Geometriae Dedicata **60:** 113–123, 1996.

On a Class of Hyperbolic 3-Manifolds and Groups with One Defining Relation

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(Received: 22 September 1994)

Abstract. We construct compact hyperbolic 3-manifolds with totally geodesic boundary, arbitrarily many of the same volume. The fundamental groups of these 3-manifolds are groups with one defining relation. Our main result is a classification of these manifolds up to homeomorphism, resp. isometry.

Mathematics Subject Classifications (1991): Primary: 57M50; Secondary: 20F32.

Key words: Hyperbolic 3-manifolds, totally geodesic boundary, fundamental polyhedron, 1-relator group.

1. Introduction

For each pair of integers n, k such that $n \geq 3$, $0 \leq k < n$ and $(n, 2-k) = 1$, we construct a compact orientable hyperbolic 3-manifold $M_{n,k}$ with totally geodesic boundary (a surface of genus $n-1$) by identifying faces of a certain hyperbolic polyhedron $\mathcal{P}_n$. For fixed n, the manifolds $M_{n,k}$ have the same volume. The fundamental group $\pi_1(M_{n,k})$ of $M_{n,k}$ is a group with n generators and one defining relation in which each generator occurs exactly three times, with exponent sum ± 1; the abelianized group is isomorphic to $\mathbf{Z}^{n-1}$. Note that the fundamental group of a closed hyperbolic 3-manifold can never be a group with one defining relation because the cohomological dimension of a torsion-free 1-relator group is 2 whereas for the fundamental group of a closed aspherical 3-manifold it is 3.

Our main result is a classification of these manifolds: $M_{n,k}$ and $M_{n,k'}$ are homeomorphic (or equivalently, isometric) if and only if $k \equiv k' \bmod n$. The manifold $M_{3,1}$ is the manifold constructed by Thurston ([8, §3.4]) by identifying faces of 2 truncated tetrahedra which was shown by Kojima and Miyamoto ([4]) to be one of the six compact manifolds which have minimal volume among all compact hyperbolic 3-manifolds with totally geodesic boundary. Our methods of classification apply more generally for classes of hyperbolic 3-manifolds (closed or with totally geodesic boundary) which are cyclic branched coverings of hyperbolic links in the 3-sphere $\mathbf{S}^3$ with two or more components (or of two or more arcs in the 3-ball $\mathbf{B}^3$) if one has enough information about the symmetry group of the link. For example, if $DM_{n,k}$ denotes the double of $M_{n,k}$ along the boundary then

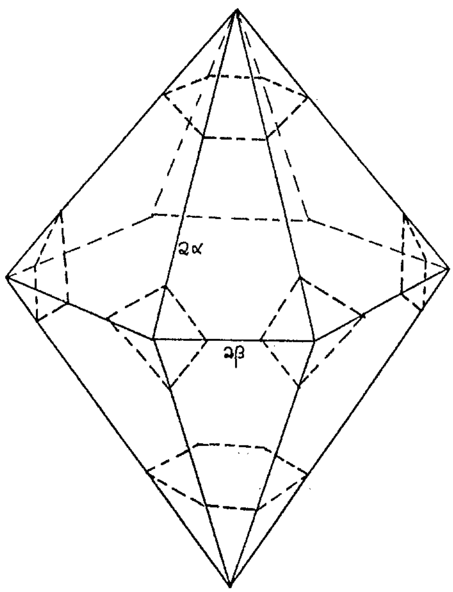

Fig. 1.

it can be shown by our methods that the classification of the closed hyperbolic 3-manifolds $DM_{n,k}$ is also given by the above conditions. What remains open is the classification up to isomorphism of the fundamental groups $\pi_1(M_{n,k})$. We suppose it corresponds to the classification of the manifolds up to homeomorphism but have no proof at the moment. As these are groups with one defining relation maybe some algebraic methods apply; however, a purely algebraic classification seems to be difficult to us.

