with all their faces, are abstractly isomorphic to h-simplexes. By abuse of language we continue to call *vertices* (resp. *h-simplexes*) the 0-balls (resp. h-balls, $h > 1$) of K. We say that K is a *contracted complex* if K has exactly $n + 1$ vertices. A *pseudodissection* (resp. *contracted triangulation*) of a compact polyhedron $\mathbb{P}$ is a pseudocomplex (resp. contracted complex) K whose underlying space $|K|$ is homeomorphic to $\mathbb{P}$.

In the following the term *graph* will be used instead of multigraph, hence loops are forbidden but multiple edges are allowed. Given a graph Γ, $V(\Gamma)$ and $E(\Gamma)$

denote the sets of vertices and edges of Γ, respectively. An *edge-coloration* on Γ is a map $\gamma : E(\Gamma) \to \Delta_n = \{0, \dots, n\}$ (Δ_n is called the *color set*) such that $\gamma(e) \neq \gamma(f)$ for any two adjacent edges e and f of $E(\Gamma)$. An $(n+1)$-*colored graph* is a pair (Γ, γ), where Γ is a regular graph of degree $n+1$, and γ is an edge-coloration on Γ. For any subset $B \subset \Delta_n$, we set $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$. The graph (Γ, γ) is said to be *contracted* if the partial subgraph $\Gamma_{\Delta_n \setminus \{i\}}$ is connected for each color $i \in \Delta_n$.

Closely related to colored graphs is the notion of colored *n-complex*, that is a pair (K, ξ), where K is an n-pseudocomplex and $\xi : K^{(0)} \to \Delta_n$ is a map (called the *vertex-coloration* of K) which is injective on each n-simplex of K (here $K^{(0)}$ denotes the 0-skeleton of K). We associate to (K, ξ) an $(n+1)$-colored graph $(\Gamma(K), \gamma_K)$ as follows:

1) Take a vertex for each n-simplex of K, and join two such vertices by an edge for every common $(n-1)$-face of the corresponding n-simplexes. Define $V(\Gamma(K))$ and $E(\Gamma(K))$ as the sets of these vertices and edges, respectively;

2) For each edge e in $\Gamma(K)$, let $\sigma(e)$ be the corresponding $(n-1)$-simplex of K, and let v be a vertex such that the join $v * \sigma(e)$ is an n-simplex of K. Then label e with color $\xi(v)$, i.e. we set $\gamma_K(e) = \xi(v) \in \Delta_n$.

The above construction can be easily reversed so that $K(\Gamma(K))$ is abstractly isomorphic to K. Note that K is contracted if and only if $\Gamma(K)$ is. If $|K|$ is a closed connected n-manifold, then $(\Gamma(K), \gamma_K)$ is said to *represent* $|K|$ and every homeomorphic space. A *crystallization* of a closed connected n-manifold M is a contracted $(n+1)$-colored graph (Γ, γ) which represents M.

Any two colored graphs representing the same manifold are proved to be joined by a finite sequence of elementary moves which translates in dimension 3 the Singer moves [41] on Heegaard diagrams in terms of graph-theoretical tools. We briefly describe these moves since we use them in our alternative proof of Theorem 3.1.

Let (Γ, γ) be an $(n+1)$-colored graph which admits a partial subgraph Θ, formed by two vertices x and y, joined by h edges ($1 \leq h \leq n$) labelled by colors $c_0, \dots, c_{h-1}$. If B is the complement of $\{c_0, \dots, c_{h-1}\}$ in Δ_n, then $C_B(x)$ and $C_B(y)$ denote the connected components of Γ_B containing x and y, respectively. We say that Θ is a *dipole of type h* if $C_B(x)$ is different from $C_B(y)$. *Cancelling* Θ means: 1) replace in Γ_B the components $C_B(x)$ and $C_B(y)$ by their connected sum with respect to x and y; 2) leave unchanged the edges colored $c_0, \dots, c_{h-1}$ which are not incident to x and y. *Adding* Θ means the inverse process.

There is a further move on 4-colored graphs (Γ, γ) representing closed 3-manifolds which reduces the *combinatorial genus* of the graph, i.e. the smallest integer g such that $|\Gamma|$ regularly embeds into a closed connected surface of genus g (see for example [4]). Suppose that for two colors c_0 and c_1 in Δ_3, there exist connected components C of $\Gamma_{\{c_0,c_1\}}$ and C' of $\Gamma_{\{c_2,c_3\}} = \Gamma_{\Delta_3 \setminus \{c_0,c_1\}}$ having only one common vertex x_0. Let $\{x_0, x_1, \dots, x_m\}$ and $\{x_0, y_1, \dots, y_n\}$ be the sets of vertices of C and C', respectively. We say that the subgraph Ω of (Γ, γ) formed by C and C' is a *generalized dipole of type* (m, n). Let x_1, x_m, y_1, and y_n be the vertices joined with x_0 by edges of colors c_0, c_1, c_2, and c_3, respectively. *Cancelling* Ω means: 1) substitute Ω with the product Ξ of the subgraphs $C \setminus \{x_0\}$ and $C' \setminus \{x_0\}$ obtaining a new graph Γ'; 2) color the edge joining vertex (x_i, y_j) with vertex

$(x_i, y_{j'})$ (resp. $(x_{i'}, y_j)$) in Ξ with the same color of the edge joining y_j with $y_{j'}$ (resp. x_i with $x_{i'}$) in Γ, for any $i, i' \in \Delta_m\backslash\{0\}$, and $j, j' \in \Delta_n\backslash\{0\}$; 3) if a vertex $v \in \Gamma\backslash\Omega$ is joined with x_i (resp. y_j) by a c_2- or c_3-colored (resp. c_0- or c_1-colored) edge, then v is joined with (x_i, y_1) or (x_i, y_n) (resp. (x_1, y_j) or (x_m, y_j)) in Γ', for any $i \in \Delta_m\backslash\{0\}$ and $j \in \Delta_n\backslash\{0\}$. *Adding* Ω means the inverse process as usual. It is easily seen that the cancellation of a generalized dipole corresponds to a Singer move of type III' on a Heegaard diagram induced by the colored graph (see [41]).

The following is the basic result in the representation theory of closed triangulated manifolds via colored graphs [10] [35].

Theorem 4.1. *(Existence) Any closed connected piecewise-linear n-manifold can be represented by a crystallization.*

(Equivalence) Two colored graphs (or crystallizations) represent homeomorphic manifolds if and only if one can be transformed into each other by a finite sequence of cancelling and/or adding (generalized) dipoles.

We now describe the algorithm given in [9] for constructing a crystallization of the 2-fold cyclic covering of $\mathbb{S}^3$ branched over a bridge-presentation of a link L. Let P denote the projection of L on the plane $z = 0$. P can always be assumed to be connected. This is immediate if L is nonsplitting. If L splits, then we isotope arcs of L on the plane $z = 0$ to pass "in and out" under bridges of different components. Let $B_1, \ldots, B_m$ denote the projections of the bridges of L. We can assume that P intersects all B_i' s at right angles. Let E_i be an ellipse on the plane $z = 0$ whose principal axis is B_i. Let V be the set of points of intersection between the ellipses E_i and P. Then V separates the part of L lying on the plane $z = 0$ into edges. Let C (resp. D) be the set of these edges which are internal (resp. external) to the ellipses. Let $\tau : V \to V$ be the involution which interchanges the end-points of the edges of C, and fixes any point of $\bigcup_i (E_i \cap B_i)$. Let $\delta : V \to V$ be the involution which interchanges the end-points of the edges of D. Note that V separates the ellipses into a set F consisting of an even number of edges. Label all the edges in D with color 2, and the edges on E_1 alternatively with colors 0 and 1, starting from an arbitrary vertex. Complete the coloring on each E_i, $i \geq 2$, by colors 0 and 1 so that each region of the planar 2-cell embedding of $F \cup D$ is bounded by edges alternatively colored only by two colors. Draw a further set D' of edges, each one connecting a pair of points of V which correspond under the involution $\tau\delta\tau$, and label these edges with color 3. Let (Γ_L, γ_L) be the 4-colored graph such that $V(\Gamma_L) = V$, $E(\Gamma_L) = D \cup D' \cup F$, and γ_L is the edge-coloration defined above. Note that the involution τ determines a unique involutory automorphism of Γ_L which interchanges 0-colored (resp. 2-colored) edges with 1-colored (resp. 3-colored) edges. For this reason, Γ_L is called the 2-*symmetric graph* associated with the bridge-presentation of L.

The following is the main result of [9].

Theorem 4.2. *Given a bridge-presentation of a link L, the associated 2-symmetric graph (Γ_L, γ_L) is a crystallization of the 2-fold cyclic covering of the 3-sphere branched over L.*

Let M be a closed connected 3-manifold, and let (Γ, γ) be a crystallization of M. Then $|\Gamma|$ can be regularly embedded into a splitting surface Σ of M, and for every cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ of Δ_3, the triple (Σ, α, β) – where α (resp. β) is the set of the cycles in Γ alternatively colored ϵ_0 and ϵ_1 (resp. ϵ_2 and ϵ_3) but one arbitrarily chosen – is a Heegaard diagram of M, denoted by $\mathcal{H}(\Gamma; \epsilon_0, \epsilon_1)$. The equivalence between the representation theories of 3-manifolds by Heegaard diagrams and crystallizations can be found in [5] and [36]. In [2] Birman and Hilden studied the relationship between Heegaard diagrams and branched coverings of the 3-sphere. Following [2], let Y_g be a solid (orientable) handlebody of genus g in the Euclidean 3-space E^3. Let Y'_g be a disjoint copy of Y_g, and let $t : Y_g \to Y'_g$ be the map that identifies a point $x \in Y_g$ with its corresponding points $x' \in Y'_g$. Let φ be an orientation reversing self-homeomorphism of ∂Y_g. The identification space $M = Y_g \cup_{t\circ\varphi} Y'_g$ is a closed (orientable) 3-manifold represented by a Heegaard splitting of genus g. Suppose that $\alpha : E^3 \to E^3$ is a homeomorphism of period p such that Y_g is left invariant under the action of α. Obviously, Y'_g is left invariant under the action of $\alpha' = t \circ \alpha \circ t^{-1}$. The Heegaard splitting $Y_g \cup_{t\circ\varphi} Y'_g$ is said to be *p-symmetric* if:

1) There is an integer p_0 $(1 \le p_0 \le p)$ such that

$$\varphi \circ (\alpha|_{\partial Y_g}) \circ \varphi^{-1} = (\alpha|_{\partial Y_g})^{p_0};$$

2) The orbit space Y_g/α of Y_g under the action of α is a 3-ball;
3) The fixed point set of α coincides with the fixed point set of α^k, for each $1 \le k < p$;
4) The image of the fixed point set of α is an unknotted set of arcs in the 3-ball Y_g/α.

The *p-symmetric Heegaard genus* of a closed 3-manifold M is the smallest integer g such that M admits a p-symmetric Heegaard splitting of genus g. We say that a crystallization (Γ, γ) of M is *p-symmetric* if the associated Heegaard diagram $\mathcal{H}(\Gamma; \epsilon_0, \epsilon_1)$ arises from a p-symmetric Heegaard splitting of M, for some distinct colors ϵ_0 and $\epsilon_1 \in \Delta_3$.

