

THREE CYCLIC BRANCHED COVERS SUFFICE TO DETERMINE HYPERBOLIC KNOTS

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Accepted 20 July 2004

ABSTRACT

Let $n > m > 2$ be two fixed coprime integers. We prove that two Conway reducible, hyperbolic knots sharing the 2-fold, m -fold and n -fold cyclic branched covers are equivalent. Using previous results by Zimmermann we prove that this implies that a hyperbolic knot is determined by any three of its cyclic branched covers.

Keywords: Hyperbolic knots; cyclic branched covers; orbifolds; Bonahon–Siebenmann decomposition.

Mathematics Subject Classification 2000: Primary 57M25; Secondary 57M12, 57M50

1. Introduction

In this paper we address the following question: Which is the minimal number of cyclic branched covers needed to determine a hyperbolic knot?

We start by giving some definitions to make the meaning of this question more precise. Let K be a knot in \mathbf{S}^3 and denote by $M(n, K)$, $n \geq 2$ the (total space of the) n -fold cyclic cover of \mathbf{S}^3 branched along K . We shall say that a finite set of covers $\{M(n_1, K), \dots, M(n_q, K)\}$ determines K if, whenever a knot K' has the property that $M(n_i, K')$ is homeomorphic to $M(n_i, K)$ for all $i = 1, \dots, q$, we have that K and K' are equivalent, i.e. the pairs (\mathbf{S}^3, K) and (\mathbf{S}^3, K') are homeomorphic.

The above question can thus be restated as follows: Let K be a hyperbolic knot. Which is the minimum q (independent of K) such that $\{M(n_1, K), \dots, M(n_q, K)\}$ determines K for all choices of pairwise distinct $n_1, \dots, n_q \geq 2$? Recall that a knot is hyperbolic if its exterior in \mathbf{S}^3 admits a complete hyperbolic structure of finite volume.

It is well-known that the minimum q must be at least 3. Examples of non-equivalent hyperbolic knots sharing two cyclic branched covers are given in [19, 14]. On the other hand, Zimmermann showed [20, Theorem 3] that the set $\{M(n_1, K), M(n_2, K)\}$ determines a hyperbolic knot K if n_1 and n_2 are not coprime

and K is $2\pi/n_i$ -hyperbolic, $i = 1, 2$. Using similar methods, we shall discuss in Sec. 2 sufficient conditions for a hyperbolic knot K to be determined by the set $\{M(n_1, K), M(n_2, K), M(n_3, K)\}$, where $n_1 > n_2 > n_3 \geq 2$. The aforementioned results can be summarised as follows: Let $n_1 > n_2 > n_3 \geq 2$. The set $\{M(n_1, K), M(n_2, K), M(n_3, K)\}$ determines a hyperbolic knot K if at least one of these three conditions is satisfied:

- n_1 and n_2 are not coprime;
- $n_3 > 2$;
- K is Conway irreducible.

Recall that a knot K is Conway irreducible if it does not admit any Conway sphere, i.e. a sphere S^2 which intersects K in four points such that $S^2 \setminus \mathcal{U}(K)$ is incompressible and ∂ -incompressible in $\mathbf{S}^3 \setminus \mathcal{U}(K)$, where $\mathcal{U}(K)$ denotes a regular neighbourhood of K in \mathbf{S}^3 .

Even if these results suggest that the minimum q should be 3, it ought to be stressed that a (Conway reducible) hyperbolic knot is highly non-determined by its 2-fold cyclic branched cover and there is a certain “freedom” in constructing new hyperbolic knots sharing the same 2-fold cyclic branched cover of a given one (see [13], and [11, 15] for the π -hyperbolic case). However we shall prove the following

Theorem 1.1. *Let $n_1 > n_2 > n_3 \geq 2$ be three integers and K and K' be two hyperbolic knots. If $M(n_i, K)$ is homeomorphic to $M(n_i, K')$ for $i = 1, 2, 3$, then K and K' are equivalent.*

No examples of non-equivalent prime knots sharing three different covers are known so far, thus it is natural to ask whether the conclusion of Theorem 1.1 holds for arbitrary prime knots. Notice, on the other hand, that it is possible to construct non equivalent composite knots such that $M(n, K) = M(n, K')$ for all integers n .

By the above discussion, it suffices to prove Theorem 1.1 under the following assumptions:

- n_1 and n_2 are coprime;
- $n_3 = 2$;
- K (and thus K') is Conway reducible.

We can furthermore assume that no pair of covers $\{M(n_i, K), M(n_j, K)\}$, where $\{i, j\} \subset \{1, 2, 3\}$, determines K .

The first step of the proof (Sec. 3.3) consists in discussing necessary conditions for a Conway reducible hyperbolic knot to fail to be determined by its 2-fold, n_1 -fold and n_2 -fold cyclic branched covers, $n_1 > n_2 > 2$. Such conditions will play a substantial role in the proof of Theorem 1.1, which will be given in Sec. 3. The idea of the proof is to show that the knots K and K' are highly symmetric and that the existence of symmetries forces (\mathbf{S}^3, K) and (\mathbf{S}^3, K') to be homeomorphic.

The reader is referred to [16, 1] for basic results and definitions in knot theory and hyperbolic geometry respectively.

2. Covers of Large Order and Conway Irreducible Knots

In this section we shall discuss two sufficient conditions under which a hyperbolic knot is determined by three of its cyclic branched covers. We start by proving two useful facts, the first of which was originally observed by Hillman [7].

Claim 2.1. *Let K be the trivial knot and G a finite (abelian) group of symmetries of K which preserve a fixed orientation on K . All non-trivial periodic symmetries of G share the same axis A , and K and A form a Hopf link.*

Remark 2.2. The expression *symmetry of a knot (link) K* stands for finite order diffeomorphism of the pair (\mathbf{S}^3, K) , preserving the orientation of \mathbf{S}^3 . A symmetry of order n is *n-periodic* if it fixes setwise each component of K and its fixed-point set is non-empty and does not intersect K . A symmetry of a knot K is a *strong inversion* if it has order 2, reverses the orientation of K , and its fixed-point set is non-empty and intersects K in exactly two points.