2. The Construction

We start with the polyhedron $\mathcal{P}''_n$ shown in Figure 1; it is a double pyramid (or double cone) whose base is a regular n-gon.

We shall denote by $\mathcal{P}'_n$ the polyhedron $\mathcal{P}''_n$ with deleted vertices and by $\mathcal{P}_n$ the polyhedron $\mathcal{P}''_n$ truncated at all vertices (as indicated in Figure 1), so $\mathcal{P}_n \subset \mathcal{P}'_n \subset \mathcal{P}''_n$. Figure 2 shows the boundary of the polyhedron $\mathcal{P}''_n$ flattened out on the 2-sphere where one of the two cone points is at infinity.

We identify the faces of $\mathcal{P}''_n$ in pairs as indicated in Figure 2: chosen $0 \leq k < n$, the face $a_i b_{i+1} b_i$ gets identified with the face $c_{i+k} a_{i+k} c_{i+k+1}$ by a transformation (homeomorphism of faces) which we shall denote by x_i (for fixed n and k). These identifications induce identifications also of the polyhedron with deleted vertices $\mathcal{P}'_n$ resp. the truncated polyhedron $\mathcal{P}_n$; we denote the resulting identification spaces by $M''_{n,k}$, $M'_{n,k}$ resp. $M_{n,k}$.

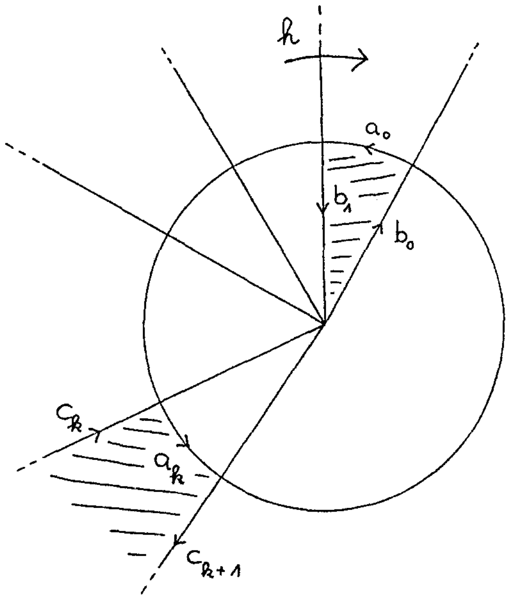

Fig. 2.

If $d = (n, 2-k)$ denotes the greatest common divisor, then the edges of $\mathcal{P}''_n$ resp. $\mathcal{P}'_n$ get identified to exactly d edges in $M''_{n,k}$ resp. $M'_{n,k}$ (we get d edge cycles). We shall only consider the case $d = 1$ in the following; then all edges of $\mathcal{P}''_n$ or $\mathcal{P}'_n$ and all vertices of $\mathcal{P}''_n$ become identified to a single edge resp. vertex, and the edge cycle relation is as follows

$$(x_0 x_1^{-1} x_{1-k}^{-1})(x_{2-k} x_{3-k}^{-1} x_{3-2k}^{-1}) \cdots (x_{k-2} x_{k-1}^{-1} x_{-1}^{-1}) = 1,$$

or

$$\prod_{i=0}^{n-1} x_{i(2-k)} x_{i(2-k)+1}^{-1} x_{(i+1)(2-k)-1}^{-1} = 1,$$

where we take indices mod n (see [5] for the notion of an edge cycle or edge cycle relation); note that in the present paper, products of maps will be always read from left to right.

For the Euler characteristic of $M''_{n,k}$ one has $\chi(M''_{n,k}) = 1 - 1 + n - 1 = n - 1$ therefore $M''_{n,k}$ is not a manifold (see [6]). Note that $M'_{n,k}$ and $M_{n,k}$ are manifolds; $M_{n,k}$ is compact and its boundary is a closed orientable surface of genus $n-1$.