The following result was proved in [2].

Theorem 4.3. *Let $g \ge 0$, $p \ge 1$, and $b \ge 1$ be integers which are related by the equation $g = (b-1)(p-1)$. Then the class of closed orientable 3-manifolds which admit p-symmetric Heegaard splittings (or equivalently, p-symmetric crystallizations) of (combinatorial) genus g coincides with the class of p-fold cyclic coverings of the 3-sphere branched over links of bridge number $\le b$.*

Proof of Theorem 3.1. Let us consider the polyhedral schemata $Q(n, k)$, $(n, k) = d$, which defines the closed orientable 3-manifold $M(n, k)$ as a quotient of a triangulated 3-ball B^3 by pairwise identification of its boundary 2-cells. The identification produces a combinatorial complex $\widetilde{Q}(n, k)$ which triangulates $M(n, k)$. This complex has exactly d vertices, $n + d$ edges, $n + 1$ 2-cells, and one 3-cell. By rotational symmetry it suffices to prove the result in case $n = 2d$. We show that $M(d) := M(2d, d)$ can be represented by a 2-symmetric crystallization $\Gamma(d)$ of

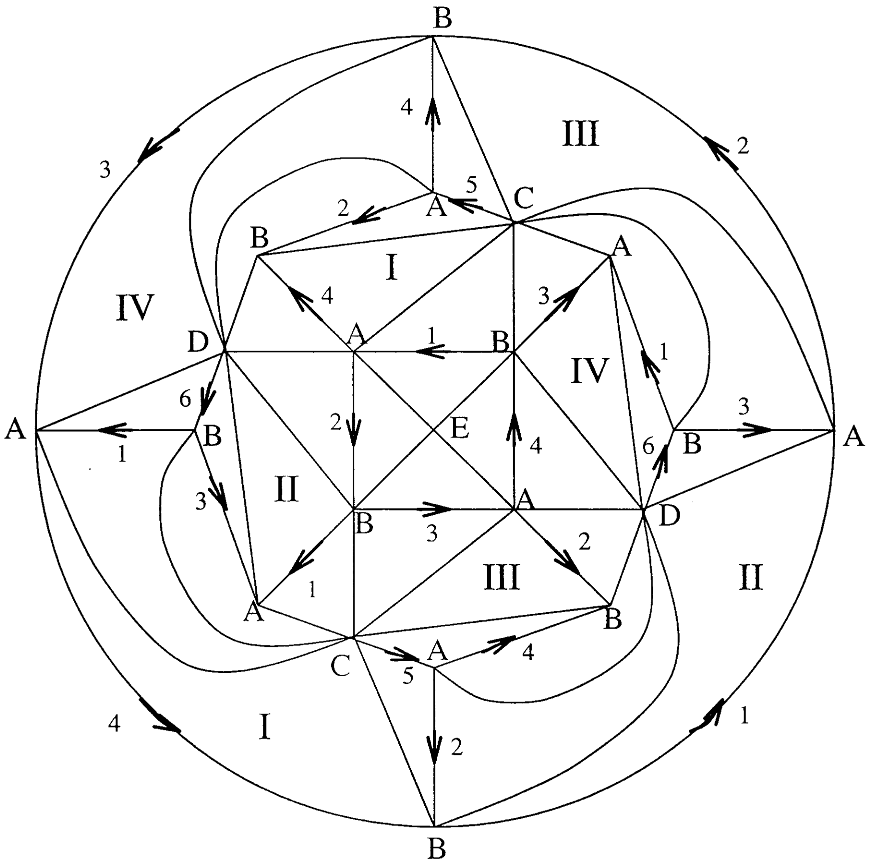


Fig. 4.1. The colored complex $\tilde{K}(2)$ triangulating $M(2)$

combinatorial genus d. Then Theorem 4.2 implies that $M(d)$ is the 2-fold covering of the 3-sphere branched over a $(d+1)$-bridge link. Now applying the algorithm described before of the statement of Theorem 4.2, it turns out that this link is equivalent, up to Reidemeister moves, to L_{d+1}. Note that L_2 and L_3 are the Whitehead link and the link 8^3_9 (according to notation of [38]), respectively. We have to treat cases d even and d odd in a different way, depending on the identification of polygons in $Q(d) := Q(2d, d)$.

(d **even**) Triangulate $Q(d)$ into a simplicial complex $K(d)$ by using stellar subdivisions as indicated in Fig. 4.1 ($d = 2$: one can immediately extend the construction by a simple iteration). Here it is understood a vertex outside of the exterior circle, and the corresponding stellar subdivision. The configuration is a simplicial tessellation of the 2-sphere ∂B^3 consisting of $10d + 2$ vertices, $30d$ edges, and $20d$ triangles. Let w be a point in the interior of B^3. Now $K(d)$ is just the simplicial join from w on the above tessellation. Identify the two copies of each triangle in $\partial K(d)$ so that the corresponding oriented edges carrying the same label are glued together. The identification produces a pseudocomplex $\widetilde{K}(d)$ which triangulates $M(d)$. It consists of $2d+2$ vertices, $12d+2$ edges, $20d$ triangles, and $10d$ tetrahedra. Moreover, it is easily seen that $\widetilde{K}(d)$ is a colored complex. The vertex of $\widetilde{K}(d)$ arising from w (also denoted by the same symbol) belongs to all the tetraedra, and we label it with color 3. The vertices of $\widetilde{K}(d)$ arising from those of the $(2d)$-gon of $Q(d)$ are alternatively colored 0 and 1 according to a circular order. For example, we label vertices A and B in Fig. 4.1 with colors 0 and 1, respectively. Any vertex of $K(d)$ which is a barycenter of an edge of $Q(d)$, and the barycenter of the above $(2d)$-gon cover vertices of $\widetilde{K}(d)$ labelled with color 2. For example, we label vertices C, D, and E in Fig. 4.1 with color 2. Let $\widetilde{\Gamma}(d)$ denote the 4-colored graph associated to $\widetilde{K}(d)$ by rules 1) and 2) discussed at the beginning of the section. It can be constructed as follows. The vertices of $\widetilde{\Gamma}(d)$ are the elements of $\widetilde{V}(d) = (\{0, 2d+1\} \times \mathbb{Z}_{2d}) \cup (\{1, \ldots, 2d\} \times \mathbb{Z}_8)$. The colored edges of $\widetilde{\Gamma}(d)$ are defined by means of the following four fixed-point-free involutions on $\widetilde{V}(d)$:

$$\nu_0(i, j) = (i, j + (-1)^j)$$

$$\nu_1(i, j) = (i, j - (-1)^j)$$

$$\nu_2(i, j) = \begin{cases} (2d+1, i-1) & \text{if} \quad j = 0 \\ (0, 1-i) & \text{if} \quad j = 4 \\ (i + (-1)^{i+1}\mu(j), j) & \text{otherwise} \end{cases}$$

$$\nu_3(i, j) = \begin{cases} (i+d, 5-j) & \text{if} \quad i \in \{1, \ldots, d-1\},\ i \quad \text{odd} \\ (i+d, 3-j) & \text{if} \quad i \in \{2, \ldots, d\}, \quad i \quad \text{even} \\ (2d+1, 1-j) & \text{if} \quad i = 0, \end{cases}$$

where $\mu : \mathbb{Z}_8 \backslash \{0, 4\} \to \{+1, -1\}$ is the function defined by

$$\mu(j) = \begin{cases} +1 & \text{if} \quad 1 \le j \le 3 \\ -1 & \text{if} \quad 5 \le j \le 7. \end{cases}$$

According to the definition of $\widetilde{V}(d)$, it is evident that the arithmetic is either mod $2d$ or mod 8 in the second coordinate of each pair (i, j). To define the 4-colored graph $\widetilde{\Gamma}(d)$ it suffices to interpret the involutions ν_i as colored edges, i.e. for each $i \in \Delta_3$ two vertices x and y in $\widetilde{V}(d)$ are joined by an edge colored i if and only if $y = \nu_i(x)$. The geometrical shape of $\widetilde{\Gamma}(d)$ can be described as follows. It consists of $2d$ $\{0, 1\}$-colored cycles C_i of length 8 cyclically set on the plane following the natural order of the set $\{1, \ldots, 2d\}$, and of two $\{0, 1\}$-colored cycles C_0 and C_{2d+1} of length $2d$. The vertex set of each C_i, $i \in \{1, \ldots, 2d\}$, consists of pairs (i, j), for any $j \in \mathbb{Z}_8$. The vertex set of C_i, $i = 0, 2d+1$, consists of pairs (i, j), for any $j \in \mathbb{Z}_{2d}$. We cyclically order the vertices (i, j) of each C_i following the natural order of j in $\mathbb{Z}_8$ (or $\mathbb{Z}_{2d}$) so that all these orderings induce the clockwise (resp. anti-clockwise) orientation of the plane when i is odd (resp. even). There are exactly three 2-colored edges between C_i and C_{i+1} (resp. C_i and C_{i-1}), for any $i = 1, \ldots, 2d$ (here $0 \equiv 2d$). There is exactly one 2-colored edge between C_i and C_{2d+1} (resp. C_i and C_0), for any $i = 1, \ldots, 2d$. Furthermore, it is possible to