Proof of Claim 2.1. Since K is fibred, there exists a G -equivariant fibration for K whose fibres are discs [6, Theorem 5.2]. Let $h \in G$ be an n -periodic symmetry for K . Since $\text{Fix}(h) \neq \emptyset$, h must fix each disc of the fibration and its axis must intersect each disc in precisely one point, i.e. $\text{Fix}(h)$ is the core of the solid torus $\mathbf{S}^3 - \mathcal{U}(K)$ and K and $\text{Fix}(h)$ form a Hopf link. Let h and h' be two periodic symmetries of K . Then, since h and h' commute, $\text{Fix}(h)$ is left invariant by h' . In particular, the unique intersection point of $\text{Fix}(h)$ with a disc of the fibration must be fixed by h' , and the conclusion follows. \square

Claim 2.3. *Let K be a Conway irreducible hyperbolic knot which is not determined by the set $\{M(2, K), M(n, K)\}$, where $n > 2$. Then n is odd, K is π -hyperbolic and admits a 2-periodic symmetry with trivial quotient, i.e. the quotient of the knot by the action of the symmetry is the trivial knot.*

Proof. According to Thurston's orbifold geometrisation theorem (see [2, 4] for a proof), the cyclic branched covers of a Conway irreducible hyperbolic knot K are geometric and the covering transformations preserve the geometric structure. There are two possible cases: either $M(2, K)$ is an atoroidal Seifert manifold and K is a 2-bridge knot or a Montesinos knot with at most three rational tangles, or $M(2, K)$ is hyperbolic and K is a π -hyperbolic knot by definition.

(a) K is π -hyperbolic.

Hodgson and Rubinstein [8] proved that 2-bridge knots are determined by their 2-fold cyclic branched covers. If K is a Montesinos knot and $M(2, K)$ does not

determine K , then K cannot be the figure-eight knot and $M(2, K)$ must be the 2-fold cyclic branched cover of a torus knot K' . However $M(n, K')$ is a Seifert fibred manifold while $M(n, K)$ is hyperbolic because of Thurston's orbifold geometrisation theorem and Dunbar's list [5] of non-hyperbolic orbifolds.

(b) n is odd.

If K is π -hyperbolic and is not determined by the set $\{M(2, K), M(n, K)\}$ where $n > 2$, it was proved by Zimmermann that n is necessarily odd [20, Theorem 3] and that K admits an n -periodic symmetry with trivial quotient [20, Corollary 1].

(c) K admits a 2-periodic symmetry with trivial quotient.

This part of the proof is less straightforward. One needs to understand the structure of $\text{Iso}(M(2, K))$. Assume that K and K' are two non-equivalent π -hyperbolic knots having the same 2-fold and n -fold cyclic branched covers, n odd. Let h respectively h' be lifts of the n -periodic symmetries of K respectively K' to the common 2-fold cyclic branched cover M . Because of Thurston's orbifold geometrisation theorem, h and h' are isometries of the hyperbolic manifold M with non-empty fixed-point set consisting of one or two components.

We want to show that h and h' are conjugate in $\text{Iso}(M)$. The reasoning will follow the lines of [19, 20]. In particular we shall often exploit the following simple fact: any finite group of isometries which leaves invariant a simple closed geodesic is a finite subgroup of $\mathbb{Z}_2 \times (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$ in which the generator of \mathbb{Z}_2 sends each element of the product $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ to its inverse.

(d) *If the groups generated by the lifts of the n -periodic symmetries of K and K' to M are not conjugate in $\text{Iso}(M)$, then the fixed-point set of the lift of the n -periodic symmetry of K is connected.*

By hypothesis, h and h' are not conjugate in $\text{Iso}(M)$. Choose any maximal (necessarily odd) prime power divisor of n -say $q > 1$ - and let H respectively H' be the cyclic subgroups of $\langle h \rangle$ respectively $\langle h' \rangle$ of order q . H and H' cannot be conjugate, else the element conjugating them would map $\text{Fix}(h)$ to $\text{Fix}(h')$ and conjugate $\langle h \rangle$ to $\langle h' \rangle$ against the hypothesis. Thus [17, Chap. 2, 1.5] assures that a group H'' , conjugate to either H or H' , lies in the normaliser of H but is different from H . Note that this implies that $H \cap H'' = \{0\}$ else H and H'' would both be groups of rotations of order q about the same axis and they would coincide. Since q is odd, the elements of H'' must leave invariant each connected component of $\text{Fix}(h)$ (which are at most two) and the elements of H commute with the elements of H'' . One deduces that h commutes with all rotations around $\text{Fix}(H'')$ and we can find a cyclic subgroup of order n , $\langle h'' \rangle$, conjugate to either $\langle h \rangle$ or $\langle h' \rangle$, which commutes with $\langle h \rangle$. Since h'' is conjugate to either h or h' , the quotient of M by the action of h'' is \mathbb{S}^3 and $\text{Fix}(h'')$ projects to a link. Such link admits a periodic symmetry induced by h , hence the projection of $\text{Fix}(h)$ must be connected and since n is odd

and the number of connected components of $\text{Fix}(h)$ is at most two, we deduce that $\text{Fix}(h)$ is indeed connected.

- (e) *The groups generated by the lifts of the n -periodic symmetry of K and K' to $M(2, K) = M(2, K')$ are conjugate in $\text{Iso}(M(2, K))$.*

Consider now the covering involution for K , τ . Both τ and h'' commute with h , thus τ and h'' commute, for $\text{Fix}(h)$ is connected. This means that h and h'' induce two n -periodic symmetries of K which is absurd because of Smith's conjecture.