3. Geometric Realization

We want to realize $\mathcal{P}'_n$ and $\mathcal{P}_n$ as hyperbolic polyhedra in hyperbolic 3-space $\mathbf{H}^3$ such that Poincaré's theorem on fundamental polyhedra can be applied ([5])

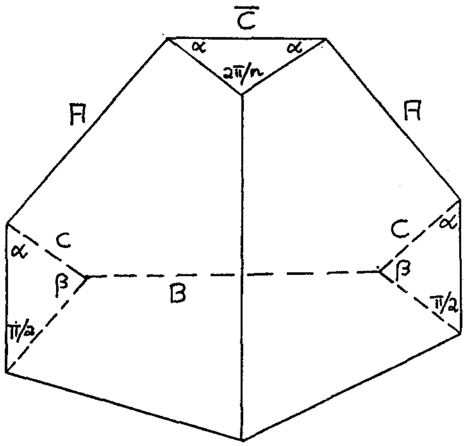

Fig. 3.

realizing $M'_{n,k}$ resp. $M_{n,k}$ as hyperbolic manifolds: the first one complete and open, the second one compact with totally geodesic boundary.

We shall denote the dihedral angles of $\mathcal{P}'_n$ at the $2n$ edges emanating from the two cone points by 2α and at the n edges of the base of the two pyramids by 2β (see Figure 1). The angles along the single edge cycle have to sum up to 2π, so we get the condition $n(2\alpha + \beta) = \pi$.

Note that all faces of $\mathcal{P}'_n$ are triangles with deleted vertices so the identifications can be obtained by hyperbolic isometries. The truncation of $\mathcal{P}'_n$ will be by hyperbolic planes intersecting the faces of $\mathcal{P}'_n$ in right angles, so all remaining dihedral angles of $\mathcal{P}_n$ will be equal to $\pi/2$. Then the faces of $\mathcal{P}_n$ which get identified are right-angled hexagons. We divide each of the two truncated pyramids into n partially truncated tetrahedra $\mathcal{T}_n$ indicated in Figure 3. The identifying transformations of $\mathcal{P}_n$ can be chosen as hyperbolic isometries in $\mathbf{H}^3$ if and only if $A = B$ and $\bar{C} = C$, where the letters in Figure 3 denote the hyperbolic lengths of the corresponding edges.

Applying the formula for right-angled hexagons in [2, §7.19], we get

$$\cosh B = \frac{\cosh^2 C + \cosh \bar{C}}{\sinh^2 C},$$

$$\cosh A = \frac{\cosh \bar{C} \cosh C + \cosh C}{\sinh C \sinh \bar{C}}.$$

We note that $\bar{C} = C$ implies $A = B$. Applying the cosine rule for hyperbolic triangles ([2, §7.12]) we get

$$\cosh C = \frac{\cos \alpha \cos \beta}{\sin \alpha \sin \beta},$$

$$\cosh \bar{C} = \frac{\cos^2 \alpha + \cos(2\pi/n)}{\sin^2 \alpha}.$$

Imposing $\cosh C = \cosh \bar{C}$ we get

$$\begin{aligned} \cot \beta &= \cot \alpha + \frac{\cos(2\pi/n)}{\cos \alpha \sin \alpha} \\ &= \cot \alpha + \frac{2\cos(2\pi/n)}{\sin 2\alpha}. \end{aligned}$$

We want to find α, β such that $n(2\alpha + \beta) = \pi$. There is a unique solution for each $n \geq 3$. For $n = 3$ resp. $n = 4$ one has $2\alpha = \beta = \pi/6$ (the case considered by Thurston in [8, §3.4]) resp. $\alpha = \beta = \pi/12$. If $n \geq 5$ then $\beta < \alpha < \pi/12$ and, letting α tend to 0, we again get a unique solution.