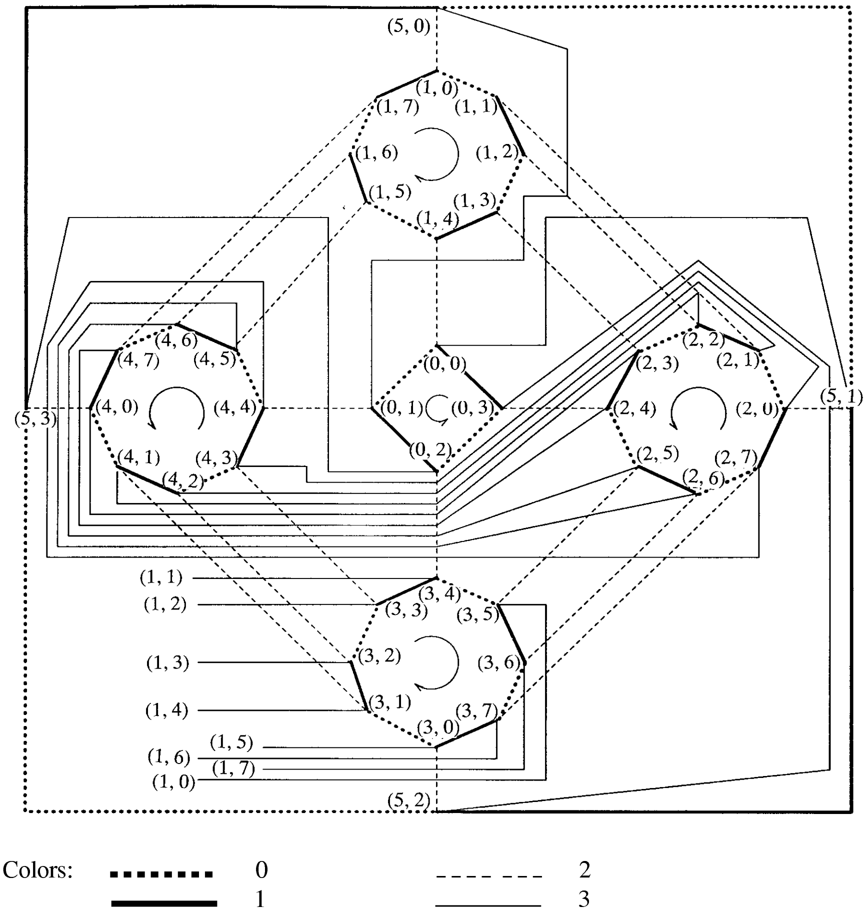
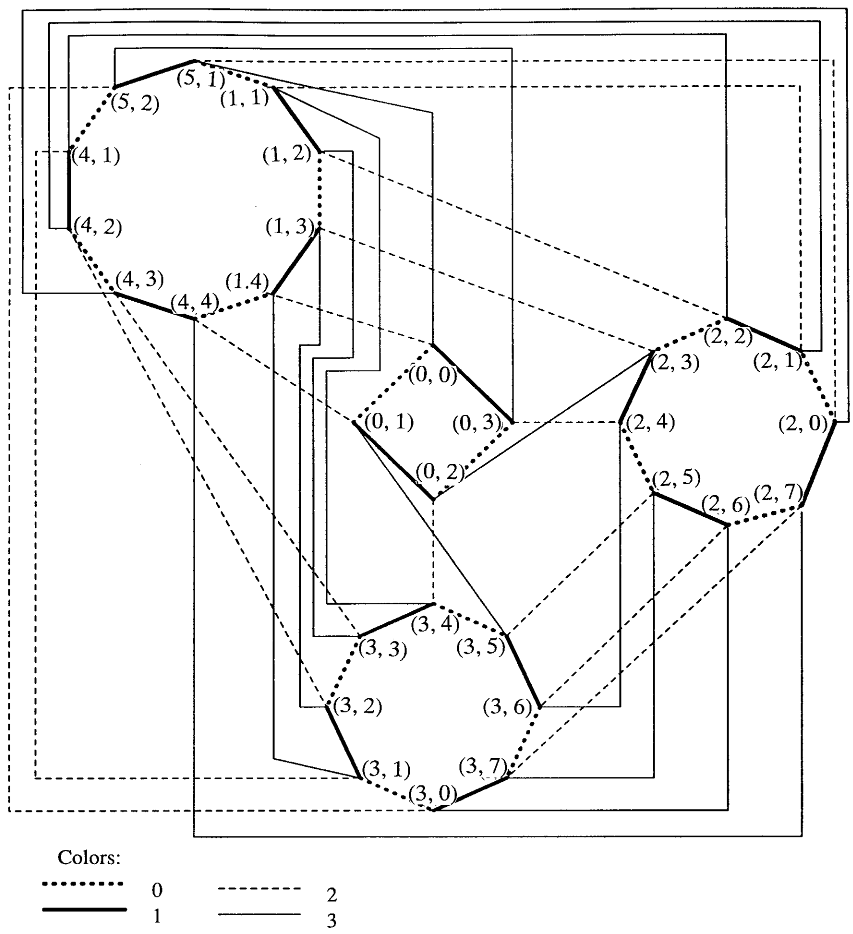


Fig. 4.2. The 4-colored graph $\widetilde{\Gamma}(2)$ of $M2$

draw these edges so that the subgraph $\widetilde{\Gamma}(d)_{\{0,1,2\}}$ is regularly embedded in the plane. There are exactly eight 3-colored edges between C_i and C_{i+d}, for any $i = 1, \dots, d$, and $2d$ 3-colored edges between C_0 and C_{2d+1}. Again as above, we can draw these edges so that the subgraph $\widetilde{\Gamma}(d)_{\{0,1,3\}}$ is regularly embedded in the plane. For example, Fig. 4.2 shows the 4-colored graph $\widetilde{\Gamma}(2)$ representing $M(2)$. Now the graph $\widetilde{\Gamma}(d)$ has combinatorial genus $2d+1$, and contains $2d-2$ dipoles of type 1. Cancelling them (and the induced dipoles of type 2) yields a crystallization $\Gamma'(d)$ of $M(d)$ whose combinatorial genus is $d+1$. We explicitly describe the moves for case $d = 2$. We successively cancel from $\widetilde{\Gamma}(2)$ the two dipoles of type 1 with vertex sets $\{(1,0),(5,0)\}$ and $\{(1,5),(4,5)\}$. These moves induce three dipoles of type 2 with vertex sets $\{(1,6),(4,6)\}$, $\{(1,7),(4,7)\}$, and $\{(5,3),(4,0)\}$ (here we maintain the same label for each vertex left unchanged under the current move). The



Colors: 0 1 2 3

Fig. 4.3. The genus 3 crystallization $\Gamma'(2)$ of $M2$

further cancellation of these dipoles yields a (combinatorial) genus 3 crystallization $\Gamma'(2)$ of $M(2)$, depicted in Fig. 4.3.

There is a generalized dipole of type $(3, 5)$ in $\Gamma'(2)$ involving vertices $x_0 = (0, 3)$, $x_1 = (0, 2)$, $x_2 = (0, 1)$, and $x_3 = (0, 0)$ of a cycle C_1 alternatively colored 0 and 1, and vertices $x_0 = (0, 3)$, $y_1 = (5, 2)$, $y_2 = (3, 0)$, $y_3 = (2, 6)$, $y_4 = (3, 6)$, and $y_5 = (2, 4)$ of a cycle C_2 alternatively colored 2 and 3. Cancelling this generalized dipole (and the induced dipole of type 2 between vertices $(5, 1)$ and (x_3, y_1)) yields a (combinatorial) genus 2 crystallization $\Gamma(2)$ of $M(2)$. As shown in Fig. 4.4 (top), there is a unique involutory automorphism of $\Gamma(2)$ which

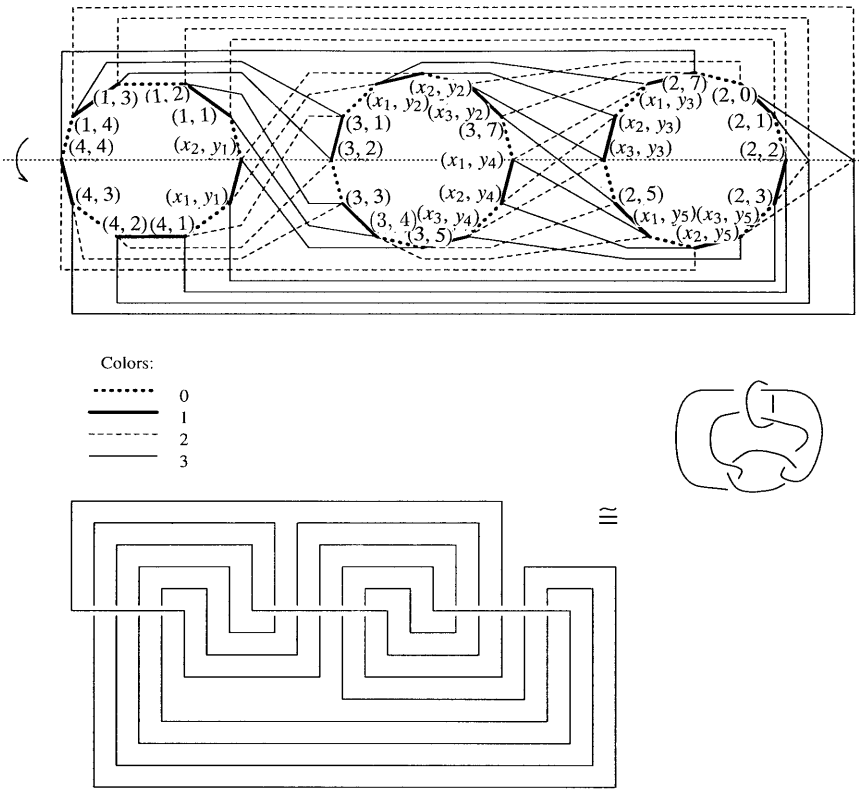


Fig. 4.4. The 2-symmetric crystallization $\Gamma(2)$ of $M(2)$ **(top)**; A 3-bridge presentation of the link $L_3 = 8^3_9$ **(bottom)**

fixes vertices (4, 4), (x_2, y_1), (3, 2), (x_1, y_4), (x_3, y_3), and (2, 2), and interchanges 0-colored (resp. 2-colored) edges with 1-colored (resp. 3-colored) edges. Thus $\Gamma(2)$ is actually 2-symmetric, and $M(2)$ is the 2-fold covering of the 3-sphere branched over the 3-bridge link of Fig. 4.4 (bottom) (use the algorithm described before of Theorem 4.2).

Using Reidemeister moves, it is immediate to verify that this link is equivalent to the link $L_3 = 8^3_9$. Extending these constructions one can easily obtain the general result. For example, Fig. 4.5 shows the 2-symmetric crystallization $\Gamma(4)$ of $M(4)$ (of combinatorial genus 4), and the branch set of the 2-fold covering $M(4) \to \mathbb{S}^3$ (which is a 5-bridge presentation of the link L_5).

(*d* **odd**) The construction is different from the previous one because doing stellar subdivision of the $(2d)$-gon of $Q(d)$ from a vertex in its interior does not produce a colored complex as a quotient (in fact, we have to do a simplicial join from a boundary vertex). To simplify the reading we illustrate the constructions only for case $d = 3$, but one can patiently extend them by a simple iteration, as

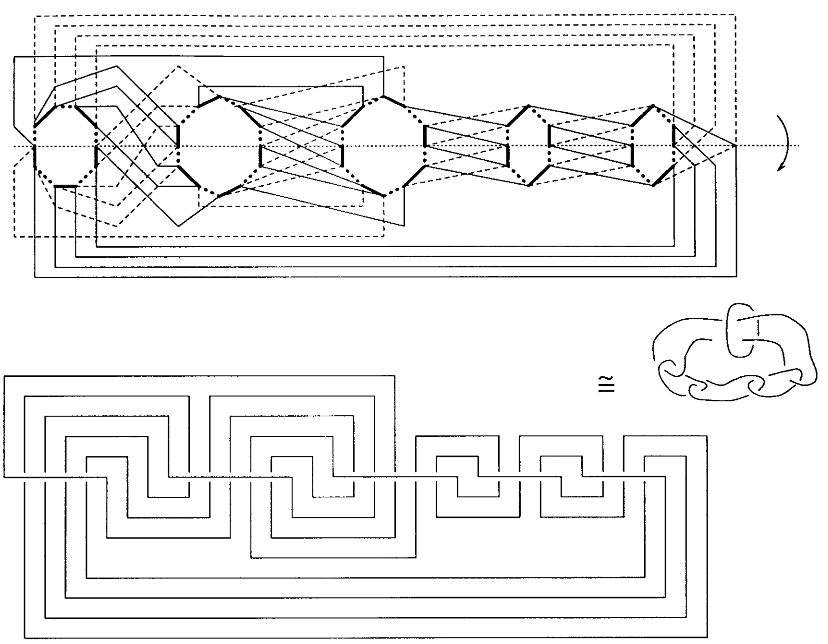


Fig. 4.5. The 2-symmetric crystallization $\Gamma(4)$ of $M(4) = M(8, 4)$ **(top)**; A 5-bridge presentation of the link L_5 **(bottom)**

done for case d even. Triangulate $Q(3)$ into a simplicial complex $K(3)$ by using stellar subdivisions as indicated in Fig. 4.6 (here it is understood, as usual, a vertex w outside of the exterior circle (or equivalently, lying in the interior of the 3-ball B^3), and the corresponding stellar subdivision.