- (f) *K admits a 2-periodic symmetry.*

Up to conjugation, we can thus assume that $h = h'$. Consider now τ' , the covering involution for K' . The group generated by τ , τ' and h in $\text{Iso}(M)$ is of the form $\mathbb{Z}_n \oplus \mathbf{D}_{2t}$. Note that the maximal cyclic group generated by $\tau\tau'$ must have even order $2t$, else τ and τ' would be conjugate and the knots K and K' would be equivalent. The element $(\tau\tau')^t\tau$ commutes with τ and is conjugate to either τ or τ' according to the parity of t . In particular, $\text{Fix}((\tau\tau')^t\tau)$ is non-empty, and $(\tau\tau')^t\tau$ induces a symmetry of K with non-empty fixed-point set. Such symmetry cannot be a strong inversion, for $(\tau\tau')^t\tau$ commutes with the n -periodic symmetry induced by h , and so must be a 2-periodic symmetry. To see that such 2-periodic symmetry has trivial quotient, reason as in [20, Corollary 1]. □

Proposition 2.4. *Let $n_1 > n_2 > n_3 \geq 2$ be three integers and K and K' be two hyperbolic knots having the same n_i -fold cyclic branched covers, $i = 1, 2, 3$. If one of the following conditions is satisfied, then K and K' are equivalent:*

- (i) $n_3 \geq 3$;
- (ii) K is Conway irreducible.

Proof. Assume, by contradiction, that K and K' are not equivalent. If K is not determined by its cyclic branched cover of order $n_i > 2$, it was proved in [20, Corollary 1] that it admits an n_i -periodic symmetry with trivial quotient. If K is Conway irreducible and is not determined by its n_i -fold, $n_i > 2$, and 2-fold cyclic branched covers, K admits a 2-periodic symmetry with trivial quotient, according to Claim 2.3. So, in all cases, K admits three distinct periodic symmetries with trivial quotients.

Let h , h' and h'' be the three distinct periodic symmetries of K with trivial quotients. We distinguish two possible cases:

- (a) $\text{Fix}(h) = \text{Fix}(h')$ and the order of h is smaller than the order of h' .

Consider the link $p_h(K \cup \text{Fix}(h))$. Such link is hyperbolic, since so is K and has two trivial components. The periodic symmetry h' induces a periodic symmetry of the trivial knot $p_h(K)$ with axis $p_h(\text{Fix}(h))$. By Claim 2.1, $p_h(K \cup \text{Fix}(h))$ is the Hopf link and we get a contradiction.

- (b) *The three fixed-point sets for h , h' and h'' are all distinct. Notice that the Smith conjecture implies that the orders of two periodic symmetries of a hyperbolic knot with disjoint axes must be coprime.*

Since h , h' and h'' commute, h' and h'' induce periodic symmetries of the trivial knot $p_h(K)$ with distinct axes, and we reach again a contradiction to Claim 2.1 which ends the proof of Proposition 2.4. □

The proof of Proposition 2.4 gives:

Scholium 2.5. *Let K be knot admitting three periodic symmetries h , h' and h'' which generate a finite cyclic group. If the two symmetries of smaller orders have trivial quotients, then K is the trivial knot.*

Proof. It is enough to note that, in the proof of Proposition 2.4, hyperbolicity of K is only needed to assure that the group generated by the three symmetries is finite. □

3. Proof of Theorem 1.1

To avoid cumbersome notation, from now on we shall write n instead of n_1 and m instead of n_2 . Let us start by recalling what can be deduced about K and K' under the assumptions we made in Sec. 1.

3.1. Existence of common quotient links for K and K' and their properties

It was proved in [19, Theorem 1] that two non-equivalent hyperbolic knots K and K' have the same n -fold cyclic branched cover, $n \geq 3$, if and only if there exists a hyperbolic link, $\bar{K} \cup \bar{K}'$, with two components which are trivial and non-exchangeable, such that K (respectively K') is the lift of \bar{K} (respectively \bar{K}') to the n -fold cyclic cover of S^3 branched along \bar{K}' (respectively \bar{K}). In particular, K and K' admit an n -periodic symmetry. Notice that [19, Theorem 1] is stated only in the case when n is not a power of 2, however the same techniques can be extended to prove the theorem for all $n > 2$. For the interested reader and for the sake of completeness, we shall give a proof of this fact in Sec. 4.

The analogue holds for the m -fold cyclic branched cover, giving another common quotient link that we shall denote $\hat{K} \cup \hat{K}'$. The n -periodic (respectively m -periodic) symmetries of K and K' induce n -periodic (respectively m -periodic) symmetries of $\hat{K} \cup \hat{K}'$ (respectively $\bar{K} \cup \bar{K}'$). By quotienting further, we obtain a hyperbolic link with three trivial components $A \cup B \cup C$, admitting a symmetry σ of order a power of 3 cyclically exchanging its components. This follows from the fact that K is mapped to $A \cup B \cup C$ in two different ways (via \bar{K} and \hat{K}) if we consider $A \cup B \cup C$ as a quotient of K' , and can be easily seen by considering the following

commuting diagrams of orbifold covers (here the second component of a pair is the singular set of the orbifold and the indices stand for the orders of ramification).

$$\begin{array}{ccc}
 M & & \\
 \downarrow & & \\
 (\mathbf{S}^3, K_2) & \longrightarrow & (\mathbf{S}^3, \bar{K}_2 \cup \bar{K}'_n) \\
 \downarrow & & \downarrow \\
 (\mathbf{S}^3, \hat{K}_2 \cup \hat{K}'_m) & \longrightarrow & (\mathbf{S}^3, A_2 \cup B_n \cup C_m) \\
 & & \\
 & & (\mathbf{S}^3, \bar{K}_n \cup \bar{K}'_2) \longrightarrow (\mathbf{S}^3, A_n \cup B_2 \cup C_m) \\
 & \uparrow & \\
 M \longrightarrow & (\mathbf{S}^3, K'_2) & \parallel \\
 & \downarrow & \\
 & (\mathbf{S}^3, \hat{K}'_m \cup \hat{K}'_2) & \longrightarrow (\mathbf{S}^3, A_m \cup B_n \cup C_2)
 \end{array}$$