Finally, Andreev's theorem ([1]; [8, Ch. 5]) applies showing that $\mathcal{P}_n$ (with the above angles) can be realized as a hyperbolic polyhedron; we also gave a short direct construction of $\mathcal{P}_n$ using continuity arguments but do not include it here (see [3] for some similar constructions).

PROPOSITION 1. *For exactly one pair of angles α, β the polyhedra $\mathcal{P}'_n$ and $\mathcal{P}_n$ can be realized as hyperbolic polyhedra (which we denote again by $\mathcal{P}'_n$ resp. $\mathcal{P}_n$) such that all identifying transformations x_i can be realized as hyperbolic isometries and such that the cycle condition $n(2\alpha + \beta) = \pi$ holds.*

Now Poincaré's theorem ([5]) applies to $\mathcal{P}'_n$. We denote by $G_{n,k}$ the group generated by the identifying transformations x_i. Then $G_{n,k}$ is a properly discontinuous or discrete group of isometries of $\mathbf{H}^3$ which has $\mathcal{P}'_n$ as a fundamental polyhedron, and the quotient $\mathbf{H}^3/G_{n,k}$ realizes our manifold $M'_{n,k}$ as a complete hyperbolic manifold. Truncating $\mathcal{P}'_n$ (or equivalently, $M'_{n,k}$) we realize also $M_{n,k}$ as a compact hyperbolic manifold with totally geodesic boundary. The preimage of $M_{n,k} \subset M'_{n,k}$ in $\mathbf{H}^3$ which we shall denote by $\mathbf{H}^3_t$ is invariant under $G_{n,k}$, and $\mathbf{H}^3_t/G_{n,k} = M_{n,k}$. We have $\pi_1(M_{n,k}) \cong \pi_1(M'_{n,k}) \cong G_{n,k}$, and by Poincaré's theorem

$$G_{n,k} = \left\langle x_0, \ldots, x_{n-1} \,\middle|\, \prod_{i=0}^{n-1} x_{i(2-k)} x^{-1}_{i(2-k)+1} x^{-1}_{(i+1)(2-k)-1} = 1 \right\rangle.$$

4. A Hyperbolic 3-Orbifold

In the following, we shall denote by $h = h_n$ the rotational symmetry (clockwise direction in Figure 2) of order n of $\mathcal{P}'_n$ resp. $\mathcal{P}_n$ around the central axis connecting the two cone points; obviously, this is a hyperbolic isometry (elliptic transformation).

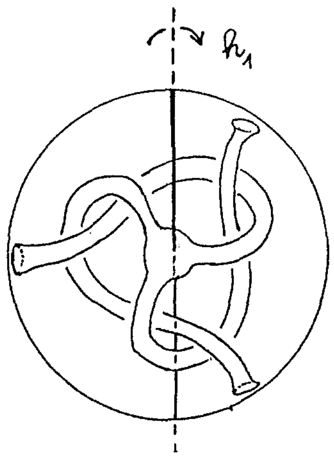

Fig. 4.

Then we have $x_i = h^{-1}x_{i+1}h$ therefore h normalizes $G_{n,k}$; we denote by E_n the group generated by $G_{n,k}$ and h (note that E_n does not depend on k). We have an exact sequence

$$1 \to G_{n,k} \hookrightarrow E_n \xrightarrow{\pi_k} \mathbf{Z}_n = \langle h_k \rangle \to 1,$$

where we denote by $h_k = h_{n,k} = \pi_k(h)$ the projection of h to $M_{n,k} = \mathbf{H}^3_t/G_{n,k}$. We denote by $\mathcal{O}_n$ the hyperbolic 3-orbifold with totally geodesic boundary $\mathcal{O}_n := \mathbf{H}^3_t/E_n = M_{n,k}/\langle h_k \rangle$.

A picture of the manifold $M_{3,1}$ is given by Thurston in [8, §3.4] and reproduced here in Figure 4; analogously, one finds a picture of $M_{n,1}$.