We label vertices B and D (resp. C and E) of Fig. 4.6 with color 0 (resp. 1). Furthermore, vertices A and F are colored 2, and w is colored 3, as usual. The identification complex $\widetilde{K}(3)$ is obviously colored, and it is represented by the 4-colored graph $\widetilde{\Gamma}(3)$ depicted in Fig. 4.7 (here any hanging edge of color 3 coming out from a vertex means that there is a 3-colored edge joining the vertex with that indicated near the free end-point).

Cancelling in $\widetilde{\Gamma}(3)$ the two dipoles of type 1 with vertex sets $\{27, 28\}$ and $\{37, 38\}$ (and the three induced dipoles of type 2 between vertices 7–8, 15–16, and 23–24, respectively) yields a (combinatorial) genus 7 crystallization $\Gamma^1(3)$ of $M(3)$ (as usual, we maintain the same label for any vertex left unchanged by the current move). There is a generalized dipole Ω_1 of type $(5, 3)$ in $\Gamma^1(3)$ involving vertices $x_0 = 4, x_1 = 3, x_2 = 31, x_3 = 34, x_4 = 43$, and $x_5 = 44$ (of a cycle alternatively colored 0 and 1), and vertices $x_0 = 4$, $y_1 = 17$, $y_2 = 18$, and $y_3 = 5$ (of a cycle alternatively colored 2 and 3). Cancelling Ω_1 (and the induced dipoles of type 2 between vertices 45–(x_5, y_3), 46–(x_1, y_1), 47–(x_2, y_1), and 48–(x_3, y_1)) yields

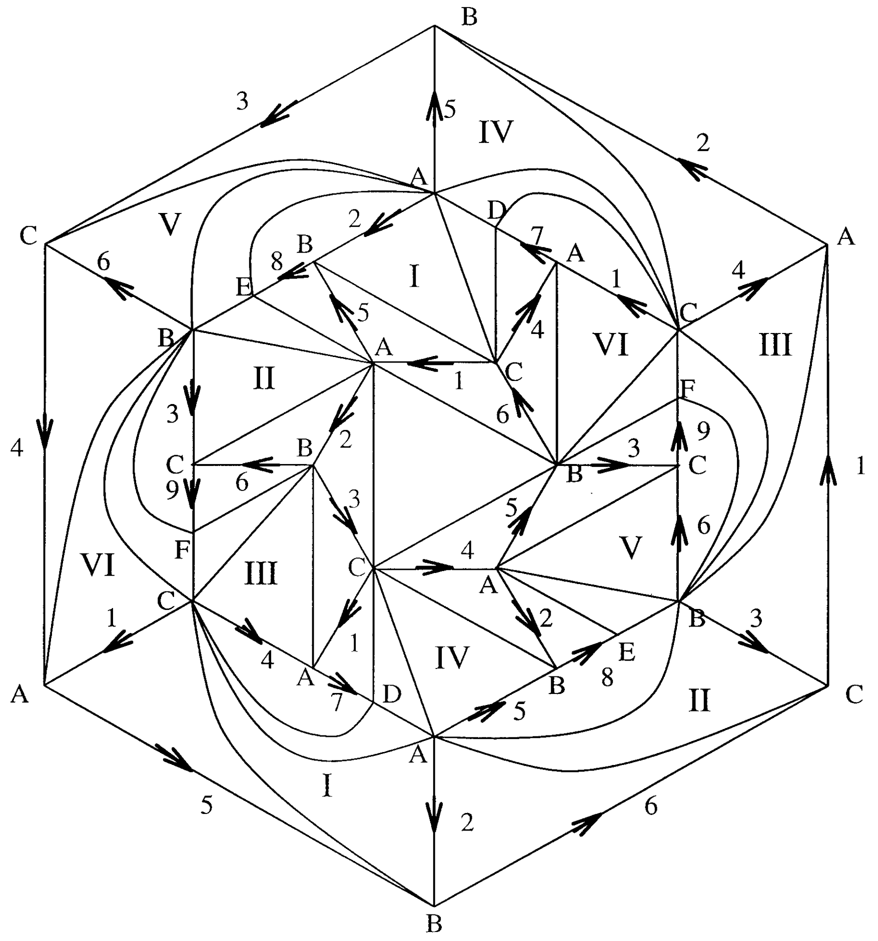


Fig. 4.6. The colored complex $\tilde{K}(3)$ triangulating $M(3)$

a (combinatorial) genus 6 crystallization $\Gamma^2(3)$ of $M(3)$. There is a generalized dipole Ω_2 of type $(5, 5)$ in $\Gamma^2(3)$ involving vertices $\alpha_0 = 9$, $\alpha_1 = 10$, $\alpha_2 = 11$, $\alpha_3 = 12$, $\alpha_4 = 13$, and $\alpha_5 = 14$ (of a cycle alternatively colored 0 and 1), and vertices $\alpha_0 = 9$, $\beta_1 = 20$, $\beta_2 = 19$, and $\beta_3 = (x_4, y_1)$, $\beta_4 = (x_4, y_2)$, and $\beta_5 = (x_4, y_3)$ (of a cycle alternatively colored 2 and 3). Cancelling Ω_2 (and the induced dipole of type 2 between vertices 21–(α_1, β_1)) yields a (combinatorial) genus 5 crystallization $\Gamma^3(3)$ of $M(3)$. There is a generalized dipole Ω_3 of type $(7, 3)$ in $\Gamma^3(3)$ formed by a $\{0, 1\}$-colored cycle with vertices $a_0 = 29$, $a_1 = 49$, $a_2 = 52$, $a_3 = 50$, $a_4 = 30$, $a_5 = 2$, $a_6 = 1$, and $a_7 = 42$, and by a $\{2, 3\}$-colored cycle of vertices $a_0 = 29$, $b_1 = (x_5, y_2)$, $b_2 = (x_5, y_1)$, and $b_3 = 26$. Cancelling Ω_3 (and the induced dipole of type 2 between vertices 25–(a_1, b_3)) gives a (combinatorial) genus 4 crystallization $\Gamma^4(3)$ of $M(3)$. There is a generalized

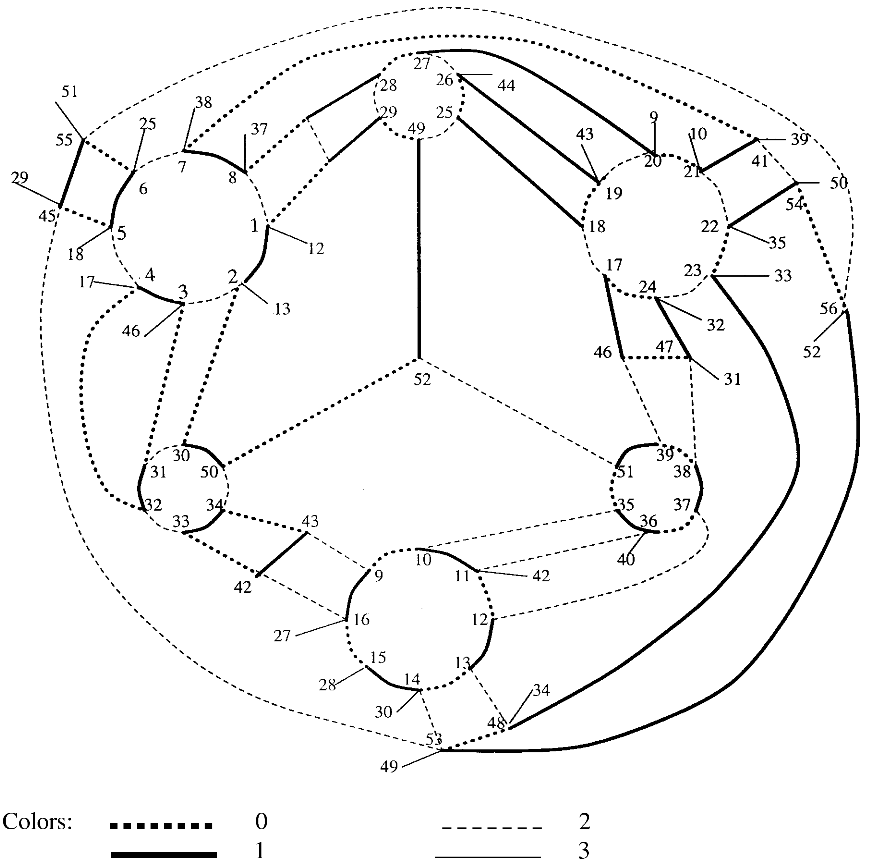
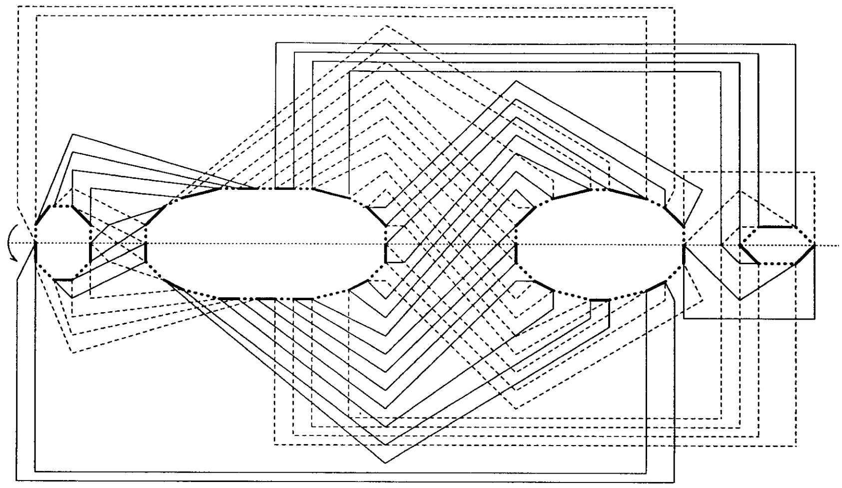


Fig. 4.7. The 4-colored graph $\tilde{\Gamma}(3)$ representing $M(3)$

dipole Ω_4 of type (3, 5) in $\Gamma^4(3)$ involving vertices $m_0 = 35, m_1 = 51, m_2 = 39$, and $m_3 = 36$ (of a cycle alternatively colored 0 and 1), and vertices $m_0 = 35$, $\ell_1 = 22$, $\ell_2 = (\alpha_1, \beta_2)$, $\ell_3 = (\alpha_1, \beta_3)$, $\ell_4 = (\alpha_1, \beta_4)$, and $\ell_5 = (\alpha_1, \beta_5)$ (of a cycle alternatively colored 2 and 3). Cancelling Ω_4 (and the induced dipoles of type 2 between vertices (α_2, β_5)– (m_3, ℓ_5), (m_3, ℓ_4)–(α_2, β_4), (m_3, ℓ_3)–(α_2, β_3), and (α_2, β_2)–(m_3, ℓ_2)) yields a (combinatorial) genus 3 crystallization $\Gamma(3)$ of $M(3)$, drawn in Fig. 4.8 (top). There is a unique involutory automorphism of $\Gamma(3)$ which interchanges 0-colored (resp. 2-colored) edges with 1-colored (resp. 3-colored) edges. Thus $\Gamma(3)$ is actually 2-symmetric, and $M(3)$ is the 2-fold covering of the 3-sphere branched over the 4-bridge link of Fig. 4.8 (bottom).