Since the two orbifolds at the far right are the same, we conclude that there must exist a symmetry of the link $A \cup B \cup C$ sending (A, B, C) to (B, C, A) . We can thus write:

$$\begin{array}{ccc}
 M & & \\
 \downarrow & & \\
 (\mathbf{S}^3, K'_2) & \longrightarrow & (\mathbf{S}^3, \bar{K}_n \cup \bar{K}'_2) \\
 \downarrow & & \downarrow \\
 (\mathbf{S}^3, \hat{K}'_m \cup \hat{K}'_2) & \longrightarrow & (\mathbf{S}^3, A_2 \cup B_m \cup C_n)
 \end{array}$$

Finally, we shall need the following relation

$$\text{lk}(A, B) = \text{lk}(B, C) = \text{lk}(C, A) = 1$$

which is a consequence of the fact that every two component sublink of $A \cup B \cup C$ is a Hopf link. Notice that this is equivalent to show that the link formed by \bar{K} and the fixed-point set of the m -periodic symmetry of $\bar{K} \cup \bar{K}'$ is a Hopf link. This last property follows from Claim 2.1.

3.2. Existence of a special hyperbolic piece N in the Jaco–Shalen–Johannson decomposition of $M = M(2, K) = M(2, K')$

Let M be the common 2-fold branched cover of K and K' . If the Jaco–Shalen–Johannson decomposition [9, 10] of M is trivial, then M is Seifert fibred and K

and K' are Montesinos knots. It is not difficult to prove that, in this case, K is determined by the set $\{M(2, K), M(n, K)\}$ for any fixed $n > 2$ (see [14, Corollary 1]).

Assume now that the Jaco–Shalen–Johannson decomposition of M is non-trivial. All the incompressible tori of the decomposition must project onto Conway spheres intersecting K which must be freely permuted by the n -periodic symmetry. This implies that the lift to M of the n -periodic symmetry of K must freely permute all geometric pieces of the decomposition, except the one which contains its fixed-point set and which is setwise fixed. Standard theory of finite actions on trees implies that the fixed-point sets of the lifts to M of the periodic symmetries of K and K' are contained in the same geometric piece N . Consider the common quotient link $\bar{K} \cup \bar{K}'$: \bar{K}' is the quotient of K' and lifts to the axis of the n -periodic symmetry of K . The Conway spheres along K' lift to closed incompressible surfaces of negative Euler characteristic contained in N , implying that N is hyperbolic. A more detailed analysis of this fact can be found in [14, Claim 5].

Notice that the lifts of the n -periodic (respectively m -periodic) symmetries of K and K' can be chosen to have order n (respectively m). This follows from the fact that these lifts can be seen as covering transformation for links in \mathbf{S}^3 obtained as lifts of one component of $\bar{K} \cup \bar{K}'$ (or of $\hat{K} \cup \hat{K}'$) in the 2-fold cover of \mathbf{S}^3 branched along the other component. Moreover the lifts of the n -periodic and m -periodic symmetries of K (respectively K') commute on M .

3.3. The action of the lifts of the symmetries on the hyperbolic piece N

Let G denote the group of isometries of N which are induced by diffeomorphisms of M preserving N . Note that the covering transformations τ and τ' for K and K' as well as the lifts h, h' and g, g' of the n -periodic and m -periodic symmetries for K and K' induce elements of G which we shall again denote by τ, τ', h, h', g and g' . Note that the fixed-point sets of h, h', g and g' , which are contained in N , consist of either one or two componets, since τ and τ' have order 2. Moreover, the number of components is the same for h and h' (respectively g and g') and depends only on the linking number of $\bar{K} \cup \bar{K}'$ (respectively $\hat{K} \cup \hat{K}'$). Note that, without loss of generality, we can assume that the order of h and h' is odd, for n and m are coprime. We distinguish several cases, according to the behaviour of h and h' .

3.3.1. The cyclic groups generated by h and h' must be conjugate in G

Recall that any finite group of isometries which leaves invariant a simple closed geodesic is a finite subgroup of $\mathbb{Z}_2 \times (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$ in which the generator of \mathbb{Z}_2 sends each element of the product $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ to its inverse.

Assume, by contradiction that the groups generated by h and h' are not conjugate in G . Let $q = p^a$ be any fixed maximal (odd) prime power dividing the order

of h . The cyclic subgroups H and H' of order q in $\langle h \rangle$ and in $\langle h' \rangle$ cannot be conjugate. Indeed, the element conjugating them would send the fixed-point set of H to the fixed-point set of H' . Since these are the fixed-point sets of the groups $\langle h \rangle$ and $\langle h' \rangle$ the given element would conjugate $\langle h \rangle$ and $\langle h' \rangle$ againsts the hypothesis.

We can thus assume that the p -Sylow subgroup of G has order strictly larger than q . Applying [17, Chap. 2, 1.5], we find a cyclic subgroup of order q , \hat{H} , which normalises H but is distinct from it. Notice that such subgroup is either H' or is conjugate to H . Note that for \hat{H} to normalise H , it must leave setwise invariant each component of the fixed-point set of H . From this, one deduces that the elements of H (and thus of $\langle h \rangle$) commute with those of \hat{H} . Moreover, since H and \hat{H} are distinct, their fixed-point sets are disjoint and $H \cap \hat{H} = \{1\}$. Consider the quotient $N/\langle h \rangle$: it admits a group of diffeomorphisms cyclic of order q induced by \hat{H} . All elements of such group fix pointwise one or two circles in $N/\langle h \rangle$ and the number of circles is the same as the number of connected components of $\text{Fix}(h)$ and $\text{Fix}(h')$. Consider the action of the elements of \hat{H} on M : they fix setwise N and freely permute the connected components of $M \setminus N$ which are knot complements. In particular they must preserve longitude-meridian systems on the boundary components of $M \setminus N$. This implies that one can preform Dehn surgery on the boundary of $N/\langle h \rangle$ in such a way that the resulting manifold is the 3-sphere and that the diffeomorphisms induced by \hat{H} extend to \mathbf{S}^3 . Since fixed-point sets of diffeomorphisms of \mathbf{S}^3 are connected, we deduce that so are $\text{Fix}(h)$ and $\text{Fix}(h')$.