The elliptic transformation h projects to the transformation h_1 indicated in Figure 4, therefore the orbifold $\mathcal{O}_n = \mathbf{H}^3_t/E_n = M_{n,1}/\langle h_1 \rangle$ is as shown in Figure 5. The underlying topological space of $\mathcal{O}_n$ is the 3-ball, the singular set consists of two arcs: one is (part of) a trefoil, the other one is unknotted (see [7], [8] for basic definitions about orbifolds). Let $I_+(\mathcal{O}_n)$ resp. $I(\mathcal{O}_n)$ denote the orientation-preserving resp. full isometry group of $\mathcal{O}_n$. Then we have

LEMMA 1. $I(\mathcal{O}_n) = I_+(\mathcal{O}_n) = \langle \tau \mid \tau^2 = 1 \rangle \cong \mathbf{Z}_2$ *where* τ *is the rotation indicated in Figure 5.*

Proof. Any isometry of $\mathcal{O}_n$ must preserve the singular set of $\mathcal{O}_n$ and does not interchange its components. Otherwise such an isometry would extend to a homeomorphism of the 3-sphere interchanging the trefoil and the unknot, which is impossible. Let λ be an isometry in $I_+(\mathcal{O}_n)$. There are two possibilities: λ can fix or interchange the two endpoints of the trefoil. Suppose λ fixes them. Then λ is the identity on the trefoil (since every point on the trefoil is determined by its distance from the endpoints). Now $I(\mathcal{O}_n)$ is a finite group (taking the double of $\mathcal{O}_n$ this follows as in the case of closed hyperbolic 3-manifolds). Therefore λ has finite

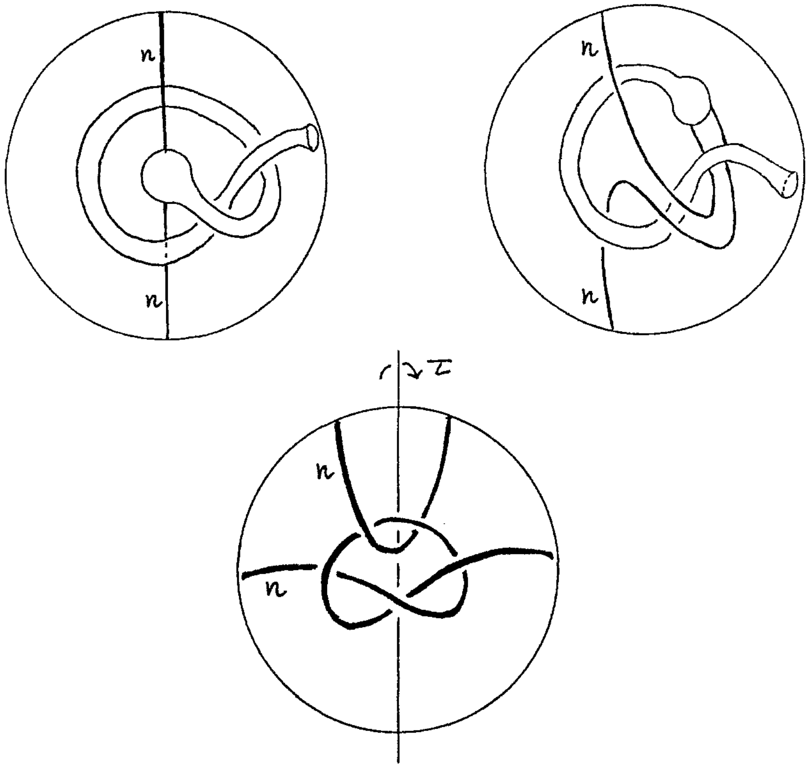

Fig. 5.

order and, by the positive solution of the Smith conjecture (in the very special case of the trefoil), λ is the identity.