Using Reidemeister moves, it is immediate to verify that this link is equivalent to the link L_4. Extending these constructions one can obtain the general case. Thus the proof is complete. □



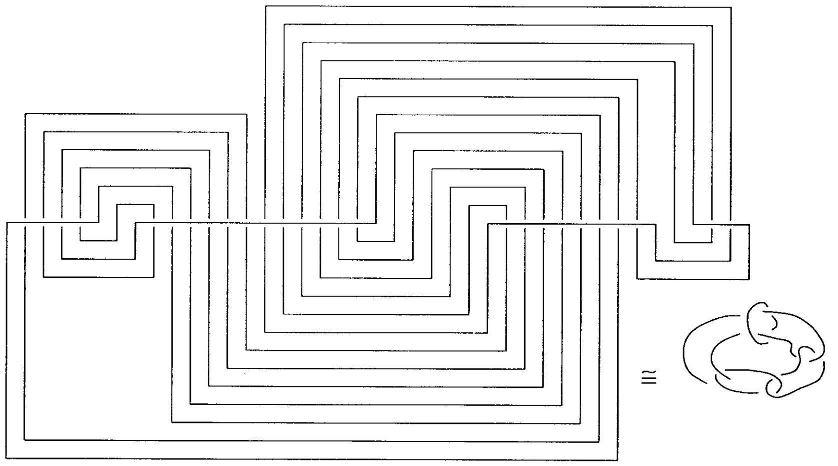


Fig. 4.8. The 2-symmetric crystallization $\Gamma(3)$ of $M(3)$ **(top)**; A 4-bridge presentation of the link L_4 **(bottom)**

It is known (see for example [3] and [11]) a combinatorial algorithm for computing a finite presentation of the fundamental group of a closed n-manifold M directly from a crystallization (Γ, γ) of M. We briefly recall the construction only for dimension 3. Let (i, j, h, k) be a permutation of the color set Δ_3. Denote by $a_1, \ldots, a_n$ (resp. $b_1, \ldots, b_n$) the connected components of the subgraph $\Gamma_{\{h,k\}}$ (resp. $\Gamma_{\{i,j\}}$), but one arbitrarily chosen. Fix a running direction and a starting vertex for each b_ℓ, and compose the word r_ℓ on generators $a_1, \ldots, a_n$ from the $\{i, j\}$-colored cycle b_ℓ by the following rule: follow the chosen direction starting from the chosen vertex, and write consecutively every generator you meet, with exponent $+1$ or -1

according to i or j being the color of the edge by which you run into the generator. This yields a finite presentation of $\Pi_1(M)$ with generators $a_1, \dots, a_n$, and relators $r_1, \dots, r_n$.

Now cyclically iterating the procedures described in the above proof, we get a (n/d)-symmetric crystallization $(\Gamma(n,k), \gamma(n,k))$ of the closed 3-manifold $M(n,k)$ whose combinatorial genus equals $n-d$ (where $d = (n,k)$, as usual). Applying the above algorithm to $\Gamma(n,k)$ yields the following result (for convenience, we treat only case d even, and leave the remaining one to the reader).

Theorem 4.4. *The fundamental group of the closed* 3*-manifold $M(n,k)$, with $d = (n,k)$ even, admits a finite presentation with $n-d$ generators $a_d, \dots, a_{n-1}$, and $n-d$ relations*

$$a_{2j-1+di}a^{-1}_{2j-1+di+n-k}a_{2j-1+n-k}a^{-1}_{2j-1}a_{2j+n-k}a^{-1}_{2j}a_{2j+di}a^{-1}_{2j+di+n-k} = 1$$
$$a^{-1}_{2j+di}a_{2j+di+n-k}a^{-1}_{2j+n-k}a_{2j}a^{-1}_{2j+n-k+1}a_{2j+1}a^{-1}_{2j+di+1}a_{2j+di+n-k+1} = 1,$$

for any $i = 1, \dots, (n/d)-1$, and for any $j = 1, \dots, d/2$. The indices are taken mod n with the additional conditions $a_i = 1$, for any $i = 1, \dots, d-1$, and $a_n = a_{n-1}a^{-1}_{n-2}\cdots a^{-1}_{d+2}a_{d+1}a^{-1}_{d}$. This presentation corresponds to a spine of $M(n,k)$, and arises from a (n/d)-symmetric Heegaard splitting (or equivalently, a (n/d)-symmetric crystallization) of $M(n,k)$ whose (combinatorial) genus equals $n-d$. In particular, one re-obtains that $M(n,k)$ is the (n/d)-fold cyclic covering of the 3*-sphere branched over a $(d+1)$-bridge presentation of a link (in fact, one equivalent to L_{d+1}) according to Theorem 4.3.*

To end the section we completely determine the topological and geometric structures of $M(2) = M(4,2)$. Recall that $M(n,k)$ is hyperbolic for any $n \geq 3$ and $d < n/2$ while it is fibered for $d = n/2$ (use the Dunbar enumeration of all closed oriented geometric 3-orbifolds which are not hyperbolic [8]).

Theorem 4.5. *Let $K \underset{\sim}{\times} [0,1]$ be the oriented skew product of the Klein bottle by a segment. Then $M(2)$ is the Nil-manifold obtained by pasting together two copies of $K \underset{\sim}{\times} [0,1]$ under the homeomorphism of their torus boundaries defined by the matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. Let $O(2)$ be the 3-orbifold with underlying space $\mathbb{S}^3$ and with singular set the link $L_3 = 8^3_9$ whose components have branched index 2. Then $M(2)$ is the 2-fold covering of $O(2)$. Since $O(2)$ is a geometrically indecomposable Nil-orbifold (see Table 3 of [8]), $M(2)$ has a geometric structure modelled on the same geometry. Let now $\Gamma(2)$ be the 2-symmetric crystallization of $M(2)$ shown in Fig. 4.4 (top). It induces a genus 2 Heegaard diagram $\mathcal{H}(2) := \mathcal{H}(\Gamma(2); 0, 1)$, depicted in Fig. 4.9.

We analyze this diagram for computing the complexity of $M(2)$ in the sense of [22], [23], and [24]. Recall that the *complexity* of a compact 3-manifold M is the smallest integer k such that M possesses an almost special spine with k vertices (or equivalently, the minimal number of 3-simplexes which have to be glued together to construct M). In [23] Matveev gave an estimate of the complexity of a manifold

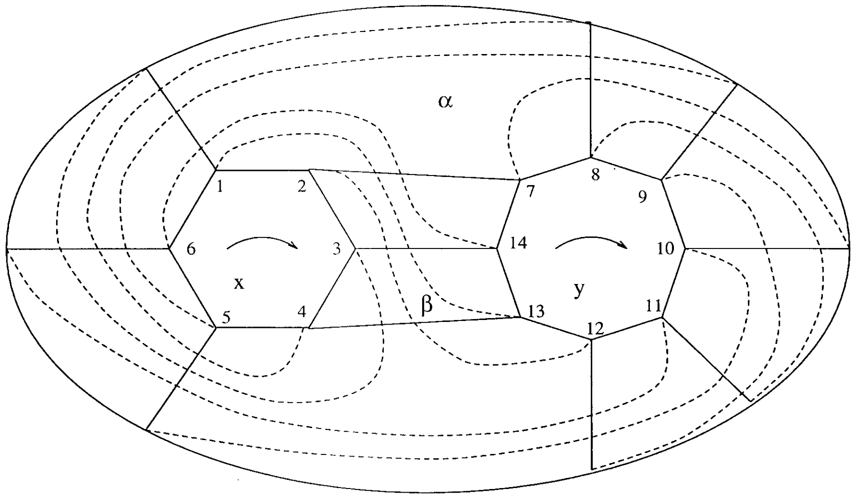


Fig. 4.9. The genus 2 Heegaard diagram $H(2)$ of $M(2)$

from a Heegaard splitting of it. More precisely, let $M = V_1 \cup V_2$ be a Heegaard splitting of M such that the meridians of the handlebody V_1 intersect the meridians of the handlebody V_2 transversally at exactly n points. Suppose that the closure of one of the domains (open 2-cells) into which these meridians split the surface ∂V_1 $(= \partial V_2)$ contains precisely m such points. Then it was proved in [23] that the complexity of M does not exceed $n - m$. Let us consider the Heegaard diagram $\mathcal{H}(2)$ of $M(2)$. The union of the splitting surface Σ and the meridional discs of the two (orientable) handlebodies of genus 2 is a special spine of $M(2)$ with two open 3-cells removed. This spine possesses 14 vertices which are numbered from 1 to 14 as indicated in Figure 4.9. An almost special spine of $M(2)$ is obtained from it by puncturing one of the 2-cells into which the meridians of Σ divide Σ. If we puncture the 2-cell α in Fig. 4.9, then, after collapsing, vertices 1, 2, 3, 4, 7, and 8 cease to be vertices of the spine. Therefore, the complexity of $M(2)$ is less or equal than 8 (use the above formula for $n = 14$ and $m = 6$). A more detailed analysis of the diagram shows that we can further puncture the adjacent 2-cell β as $\alpha \cup \beta$ is still a 2-cell of Σ (in fact, $\alpha \cap \beta$ is precisely the arc of Σ joining vertices 3 and 4). Then, after the second collapsing, vertices 13 and 14 cease to be vertices of a spine of $M(2)$. This implies that the complexity of $M(2)$ does not exceed 6. Simplifying the presentation of $\Pi_1(M(2))$, given by Theorem 4.4, we get a new presentation with two generators x $(= a_2)$ and y $(= a_3)$, and two relations $x^2 = (y^2x)^2$ and $y^2 = (x^2y^{-1})^2$. Then the first integral homology group of $M(2)$ is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. Now a complete list of all closed orientable prime 3-manifolds of complexity ≤ 6 was given in [23] (see also [24]). Looking through the list, it follows that the only 3-manifold which admits a Nil-geometry and has homology group $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ is that of the statement (in particular, the complexity of $M(2)$ equals

6). A partial alternative proof of the result can be obtained as follows. The normal subgroup $N =< x^2, y^2 >$ of $\Pi_1 = \Pi_1(M(2))$ is commutative, and isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. Since the factor group $Q = \Pi_1/N$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$, we have the exact sequence $1 \to N \cong \mathbb{Z}\oplus\mathbb{Z} \to \Pi_1 \to Q \cong \mathbb{Z}_2 * \mathbb{Z}_2 \to 1$. Now one can apply Theorem 11.8 of [14]. □

5. Dehn surgery

Let L_{d+1} be the oriented link of $d + 1$ components (in the right-oriented 3-sphere $\mathbb{S}^3$) illustrated in Fig. 5.1. This link is formed by a chain of d unknotted circles K_i, for $i = 1, 2, \dots, d$, each of which is linked with exactly two adjacent components of the chain, plus a further circle Λ transversally linked to the chain.