Consider now the element τ : it commutes with h by construction. Since the fixed-point set of h is connected, τ must commute with the elements of \hat{H} . In particular, \hat{H} must freely permute the H -orbits of connected components of ∂N .

By performing again \hat{H} -, H -equivariant hyperbolic Dehn surgery on N , one can construct two distinct q -periodic symmetries (induced by \hat{H} and H) for the hyperbolic knot which is the image of $\text{Fix}(\tau)$ in $\mathbf{S}^3 = \tilde{N}/\langle \tau \rangle$, which is absurd, (here \tilde{N} denotes the manifold obtained by Dehn surgery on N). Remark that we can choose the surgery in such a way that the image of $\text{Fix}(\tau)$ in $\mathbf{S}^3 = \tilde{N}/\langle \tau \rangle$ is connected and note that any surgery is τ -equivariant.

The above discussion shows that the groups $\langle h \rangle$ and $\langle h' \rangle$ must be conjugate, so that we can assume $h = h'$.

3.3.2. The groups generated by g and g' must coincide

Since $h = h'$ we see that both g and g' commute with h . Since the number of connected components of $\text{Fix}(h)$ is one or two and since the order of g and g' is strictly larger than 2, up to taking a power, we can assume that both g and g' leave setwise invariant each connected component of $\text{Fix}(h)$. This implies that (some non-trivial powers of) g and g' commute. Moreover, since g induces a symmetry with non-empty fixed-point set of $\mathbf{S}^3 = M/\langle h \rangle$ and the order of h is odd, we see that $\text{Fix}(g)$ must be connected. Reasoning as in the previous case, we see that both (non-trivial powers of) g and g' commute with τ . By performing again equivariant

hyperbolic Dehn surgery, we get that g and g' induce periodic symmetries of the same order of a hyperbolic knot, which contradicts Smith's conjecture.

The above discussion shows that the subgroup of G generated by $\tau, \tau', h = h', g = g'$ is of the form $\mathbb{Z}_n \oplus \mathbb{Z}_m \oplus \mathbf{D}_t$, where \mathbf{D}_t denotes the dihedral group of order $2t$.

3.3.3. t cannot be even

Assume, by contradiction, that t is even. Under this hypothesis, the normaliser of τ contains a group of the form $\mathbb{Z}_n \oplus \mathbb{Z}_m \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle \tau, (\tau\tau')^{t/2}, h, g \rangle$ which implies that both n and m are odd. We want to see that the fixed-point sets of h and g are connected. This is proved as in 3.3.2, using the fact that g (respectively h) induces a finite order diffeomorphism with non-empty fixed-point set of $\mathbf{S}^3 = M/\langle h \rangle$ (respectively $\mathbf{S}^3 = M/\langle g \rangle$). It is now easy to see that one can perform hyperbolic Dehn surgery on $N/\langle h \rangle$ which is equivariant by the action of the elements induced by g, τ and τ' (note that the element induced by g must freely permute the boundary components of $N/\langle h \rangle$) and in such a way that the resulting manifold is \mathbf{S}^3 , thus we see that the group $\mathbb{Z}_m \oplus \mathbf{D}_t, t \geq 2$ even, is contained in the group of symmetries of the hyperbolic knot which is the image of $\text{Fix}(h)$ in $\mathbf{S}^3 = \tilde{N}/\langle h \rangle$. The remark at the beginning of 3.3.1 implies that $t = 2$ and the involutions induced by τ and τ' are 2-periodic symmetries. This is however absurd because of Smith's conjecture (see Remark 3.1).

The above discussion shows that t must be odd. In this case τ and τ' are conjugate, so that, after a change of generators, we can assume that $\tau = \tau'$ on N .

3.4. The lifts to M of the n -periodic (respectively m -periodic) symmetries of K and K' can be chosen to coincide on M

This is in fact one of the crucial points of the proof. Let τ (respectively τ') be the covering involution for K (respectively K') and h, g (respectively h', g') the lifts of the n -periodic and m -periodic symmetries of K (respectively K'). Let $\{M_j\}_{j \in J}$ be the finite collection of connected components of $M \setminus N$. Notice that, since the characteristic graph of the Jaco–Shalen–Johannson decomposition is a tree, the indices can be chosen to vary over the boundary components of N . It was observed in [14] that the lifts of the periodic symmetries of the two knots must act freely on $\{M_j\}_{j \in J}$, moreover, since n and m are coprime and h and g commute, h must act freely on the orbits of g and vice versa, and the same holds for h' and g' . Observe that h and h' (respectively g and g') act in the same way on the set $\{M_j\}_{j \in J}$, for they coincide on N . On the other hand τ and τ' fix M_j for all j 's.

Let us start by showing that, up to conjugation, τ can be chosen to commute with h' and g' . Let M_{j_0} be a fixed connected component and $M_{j_1} = h(M_{j_0}) = h'(M_{j_0})$. Let $\tau_{j_0} := \tau|_{M_{j_0}}$ and $\tau_{j_1} := \tau|_{M_{j_1}}$. We know that $\tau_{j_1} = h\tau_{j_0}h^{-1}$, for h and τ commute, so that τ_{j_1} and $h'\tau_{j_0}h'^{-1}$ are conjugate on M_{j_1} . Choose now, from

each orbit of the action of the group $\langle h', g' \rangle$ on $\{M_j\}_{j \in J}$, a representative, i.e. a connected component M_{j_i} , $i = 1, \dots, k$. Define, for each $j \in J$, an involution u_j of M_j in the following way: if $j \in \{j_1, \dots, j_k\}$ then $u_j := \tau|_{M_j}$, else there exist a unique $j_s \in \{j_1, \dots, j_k\}$ and unique exponents a, b such that $M_j = g'^a h'^b(M_{j_s})$ (this is well defined since h' and g' commute and their action is free on $\{M_j\}_{j \in J}$). In this case let $u_j := g'^a h'^b \tau|_{M_{j_s}} (g'^a h'^b)^{-1}$. As in [13, Proposition 2.2], we can find an involution u of M , conjugate to τ , which coincides with u_j on M_j for all $j \in J$ and with $\tau|_N = \tau'_{|N}$ on N . By construction, u commutes with h' and g' .