It follows that there exists at most one orientation-preserving nontrivial isometry. Such an isometry interchanges the two endpoints of the trefoil and has order 2: this is shown in Figure 5.

Finally there do not exist orientation-reversing isometries because the trefoil is not amphicheiral.

Finally we derive a presentation of the group E_n. A fundamental polyhedron of E_n consists of two copies of the tetrahedron $\mathcal{T}_n$ (a slice of an angle $2\pi/n$ of $\mathcal{P}_n$). We have two sidepairing transformations: one is the rotation h, the other one, denoted by x, identifies the two other faces (which are right-angled hexagons when truncated). We have two edge cycles which give the relations $h^n = 1$ and $(hxhx^{-2})^n = 1$. The axes of the two elliptic elements h and $hxhx^{-2}$ of order n project to the two components of the singular set of the orbifold $\mathcal{O}_n = \mathbf{H}_t^3/E_n$ (the axis of h to the trefoil part), therefore the subgroups $\langle h \rangle$ and $\langle hxhx^{-2} \rangle$ generated by these elements are not conjugate in E_n and every maximal finite (elliptic, cyclic) subgroup of E_n is conjugate to one of these two subgroups.

LEMMA 2. *The group E_n has exactly two conjugacy classes of maximal elliptic subgroups represented by the subgroups $\langle h\rangle$ and $\langle hxhx^{-2}\rangle$. The elliptic elements h and $hxhx^{-2}$ are rotations of minimal angle $2\pi/n$ in these subgroups. For every hyperbolic isometry σ which normalizes E_n, conjugation by σ fixes both conjugacy classes. So, up to conjugation, we can assume that it fixes both subgroups $\langle h\rangle$ and $\langle hxhx^{-2}\rangle$; then conjugation by σ fixes both generators h and $hxhx^{-2}$ or maps them to their inverses.*

Proof. We still have to prove that conjugation by σ does not exchange the two conjugacy classes. But this follows from the preceding lemma because σ projects to an isometry of $\mathcal{O}_n = \mathbf{H}^3_t/E_n$ which cannot exchange the two components of the singular set.

By Poincaré's theorem, we have the presentation

$$E_n = \langle x, h \mid h^n = (hxhx^{-2})^n = 1\rangle.$$

On the other hand, using the above exact sequence, we get

$$E_n = \left\langle x_0, \ldots, x_{n-1}, h \,\middle|\, \prod_{i=0}^{n-1} x_{i(2-k)} x^{-1}_{i(2-k)+1} x^{-1}_{(i+1)(2-k)-1} = 1, \right.$$
$$\left. h^n = 1,\ hx_ih^{-1} = x_{i+1} \right\rangle.$$

The connection between the two presentations is given by

$$x_i = h^i x h^{-i-k}, \quad i = 0, \ldots, n-1,$$
$$x = x_0 h^k.$$

For the projection

$$\pi_k \colon E_n \to \mathbf{Z}_n = \langle h_k\rangle$$

we have

$$\pi_k(G_{n,k}) = 1, \quad \pi_k(h) = h_k, \quad \pi_k(x) = h_k^k, \quad \pi_k(hxhx^{-2}) = h_k^{2-k}.$$

5. The Classification

The main result of the paper is the following

THEOREM 1. *The manifolds $M_{n,k}$ and $M_{n',k'}$ are homeomorphic (or equivalently, isometric) if and only if $n = n'$ and $k \equiv k'$* mod n.

The fact that homeomorphism and isometry are equivalent notions for these manifolds is due to the following consequence of Mostow's rigidity theorem.