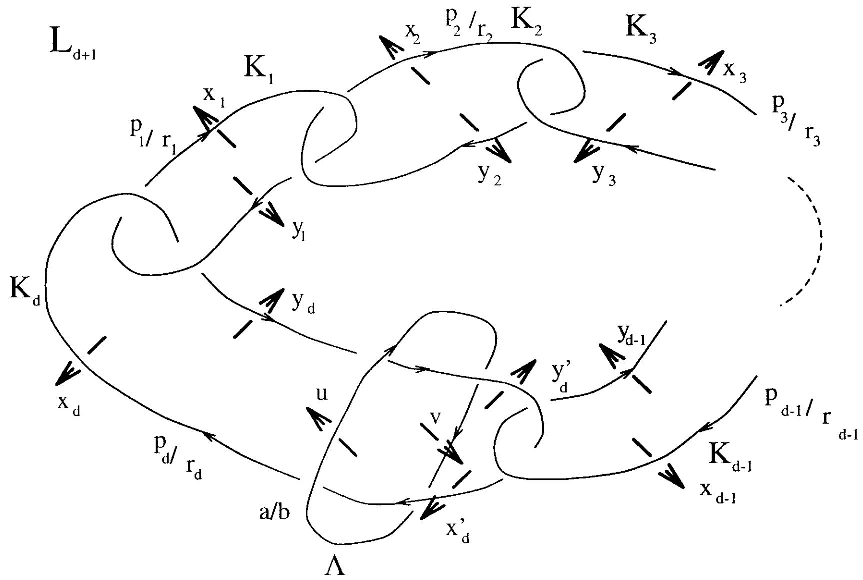


Fig. 5.1. The oriented link L_{d+1} in S^3

Let p_i/r_i be the surgery coefficient along the i-th component K_i of the chain, and let a/b be the surgery coefficient along the transversal component Λ, where $(p_i, r_i) = (a, b) = 1$. Let $M(p_1/r_1, \dots, p_d/r_d; a/b)$ denote the closed connected orientable 3-manifold obtained by Dehn surgeries along the components of L_{d+1} with surgery coefficients p_i/r_i and a/b. We now obtain finite presentations of the fundamental group of these manifolds as follows. Taking the generators x_i, y_i, x'_d, y'_d, u, and v according to Fig. 5.1 yields a Wirtinger presentation of the link group of L_{d+1}:

$$\Pi_1(\mathbb{S}^3\backslash L_{d+1}) \cong < x_1, y_1, \ldots, x_d, y_d, x'_d, y'_d, u, v : \quad \begin{array}{l} y_i x_{i-1} = x_{i-1} x_i \\ x_{i-1} y_i = y_i y_{i-1} \\ (i = 1, \ldots, d-1) \\ y'_d x_{d-1} = x_{d-1} x'_d \\ x_{d-1} y'_d = y'_d y_{d-1} \\ u y_d = y'_d u = v y'_d \\ v x'_d = x'_d u = u x_d \\ (\text{indices mod } d) > \end{array}$$

The meridian m_i and the longitude ℓ_i of each component K_i, and the meridian m and the longitude ℓ of the transversal circle Λ are:

$$\begin{aligned} & m_i = x_i \qquad (i = 1, \ldots, d) \\ & m = u \\ & \ell_i = y_{i+1} x_{i-1} \qquad (i = 1, \ldots, d-2; \quad \text{indices mod } d) \\ & \ell_{d-1} = y'_d x_{d-2} \\ & \ell_d = y_1 u^{-1} x_{d-1} u \\ & \ell = (y'_d)^{-1} x'_d \\ & [m_i, \ell_i] = [m, \ell] = 1 \qquad (i = 1, \ldots, d). \end{aligned}$$

A presentation of the fundamental group of $M(p_i/r_i; a/b)$ is obtained from that of the link group of L_{d+1} by adding relations:

$$\begin{aligned} & m_i^{p_i} \ell_i^{r_i} = 1 \qquad (i = 1, \ldots, d) \\ & m^a \ell^b = 1. \end{aligned}$$

We improve this presentation, and obtain a new one which arises from a Heegaard diagram of the considered manifold (i.e. it is *geometric*). Since p_i and r_i (resp. a and b) are coprime, there exist integers s_i and q_i (resp. s and q) such that $r_i s_i - p_i q_i = 1$, for any $i = 1, \ldots, d$, and $bs - aq = 1$.

Setting

$$w_i := m_i^{s_i} \ell_i^{q_i}$$

and

$$w := m^s \ell^q,$$

we get

$$\begin{aligned} w_i^{r_i} &= m_i, \qquad & w^b &= m \\ w_i^{-p_i} &= \ell_i, \qquad & w^{-a} &= \ell. \end{aligned}$$

We can now express generators x_i, y_i, x'_d, y'_d, u, and v in terms of w_i and w. Indeed, we have

$$\begin{aligned} y_1 &= \ell_d u^{-1} x_{d-1}^{-1} u = w_d^{-p_d} w^{-b} w_{d-1}^{-r_{d-1}} w^b \\ y_{i+1} &= \ell_i x_{i-1}^{-1} = w_i^{-p_i} w_{i-1}^{-r_{i-1}} \qquad (i = 1, \dots, d-2) \\ y'_d &= \ell_{d-1} x_{d-2}^{-1} = w_{d-1}^{-p_{d-1}} w_{d-2}^{-r_{d-2}} \\ y_d &= u^{-1} y'_d u = w^{-b} w_{d-1}^{-p_{d-1}} w_{d-2}^{-r_{d-2}} w^b \\ x'_d &= y'_d \ell = w_{d-1}^{-p_{d-1}} w_{d-2}^{-r_{d-2}} w^{-a} \\ v &= y'_d u (y'_d)^{-1} = w_{d-1}^{-p_{d-1}} w_{d-2}^{-r_{d-2}} w^b w_{d-2}^{r_{d-2}} w_{d-1}^{p_{d-1}}. \end{aligned}$$

Substituting these formulae in

$$y_1 x_d x_1^{-1} x_d^{-1} = 1$$

and

$$x_d y_1 y_d^{-1} y_1^{-1} = 1$$

yields relations

$$w_d^{p_d + r_d} w_1^{r_1} w_d^{-r_d} w^{-b} w_{d-1}^{r_{d-1}} w^b = 1$$

and

$$w_{d-1}^{p_{d-1} + r_{d-1}} w^b w_d^{r_d} w^{-b} w_{d-1}^{-r_{d-1}} w_{d-2}^{r_{d-2}} = 1,$$

respectively. For any $i = 1, \dots, d-2$, relations $y_{i+1} x_i x_{i+1}^{-1} x_i^{-1} = 1$ (or equivalently, $x_i y_{i+1} y_i^{-1} y_{i+1}^{-1} = 1$) become

$$w_i^{p_i + r_i} w_{i+1}^{r_{i+1}} w_i^{-r_i} w_{i-1}^{r_{i-1}} = 1.$$

Relation $v x'_d u^{-1} (x'_d)^{-1} = 1$ becomes an identity, while from relation

$$u x_d u^{-1} (x'_d)^{-1} = 1$$

we get

$$w^b w_d^{r_d} w^{a-b} w_{d-2}^{r_{d-2}} w_{d-1}^{p_{d-1}} = 1.$$

Summarizing we have obtained the following result

Theorem 5.1. *The fundamental group of the closed connected orientable* 3*-manifold* $M(p_1/r_1, \dots, p_d/r_d; a/b)$ *obtained by Dehn surgery along the oriented link* L_{d+1} *(*$d \geq 2$*) with surgery coefficients* p_i/r_i *and* a/b *admits the finite presentation*

$$\begin{aligned} < w_1, \dots, w_d, w : \quad & w_i^{p_i + r_i} w_{i+1}^{r_{i+1}} w_i^{-r_i} w_{i-1}^{r_{i-1}} = 1 \\ & (i = 1, \dots, d-2) \\ & w_{d-1}^{p_{d-1} + r_{d-1}} w^b w_d^{r_d} w^{-b} w_{d-1}^{-r_{d-1}} w_{d-2}^{r_{d-2}} = 1 \\ & w_d^{p_d + r_d} w_1^{r_1} w_d^{-r_d} w^{-b} w_{d-1}^{r_{d-1}} w^b = 1 \\ & w^b w_d^{r_d} w^{a-b} w_{d-2}^{r_{d-2}} w_{d-1}^{p_{d-1}} = 1 \\ & (\textit{indices mod} \quad d) > . \end{aligned}$$

For $d = 2$ there are only the last three relations. For $d = 1$, the link L_2 is just the Whitehead link, and the fundamental group of $M(p_1/r_1; a/b)$ has the finite presentation

$$
\begin{aligned}
< w_1, w : \quad & w_1^{p_1+r_1} w^b w_1^{r_1} w^{-b} w_1^{-r_1} w^{-b} w_1^{r_1} w^b = 1 \\
& w^b w_1^{r_1} w^{a-2b} w_1^{r_1} w^b w_1^{p_1} = 1 > .
\end{aligned}
$$

In [25] Mednykh and Vesnin studied the manifolds obtained by Dehn surgeries on the two components of the Whitehead link, and proved that they are 2-fold branched coverings of the 3-sphere by using a well known theorem of Montesinos [27]. As a consequence, a list of the ten smallest known (with respect to volumes) closed orientable hyperbolic 3-manifolds was given in [25] (see also [26]).

We now prove that the finite group presentations, obtained above, correspond to spines of the considered manifolds (and hence these presentations are geometric – that is, they arise from Heegaard splittings (diagrams) of genus $d+1$). In particular, the genus of the manifolds $M(p_1/r_1, \ldots, p_d/r_d; a/b)$ does not exceed $d+1$; and in the case of the Whitehead link, the genus is equal or less than 2 (this confirms that the manifolds $M(p_1/r_1; a/b)$ are 2-fold branched coverings of the 3-sphere by Viro's theorem).