If h' and g' commute with τ they induce an n -periodic and an m -periodic symmetry of K . However, since K is hyperbolic, the Smith conjecture [12] implies that its periodic symmetries are unique, thus showing that h' and h (respectively g' and g) are lifts of the same periodic symmetry.

3.5. Let $\tilde{B} \cup \tilde{C}$ be the lift of $B \cup C$ to the 2-fold cover of \mathbf{S}^3 branched along A . There exists a map $\tilde{f} : (\mathbf{S}^3, \tilde{B} \cup \tilde{C}) \rightarrow (\mathbf{S}^3, \tilde{B} \cup \tilde{C})$ which exchanges \tilde{B} and \tilde{C}

We have two orbifold covers:

$$\begin{aligned} (\mathbf{S}^3, \tilde{B}_n \cup \tilde{C}_m) &\rightarrow (\mathbf{S}^3, A_2 \cup B_n \cup C_m) \\ (\mathbf{S}^3, \tilde{B}_m \cup \tilde{C}_n) &\rightarrow (\mathbf{S}^3, A_2 \cup B_m \cup C_n) \end{aligned}$$

However, the two orbifolds $(\mathbf{S}^3, \tilde{B}_n \cup \tilde{C}_m)$ and $(\mathbf{S}^3, \tilde{B}_m \cup \tilde{C}_n)$ are the same, because they are the orbifold obtained by quotienting M via the action of $\langle h, g \rangle$. This means that there must exist a symmetry of $\tilde{B} \cup \tilde{C}$ exchanging its two components.

We want to show that \tilde{f} induces a symmetry of $A \cup B \cup C$ which fixes A and exchanges B and C .

Remark 3.1. Smith’s conjecture and Mostow’s rigidity assure uniqueness of periodic symmetries of a two component hyperbolic link (compare Sec. 3.3.2). On its turn, uniqueness implies that the restriction of \tilde{f} to $N/\langle h, g \rangle$ must commute with the restriction of the involution induced by $\tau = \tau'$ on $N/\langle h, g \rangle$. Indeed, up to hyperbolic Dehn surgery, $N/\langle h, g \rangle$ embeds in \mathbf{S}^3 in such a way that $\tilde{B} \cup \tilde{C}$ is a hyperbolic link and the restrictions of both \tilde{f} and $\tau = \tau'$ extend to \mathbf{S}^3 , the latter as a 2-periodic symmetry of $\tilde{B} \cup \tilde{C}$. Thus \tilde{f} induces a symmetry f of $N/\langle h, g, \tau \rangle$.

We need to understand the behaviour of a special family of Conway spheres for $A \cup B \cup C$ under the action of a symmetry. Let us start with a definition. Let $L = L_1 \cup \dots \cup L_r$ be an r component hyperbolic link. Denote by $\mathcal{O}_i(p_1, \dots, p_r)$ the orbifold $(\mathbf{S}^3, L_{1p_1} \cup \dots \cup L_{rp_r})$ where $p_i = 2$ and $p_j \geq 3$ if $i \neq j$. Consider the Bonahon–Siebenmann family for $\mathcal{O}_i(p_1, \dots, p_r)$ [3]: since L is hyperbolic, it consists of Conway spheres intersecting L_i . Assume that for all i ’s and all choices (p_1, \dots, p_r) , $p_i = 2$ and $p_j \geq 3$, each Conway sphere of the family does not separate

the sublink $L_1 \cup \dots \cup L_{i-1} \cup L_{i+1} \cup \dots \cup L_r$. Consider now the orbifold $\mathcal{R}_i(p_1, \dots, p_r)$ which is the complement in $\mathcal{O}_i(p_1, \dots, p_r)$ of the geometric pieces which contain only singular points of order 2. If there exist $P_1, \dots, P_r \geq 3$ such that for all i 's and all choices (p_1, \dots, p_r) , $p_i = 2$ and $p_j \geq P_j$, $\mathcal{R}_i(p_1, \dots, p_r)$ is hyperbolic, we shall say that L is *well-built*. Notice that, in this case and for fixed i , the topological type of $\mathcal{R}_i(p_1, \dots, p_r)$ and the remaining geometric pieces of the decomposition of the $\mathcal{O}_i(p_1, \dots, p_r)$'s do not depend on the choice of the p_j 's, $j \neq i$ and $p_j \geq P_j$. Let L be a well-built link. We shall say that a Conway sphere for L is *fat* if it is a boundary component of some $\mathcal{R}_i(p_1, \dots, p_r)$, $p_i = 2$ and $p_j \geq P_j$, up to isotopy. Notice that fat spheres form a well defined family of disjoint Conway spheres up to isotopy. Moreover we have:

Lemma 3.2. *Let L be a well-built link. Any symmetry of L must preserve the family of fat spheres up to isotopy, in particular it preserves the 2-tangles they bound.*

Proof. Let ψ be a symmetry of L and S a fat sphere intersecting the i th component of L . Assume that ψ sends L_i onto L_j . We only need to show that $\psi(S)$ is a fat sphere on L_j . Let us fix an integer $p \geq \max\{P_1, \dots, P_j\}$. ψ induces an orbifold map between $\mathcal{O}_i(p_1, \dots, p_r)$ and $\mathcal{O}_j(p'_1, \dots, p'_r)$, where $p_i = p'_j = 2$ and $p_l = p'_s = p$ if $l \neq i$ and $s \neq j$. Since the induced map must preserve the geometric pieces of the Bonahon–Siebenmann decomposition, the assertion follows. \square