PROPOSITION 2. *Let M and M' be compact hyperbolic 3-manifolds with totally geodesic boundary. Then M and M' are homeomorphic if and only if they are isometric. The isometry groups $I(M)$ resp. $I(M')$ of M resp. M' are finite.*

Proof. Suppose $f: M \to M'$ is a homeomorphism. Let $DM = M \cup_{\partial M} M$ be the double of M along the boundary. Then DM is a closed hyperbolic 3-manifold which admits an isometric involution τ interchanging the two copies of M. Let $Df: DM \to DM'$ be the 'double' of the homeomorphism f. Then $(Df)\tau'(Df)^{-1} = \tau$, where τ' is the corresponding involution of the double DM' of M'. By Mostow rigidity, the homeomorphism Df is homotopic to an isometry $I: DM \to DM'$. Then $I\tau' I^{-1}$ is homotopic and therefore equal to τ, and I maps the fixed point set ∂M of τ to the fixed point set $\partial M'$ of τ'. Therefore I splits along ∂M restricting to an isometry between M and M'. For the remaining assertion, note that, by doubling isometries, the isometry group $I(M)$ of M can be considered as subgroup of the isometry group $I(DM)$ of the closed hyperbolic 3-manifold DM which is finite ([7, §5.7]).

PROOF OF THEOREM 1. Suppose $M_{n,k}$ and $M_{n',k'}$ are isometric; then $n = n'$. Let $\delta: M_{n,k} \to M_{n,k'}$ be an isometry. We distinguish two cases:

Case 1. The isometry $\delta: M_{n,k} \to M_{n,k'}$ can be chosen such that $\delta\langle h_{k'}\rangle\delta^{-1} = \langle h_k\rangle$. Let $\tilde{\delta}$ be a lift of δ to $\mathbf{H}^3_t$. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{n,k} & \hookrightarrow & E_n & \xrightarrow{\pi_k} & \mathbf{Z}_n = \langle h_k\rangle & \longrightarrow & 1 \\ & & \downarrow{\cong\tilde{\delta}_*} & & \downarrow{\cong\tilde{\delta}_*} & & \downarrow{\delta_*} & & \\ 1 & \longrightarrow & G_{n,k'} & \hookrightarrow & E_n & \xrightarrow{\pi_{k'}} & \mathbf{Z}_n = \langle h_{k'}\rangle & \longrightarrow & 1 \end{array}$$

where the isomorphisms $\tilde{\delta}_*$ resp. δ_* are induced by conjugation with $\tilde{\delta}$ resp. δ. By Lemma 2, up to conjugation, we have

$$\tilde{\delta}_*(h) = h^\varepsilon, \quad \tilde{\delta}_*(hxhx^{-2}) = (hxhx^{-2})^\varepsilon, \quad \delta_*(h_k) = h_{k'}^\varepsilon,$$

where $\varepsilon = \pm 1$. Then

$$\begin{aligned} h_{k'}^{\varepsilon(2-k')} &= \delta_*(h_k^{2-k}) = \delta_*(\pi_k(hxhx^{-2})) \\ &= \pi_{k'}(\tilde{\delta}_*(hxhx^{-2})) = \pi_{k'}(hxhx^{-2})^\varepsilon = h_{k'}^{\varepsilon(2-k)}, \end{aligned}$$

therefore $k \equiv k' \bmod n$ and the theorem is proved in this case.

It remains the following

Case 2. The subgroups $\delta^{-1}\langle h_{k'}\rangle\delta$ and $\langle h_k\rangle$ are not conjugate in $I(M)$. We want to show that this case really does not occur considering again three cases:

(i) $n = p^\sigma$, p prime, $p \neq 2$. We need the following

LEMMA 3. *Let* G *be a finite p-group and let* H *be a proper subgroup of* G. *The normalizer* N(H) *of* H *in* G *contains* H *as a proper subgroup.*

Proof. Let $Z(G) \neq 1$ be the center of the p-group G. We can assume $Z(G) \subset H$ because otherwise the lemma is certainly true. Now the Lemma follows by dividing out $Z(G)$ and applying the induction hypothesis to the subgroup $H/Z(G)$ of $G/Z(G)$.