To prove that the presentations are geometric, we use a result of Osborne and Stevens on the representation of closed orientable 3-manifolds by *RR-systems* ([29], [30], [31], and [42]). For this, we first recall some basic definitions (for more details see the quoted papers). A *RR-system* (*rail-road system*) is a simple planar graph-like object defined as follows. Let D be a regular hexagon in Euclidean plane E^2. For each pair of opposite faces construct a finite set (possibly empty) of parallel line segments, called *tracks*, through D with endpoints on these opposite faces. These sets of parallel segments are called the *stations*. Let $\{D_i : i = 1, \ldots, n\}$ be a set of disjoint regular hexagons in E^2. A *route* is an arc whose interior lies in the complement of $\cup_i D_i$ in E^2, and connects endpoints of tracks. A *RR-system* is the union in $E^2 \subset \mathbb{S}^2 = E^2 + \infty$ (the 2-sphere) of a finite set of disjoint routes in the complement of $\cup_i D_i$ in $\mathbb{S}^2$ such that each endpoint of every track intersects exactly one route in one of its endpoints. A RR-system gives rise to a family of group presentations in the following way. The generators x_i, for $i = 1, \ldots, n$, are in one-to-one correspondence with the hexagons D_i, and hence D_i can be labelled by x_i. In each D_i we start at some vertex of ∂D_i and proceed clockwise (according to an orientation of $\mathbb{S}^2$) along an edge. This edge corresponds to a station p_i. Orient the tracks of this station so that the positive direction is toward the above edge. Label the stations corresponding to the second and third edges of D_i encountered by r_i and $p_i + r_i$, respectively, and orient the tracks of these stations toward the corresponding edges. Beginning at some point on some route we write a word on generators x_i as follows. As we enter in each hexagon D_i we give the label of the station as exponent of x_i with sign $+1$ (resp. -1) if our direction of travel concordes (resp. opposes) the orientation of the tracks. When we have completed our travels on routes, we obtain the relations of the group presentations induced by the RR-system.

The following is the fundamental result in the theory of RR-systems (see [30] and [31]).

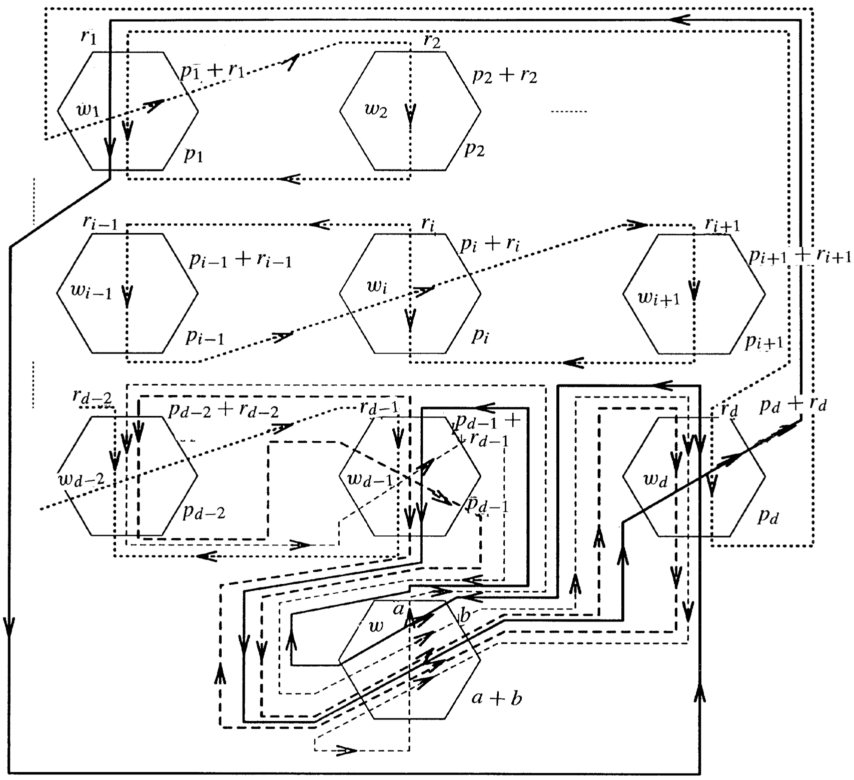


Fig. 5.2. An RR-system inducing a presentation of the fundamental group of $M(p_1/r_1, \ldots, p_d/r_d; a/b)$

Theorem 5.2. *Let $\mathcal{R}$ be a RR-system and let $\Phi_{\mathcal{R}}$ be a group presentation induced by $\mathcal{R}$. If $(p_i, r_i) = 1$ for any $i = 1, \ldots, n$, then $\Phi_{\mathcal{R}}$ corresponds to a spine of a closed orientable 3-manifold M – i.e. $M \setminus$ (open 3-cell) collapses onto the canonical cell complex of dimension 2 uniformized by $\Phi_{\mathcal{R}}$.*

Let us consider the RR-system depicted in Fig. 5.2. One can easily verify that it induces the group presentations of Theorem 5.1. So Theorem 5.2 directly implies the following result.

Theorem 5.3. *The group presentation of Theorem 5.1 corresponds to a spine of the closed 3-manifold $M = M(p_1/r_1, \ldots, p_d/r_d; a/b)$ (and it also arises from a Heegaard diagram of M).*

We remark that our link L_{d+1} is hyperbolic (see [1]) in the sense that it has hyperbolic complement. So the Thurston–Jorgensen theory [43] of hyperbolic surgery yields the following

Theorem 5.4. *For any integer $d \geq 1$, and for almost all pairs of surgery coefficients p_i/r_i ($i = 1, \ldots, d$) and a/b, the closed connected orientable* 3*-manifolds $M(p_1/r_1, \ldots, p_d/r_d; a/b)$ are hyperbolic.*

We now apply a theorem of Montesinos [27] to describe the closed manifolds $M(p_1/r_1, \ldots, p_d/r_d; a/b)$, where $a = \pm 1$ and $d \geq 2$, as 2-fold branched coverings of the 3-sphere, and to find the corresponding branch sets. It turns out that these manifolds for $a/b = 1$ and $d = 2$ are homeomorphic to the manifolds obtained by Dehn surgeries on the two components of the Whitehead link. So our result includes the main theorem of [25].

Recall that a link in the oriented 3-sphere $\mathbb{S}^3$ is *strongly invertible* if there exists an orientation-preserving involution of $\mathbb{S}^3$ which induces on each component of L an involution with exactly two fixed points. Such an involution is called a *strongly invertible involution* of L.

The following result, due to Montesinos [27], relates two different approaches for describing closed orientable 3-manifolds: Dehn surgery and branched coverings (see for example [2], [17], [18], [19], [28], [38], and [44]).

Theorem 5.5. *Let M be a closed orientable* 3*-manifold obtained by Dehn surgery on a strongly invertible link L of n components. Then M is a* 2*-fold covering of the* 3*-sphere branched over a link of at most $n+1$ components. Conversely, every* 2*-fold cyclic branched covering of the* 3*-sphere can be obtained in this fashion.*

The proof of Theorem 5.5, given in [27], is constructive, and gives an effective algorithm for describing the branch set of the considered 2-fold covering.

Let now consider the link $L_{d+1} = K_1 \cup \cdots \cup K_d \cup \Lambda$ with surgery coefficients $p_1/r_1, \ldots, p_d/r_d$, and a/b for $a = 1$, respectively. Twist the solid torus $\mathbb{S}^3 \setminus \operatorname{int} N$, where N is a tubular neighborhood of the transversal circle Λ. The meridian m of N is carried to $\tau\ell + m$ (ℓ being the longitude of N), where τ represents the number of twists which is positive (resp. negative) if the twist is in the right-hand (resp. left-hand) sense. Let $L'_{d+1} = K'_1 \cup \cdots \cup K'_d \cup \Lambda'$ be the link obtained from L_{d+1} by twisting around Λ. The following surgery coefficients on the components of L'_{d+1} do not change (up to homeomorphism) the represented 3-manifold according to the Kirby–Rolfsen calculus [17] [38]:

$$\text{(twisted component } \Lambda'\text{)}: \quad \frac{a'}{b'} = \frac{1}{\tau + b}$$

$$\text{(other components } K'_i\text{)}: \quad \frac{p'_i}{r'_i} = \frac{p_i}{r_i} + \tau[\mathrm{lk}(\Lambda, K_i)]^2 = \frac{p_i}{r_i},$$

since the linking number between the transversal circle Λ and each component K_i of the chain vanishes. These formulae are consistent with the convention $\pm 1/0 = \infty$, which corresponds to the trivial surgery. Thus the surgery manifold is unchanged by erasing all components with coefficient ∞. Setting $\tau := -b$, we can delete the component Λ' from L'_{d+1} obtaining the chain L''_d of d component illustrated in Fig. 5.3 (for convenience, we will assume b odd and positive; one can obtain the general case by slight modifications).

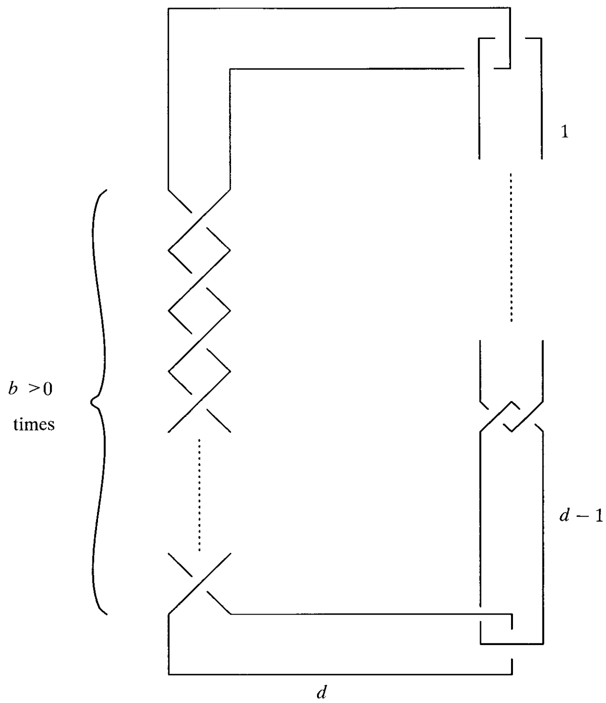


Fig. 5.3. The link L''_d in S^3

This chain is strongly invertible, and a strongly invertible involution ρ of L''_d is shown in Fig. 5.4 (top). We choose meridians μ_i and longitude λ_i according to Fig. 5.4b.

Let V be a regular neighborhood of L''_d in $\mathbb{S}^3$. Without loss of generality, we can choose neighborhood V, meridians μ_i, and longitudes λ_i on ∂V to be invariant under the involution ρ (see Fig. 5.4 (bottom)). The quotient space $\mathbb{S}^3/\rho$ of $\mathbb{S}^3$ under ρ is shown in Fig. 5.5.

The image of V under the canonical projection $\pi : \mathbb{S}^3 \to \mathbb{S}^3/\rho$ consists of d 3-balls B_i. Let θ denote the axis of the involution ρ in $\mathbb{S}^3$. For each 3-ball B_i, the set $B_i \cap \pi(\theta)$ consists of two arcs. By isotopy of B_i along the image $\pi(\lambda_i)$ of longitude λ_i for any $i = 1, \dots, d$, we get Fig. 5.6 (top). Each 3-ball B_i with arcs $B_i \cap \pi(\theta)$ is a trivial tangle in the sense of [8]. By the Montesinos algorithm for describing the branch link we need to replace these trivial tangles B_i by p_i/r_i-rational tangles for any $i = 1, \dots, d$ [8] [27] (see Fig. 5.6, bottom).