3.6. The link $A \cup B \cup C$ is well-built

We know that, for the orbifold $(\mathbf{S}^3, A_2 \cup B_n \cup C_m) = \mathcal{O}_1(2, n, m)$, the associated orbifold $\mathcal{R}_1(2, n, m) = N/\langle h, g, \tau \rangle$ is hyperbolic. Thurston’s orbifold geometrisation theorem (see [2] or [4] for a proof) implies that $\mathcal{R}_1(2, p, q)$ is hyperbolic for all choices of $p, q \geq n$. The existence of the cyclic symmetry σ (see Sec. 2.1), exchanging the three components of $A \cup B \cup C$, implies that $\mathcal{R}_2(p, 2, q)$ and $\mathcal{R}_3(p, q, 2)$ are hyperbolic for all choices of $p, q \geq n$ and $A \cup B \cup C$ is well-built.

Let us now consider (X, T) the complement in $(\mathbf{S}^3, A \cup B \cup C)$ of the 2-tangles bounded by the fat spheres of $A \cup B \cup C$.

Remark 3.3. The orbifold (X, T_p) , $p \geq n$, is hyperbolic. This follows from the fact that the boundary components of (X, T_p) are hyperbolic incompressible 2-orbifolds and that $(\mathbf{S}^3, A_p \cup B_p \cup C_p)$ is hyperbolic.

Fix an arbitrary orientation on A and orientations on B and C in such a way that they are preserved by σ . The orientations thus obtained define an orientation on each arc of T and σ induces a symmetry σ' of (X, T) of order a power of 3, which again preserves the orientations on the arcs of T . Label the arcs of T by A , B or C according to the component of the link to which they belong. Observe now that, by Remark 3.1, \tilde{f} induces a symmetry f of $N/\langle h, g, \tau \rangle$, which we can assume

to be of order a power of 2. Let f' be the restriction of such symmetry to (X, T) . We want to show that f' extends to the 2-tangles of $A \cup B \cup C$ bounded by the fat spheres of A . We know that f' extends to the 2-tangles of $A \cup B \cup C$ bounded by the fat spheres of B and C , the extension being f .

3.7. f' either preserves or reverses orientations on all arcs of (X, T)

Glue on each boundary component of (X, T) a totally symmetric tangle which connects the arcs of T adjacent to the component in the same way as the original tangle used to. We obtain a three component link on which both f'^a and σ' extend. Moreover, arcs of T have the same label if and only if they belong to the same component in this new link and orientations match by construction. The linking number of two components of $A \cup B \cup C$ coincides with the linking number of the corresponding components in the new link. Since $\text{lk}(A, B) = \text{lk}(B, C) = \text{lk}(C, A) \neq 0$ the assertion follows.

Let us consider the map $\eta' = f' \circ \sigma' \circ f' \circ \sigma'$.

3.8. η' extends to a map η of $(S^3, A \cup B \cup C)$

Notice, first of all, that η' respects the labels and orientations of the arcs. Just like in Sec. 3.7, construct a three component link whose components are not trivial and which satisfies the same requirements of Sec. 3.7. Clearly η' extends to this link, it has finite order and fixes all components, preserving their orientations. If η' fixes a component pointwise, it must be the identity because of Smith's conjecture. If η' does not fix any component pointwise, it must act on each of them as a rotation (i.e. rational translation along the circle). We want to show η' is normalized by σ' . Indeed, the conjugate of η' would be a symmetry of the link which fixes each component and has the same order as η' . Since the group generated by σ' and f' is finite and since the components of the link are not trivial, if $\sigma' \eta' \sigma'^{-1} \notin \langle \eta' \rangle$ we get a contradiction to Smith's conjecture. Note now that η' extends on the tangles determined by the fat spheres of B and since η' is normalized by σ' it extends on $(S^3, A \cup B \cup C)$. We wish to remark that, if η' is not the identity then it must act freely. Indeed, if η' does not act freely, we can find a non trivial power of η' whose fixed-point set is non-empty. Consider a Seifert surface for one of the components of the new link which is equivariant by such non-trivial power of η' [18]: the surface must in fact be invariant. Since η' acts freely on each component, it must permute freely the points of intersection of each of the remaining two components with the Seifert surface. Since the algebraic intersection number is 1, this is impossible.

^aRemark that on each connected component of $S^3 \setminus N/\langle h, g \rangle$ the fixed-point set of any involution, which acts as a strong inversion on all boundary tori, consist of two arcs which connect the same pairs of points on $\partial N/\langle h, g \rangle$.

3.9. f' extends to a map φ of $(S^3, A \cup B \cup C)$

It suffices to define φ on the tangles determined by the fat spheres along A . Let D be any such tangle and define $\varphi(D) = \sigma \circ f \circ \sigma \circ \eta^{-1}(D)$. Since $\sigma \circ f \circ \sigma \circ \eta^{-1} = f$ on (X, T) this is a well-defined extension of f' .

To complete the proof, it is now sufficient to remark that φ lifts to a symmetry of $\bar{K} \cup \bar{K}'$ (and $\hat{K} \cup \hat{K}'$) which exchanges the two components thus proving that K and K' are equivalent.

4. Covers of Order a Power of 2

In this section we shall show that the conclusion of [19, Theorem 1] holds for all $n \neq 2$, by adapting the proof to the case $n = 2^d > 2$. We shall use the notation of [19] and the reader is referred Zimmermann’s paper for details. Remark that we only need to show that case ii) does not occur. From now on we shall assume that $n = 2^d > 2$ (so that $p = 2$).