Now, because $\langle h_k \rangle$ and $\delta \langle h_{k'} \rangle \delta^{-1}$ are not conjugate, $\langle h_k \rangle$ is a proper subgroup of some p-Sylow subgroup Σ_p of $I(M_{n,k})$. By Lemma 3 we find an isometry in Σ_p which normalizes $\langle h_k \rangle$ and does not lie in $\langle h_k \rangle$. This isometry projects to an isometry of $\mathcal{O}_n = M_{n,k}/\langle h_k \rangle$ of order some nontrivial power of p which is impossible because $I(\mathcal{O}_n) \cong \mathbf{Z}_2$.

(ii) $n = p_1^{\sigma_1} p_2^{\sigma_2} \ldots p_d^{\sigma_d}$, a product of at least two different primes.

Then $\langle h_k \rangle \cong \mathbf{Z}_n \cong \bigoplus_{j=1}^{d} \mathbf{Z}_{p_j^{\sigma_j}}$.

LEMMA 4. (a) *An element $y \in I(M_{n,k})$ which normalizes a nontrivial subgroup of $\langle h_k \rangle$ lies in the normalizer of the whole group $\langle h_k \rangle$.*

(b) *If some nontrivial subgroups of $\langle h_k \rangle$ and $\delta \langle h_{k'} \rangle \delta^{-1}$ are conjugate, then the whole group is conjugate.*

Proof. The fixed point set of h_k and of all of its nontrivial powers consists of two axes and is invariant under y; therefore y is in the normalizer of the whole group $\langle h_k \rangle$. The second part follows similarly.

By Lemma 4(b) the p_j-Sylow subgroups of $\langle h_k \rangle$ and $\delta \langle h_{k'} \rangle \delta^{-1}$ are not conjugate, therefore the p_j-Sylow subgroup of $\langle h_k \rangle$ is properly contained in some p_j-Sylow subgroup Σ_{p_j} of $I(M_{n,k})$. By Lemma 3 and Lemma 4(a), there exists an element $y \in \Sigma_{p_j}$ which lies in the normalizer of $\langle h_k \rangle$ but not in $\langle h_k \rangle$. This projects to an isometry of $\mathcal{O}_n$ and we get a contradiction as in case (i).

There remains the last case

(iii) $n = 2^\sigma$, $\sigma \geq 2$.

We can assume that $\langle h_k \rangle$ and $\delta \langle h_{k'} \rangle \delta^{-1}$ are proper subgroups of the same 2-Sylow subgroup Σ_2 of $I(M_{n,k})$. If the index of $\langle h_k \rangle$ in its normalizer $N(\langle h_k \rangle)$ in Σ_2 is larger than 2 we get a contradiction as in the previous cases, so by Lemma 3 we can assume that the index is exactly 2. Let y be an element in the normalizer of $N(\langle h_k \rangle)$ in Σ_2. Then $y h_k^2 y^{-1} \in \langle h_k \rangle$ (because $\langle h_k \rangle$ has index 2 in $N(\langle h_k \rangle)$) therefore y normalizes a nontrivial subgroup of $\langle h_k \rangle$ and by Lemma 4(a) also the whole group $\langle h_k \rangle$. Then $y \in N(\langle h_k \rangle)$; by Lemma 3 applied to $N(\langle h_k \rangle) \subset \Sigma_2$, one has $N(\langle h_k \rangle) = \Sigma_2$. Then, by symmetry, also the index of $\delta \langle h_{k'} \rangle \delta^{-1}$ in Σ_2 is 2. But then the index of $\langle h_k \rangle \cap \delta \langle h_{k'} \rangle \delta^{-1}$ in Σ_2 is at most 4 and therefore $\langle h_k \rangle \cap \delta \langle h_{k'} \rangle \delta^{-1}$ is nontrivial. By Lemma 4(b), $\langle h_k \rangle$ and $\delta \langle h_{k'} \rangle \delta^{-1}$ are conjugate and we have again a contradiction. This finishes the proof of the Theorem.

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