Summarizing we have proved the following

Theorem 5.6. *Let $M = M(p_1/r_1, \dots, p_d/r_d; a/b)$, $d \geq 2$, $a = \pm 1$, be the closed orientable 3-manifold obtained by Dehn surgery on the link L_{d+1} with surgery coefficients p_i/r_i and $\pm 1/b$. Then M is a 2-fold covering of the 3-sphere branched over the link pictured in Fig. 5.6 (bottom) (case $a = 1$, and b odd and positive).*

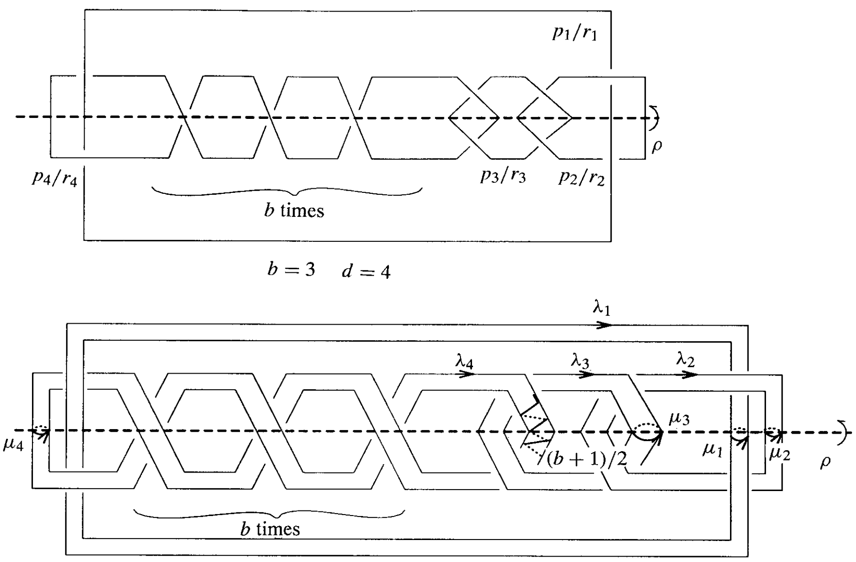


Fig. 5.4. A strongly invertible involution ρ of L''_d

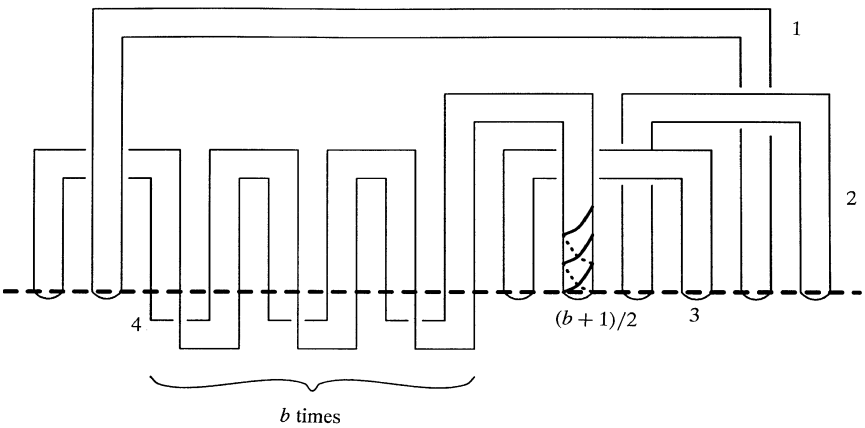


Fig. 5.5. The quotient space S^3/ρ

Appendix

As announced in the introduction, we treat here the cyclic branched coverings of the Whitehead link L_2 which are not included in the family studied above. For example, the referee pointed out the following case. Consider the monodromy $\omega : \pi_1(\mathbb{S}^3\backslash L_2) \to \Sigma_2$ defined by $\omega(\mathfrak{m}_0) = (1\,3\,5)(2\,4\,6)$ and $\omega(\mathfrak{m}_1) = (1\,2)(3\,4)$

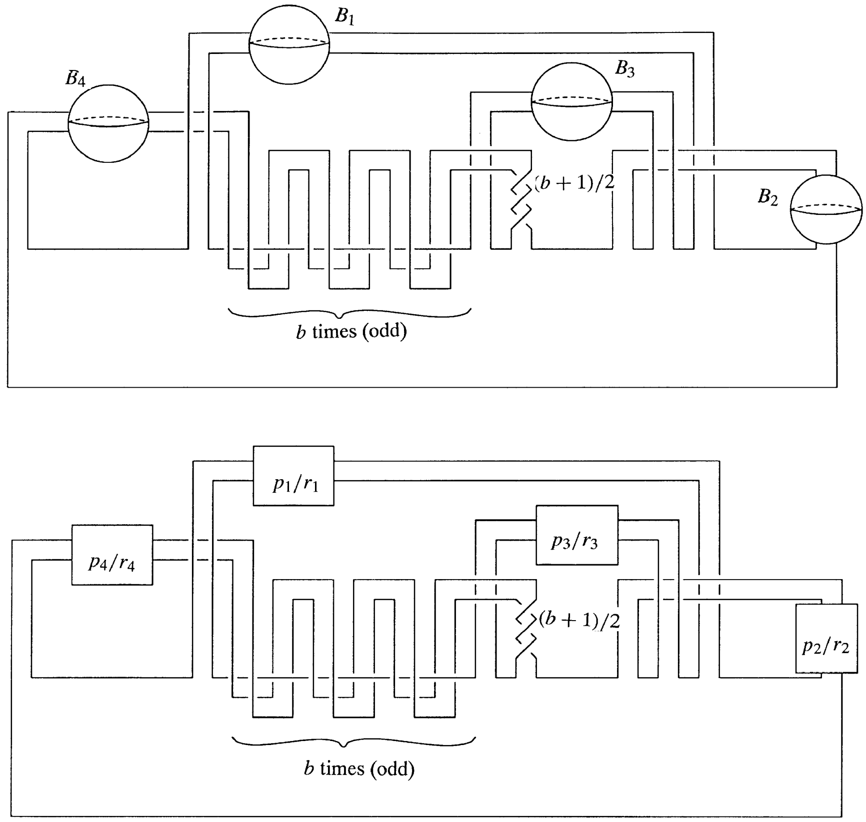


Fig. 5.6. The branched link obtained by the Montesinos algorithm

(5 6). It corresponds to a 6-fold cyclic covering of $\mathbb{S}^3$ branched over L_2, which is not included in the family $M(n,k)$. Let now $L = K_0 \cup K_1$ be an oriented link with two components in the right-hand oriented 3-sphere. As usual, we denote by $\mathcal{O}_{n_0,n_1}(L)$ the 3-orbifold whose underlying topological space is the 3-sphere and whose singular set is the link L with branching index n_i on K_i, for any $i = 0, 1$. Let $M(n, k_0, k_1)$ be the closed orientable 3-manifold defined as in Section 2, i.e., the n-fold cyclic branched covering (or covering in the sense of orbifolds) of $\mathcal{O}_{n_0,n_1}(L)$. It corresponds to the kernel of the map ψ_{n,k_0,k_1} from $\pi_1(\mathcal{O}_{n_0,n_1}(L))$ to $\mathbb{Z}_n$ which sends $\mathbf{m}_i$ to k_i, where $(n, k_i) = d_i = n/n_i$. Suppose now that n is the least common multiple of n_0 and n_1, and let m denote the greatest common divisor of n_0 and n_1 (hence we have of course $(n_0 n_1)/m = n$). The manifolds $M(n,k)$ are simply the manifolds $M(n, k_0, k_1)$, where $k_0 = 1$, $k_1 = k$, $n_0 = n$, $n_1 = n/d = m$, where $d = (n,k)$. Assume that both components of L are trivial knots (as for example in the case of the Whitehead link). Then the (n_0/m)-fold cyclic branched covering of K_0 is the 3-sphere again. Let $\mathrm{lk}(K_0, K_1)$ denote the linking number of K_0 and K_1.

If the greatest common divisor of $\mathrm{lk}(K_0, K_1)$ and n_0/m is r, then the preimage $\overline{K}_1$ of K_1 is a link with exactly r components. Similarly, we get a link $\overline{K}_0$ in $\mathbb{S}^3$ as the preimage of K_0 in the (n_1/m)-fold cyclic branched covering of K_1. The number of components of $\overline{K}_0$ equals the greatest common divisor of $\mathrm{lk}(K_0, K_1)$ and n_1/m. Of course, the links $\overline{K}_0$ and $\overline{K}_1$ have the same n_1-fold and n_0-fold cyclic branched covering, respectively, which is in fact the n-fold cyclic covering of $\mathbb{S}^3$ branched over the initial link L (since the components of L are equivalent). Suppose now that L is the Whitehead link L_2. Since the linking number of its components vanishes, the preimage $\overline{K}_1$ is a link with exactly n_0/m components; eachone of them has branching index n_1. Indeed, $\overline{K}_1$ is the chain with n_0/m rings with alternating crossings. Thus the preimage $\overline{L}_2$ of L_2 is a link with $(n_0/m) + 1$ components. It is formed by the chain $\overline{K}_1$ plus an extra unknotted component $\overline{K}_0$ which is the axis of n_0/m symmetry of the chain. According to notation in Section 2, $\overline{L}_2$ is in fact the link $L_{(n_0/m)+1}$. Let $\mathcal{O}_{m,n_1}(\overline{L}_2)$ be the orbifold whose underlying space is $\mathbb{S}^3$ and whose singular set is $\overline{L}_2$, where the extra unknotted component has branching index m, and any other component has branching index n_1. The n-fold cyclic branched coverings of L_2 are actually the n_1-fold cyclic coverings of $\mathbb{S}^3$ branched along the components of $\overline{L}_2$. But the order of these coverings equals the branching index of at least one component of the branch set $\overline{L}_2$. If the orbifold is hyperbolic, then we can apply Theorem 2.2 to classify the homeomorphism (isometry) type of such manifolds (otherwise, if n_0 or n_1 equals 2, then the considered manifold will be fibered, as discussed in Section 2). One has to compute the liftings in $\pi_1(\mathcal{O}_{m,n_1}(\overline{L}_2))$ of the meridians of the components of L_2, and their images in the cyclic group of order n. Let us consider now an n'-fold cyclic branched covering of $\mathcal{O}_{n_0,n_1}(L_2)$, where n' is a multiple of $n = l.c.m.(n_0, n_1)$. It suffices to note that the above manifold is an (n'/n)-fold unbranched cyclic covering of a manifold of type $M(n, k_0, k_1)$, for some k_0 and k_1. Hence, the classification of its homeomorphism (isometry) type follows from Theorem 2.2 again. The description of these branched coverings as polyhedral 3-balls, whose finitely many boundary faces are glued together in pairs, may be quite different from that given for $M(n, k)$. However, the manifolds $M(n, k_0, k_1)$ can be constructed in a geometric way as follows. Choose a Seifert surface for each component of L (the Seifert surface for one component ignores the other component– the two surfaces will intersect transversely). The manifold $M(n, k_0, k_1)$ can then be constructed explicitly by taking n copies of the complement in $\mathbb{S}^3$ of the union of the Seifert surfaces, numbering them $0, \ldots, n-1$, and gluing them together according to the following rule. When you pass through the first (resp. second) Seifert surface in the "positive" direction you pass from copy labelled i to that labelled $i + k_0$ (resp. $i + k_1$) modulo n.

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