Assume that $C_2 \cong \mathbb{Z}_n$ is not normal in the 2-Sylow subgroup. Then, just like in [19, p. 669], we have a group $C_2 \oplus gC_2g^{-1}$. Notice that this groups contains exactly two maximal cyclic subgroups with non-empty fixed point sets. Moreover these fixed point sets are connected and do not intersect. Applying [17, Chap. 2, 1.5] we see that either $C_2 \oplus gC_2g^{-1}$ is normal (in particular D_2 normalises it) or its normalizer contains a subgroup conjugate to $C_2 \oplus gC_2g^{-1}$ but different from it.

4.1. $C_2 \oplus gC_2g^{-1}$ is normal

Notice first of all that, since C_2 and D_2 are not conjugate, the intersection $C_2 \oplus gC_2g^{-1} \cap D_2$ is trivial, moreover a subgroup H of D_2 of index at most 2 must fix the two maximal cyclic subgroups with non-empty fixed point sets and normalize C_2 . If the order of H is at least 4, reasoning as in [19] we obtain a group $C_2 \oplus gC_2g^{-1} \oplus H$ which contradicts the fact that the normalizer of C_2 must be a subgroup of $\mathbb{Z}_2 \times (\mathbb{Z}_{2^a} \oplus \mathbb{Z}_{2^b})$. We can thus assume that $C_2 \cong \mathbb{Z}_4$ and any generator of the group exchanges the two maximal cyclic subgroups with non-empty fixed point sets, so that $H \cong \mathbb{Z}_2$. We want to show that the group generated by H and $C_2 \oplus gC_2g^{-1}$ cannot be of the form $H \times (C_2 \oplus gC_2g^{-1})$. Indeed, assume by contradiction that the generator of H sends each element of $C_2 \oplus gC_2g^{-1}$ to its inverse, then the fixed-point set of H would intersect the fixed-point sets of the two maximal cyclic subgroups, on which the generator of H acts as a strong inversion. However the fixed point set of H coincides with that of D_2 and so the generator of D_2 could not exchange the two maximal cyclic subgroups with non-empty fixed point sets of $C_2 \oplus gC_2g^{-1}$ for their fixed-point sets are disjoint, which contradicts the hypothesis.

4.2. $C_2 \oplus gC_2g^{-1}$ is not normal

In this case the normalizer of $C_2 \oplus gC_2g^{-1}$ contains a subgroup $hC_2h^{-1} \oplus (hg)C_2(hg)^{-1}$ conjugate to $C_2 \oplus gC_2g^{-1}$ but distinct from it. For this reason we may assume that at least one of the two maximal cyclic subgroups with non-empty fixed point sets of $hC_2h^{-1} \oplus (hg)C_2(hg)^{-1}$ intersects trivially $C_2 \oplus gC_2g^{-1}$. Notice that such subgroup is isomorphic to \mathbb{Z}_n and the argument seen in Sec. 3.1 gives the final contradiction.

References

- [1] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Universitext (Springer-Verlag, Berlin, 1992).
- [2] M. Boileau and J. Porti, *Geometrization of 3-Orbifolds of Cyclic Type (Appendix A by M. Heusener and J. Porti)*, Astérisque, Vol. 272 (Soc. Math. France, Paris, 2001).
- [3] F. Bonahon and L. C. Siebenmann, The characteristic toric splitting of irreducible compact 3-orbifolds, *Math. Ann.* **278** (1987) 441–479.
- [4] D. Cooper, C. D. Hodgson and S. P. Kerckhoff, *Three-Dimensional Orbifolds and Cone-Manifolds (with a postface by S. Kojima)*, MSJ Memoirs, Vol. 5 (Mathematical Society of Japan, Tokyo, 2000).
- [5] W. Dunbar, Geometric orbifolds, *Rev. Mat. Univ. Complut. Madrid* **1** (1988) 67–99.
- [6] A. L. Edmonds and C. Livingston, Group actions on fibered three-manifolds, *Comment. Math. Helv.* **58** (1983) 529–542.
- [7] J. A. Hillman, Links with infinitely many semifree periods are trivial, *Arch. Math.* **42** (1984) 568–572.
- [8] C. Hodgson and J. H. Rubinstein, Involutions and isotopies of lens spaces, in *Knot Theory and Manifolds (Vancouver 1983)*, ed. D. Rolfsen, Lecture Notes in Mathematics, Vol. 1144 (Springer-Verlag, Berlin 1985), pp. 60–96.
- [9] W. H. Jaco and P. B. Shalen, Seifert fibred spaces in 3-manifolds, *Mem. Amer. Math. Soc.* **220** (1979).
- [10] K. Johannson, *Homotopy Equivalence of 3-Manifolds with Boundary*, Lecture Notes in Mathematics, Vol. 761 (Springer-Verlag, Berlin, 1979).
- [11] M. Mecchia and M. Reni, Hyperbolic 2-fold branched coverings of links and their quotients, *Pacific J. Math.* **202** (2002) 429–447.
- [12] J. Morgan and H. Bass, *The Smith Conjecture* (Academic Press, New York, 1984).
- [13] L. Paoluzzi, On hyperbolic type involutions, *Rend. Istit. Mat. Univ. Trieste* **32**(1) (2001) 221–256.
- [14] L. Paoluzzi, Non-equivalent hyperbolic knots, *Topol. Appl.* **124** (2002) 85–101.
- [15] M. Reni, On π -hyperbolic knots with the same 2-fold branched coverings, *Math. Ann.* **316** (2000) 681–697.
- [16] D. Rolfsen, *Knots and Links* (Publish or Perish, Berkeley, 1976).
- [17] M. Suzuki, *Group Theory I*, Grundlehren der Mathematischen Wissenschaften, Vol. 247 (Springer-Verlag, Berlin, 1982).
- [18] J. L. Tollefson, Innermost disk pairs in least weight normal surfaces, *Topol. Appl.* **65** (1995) 139–154.
- [19] B. Zimmermann, On hyperbolic knots with the same m -fold and n -fold cyclic branched coverings, *Topol. Appl.* **79** (1997) 143–157.
- [20] B. Zimmermann, On hyperbolic knots with homeomorphic cyclic branched coverings, *Math. Ann.* **311** (1998) 665–673.