Conway spheres as ideal points of the character variety

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Abstract

We prove that any Bonahon-Siebenmann family of Conway spheres for a hyperbolic link is associated to an ideal point of the character variety of the link.

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1 Introduction

Essential surfaces play a central role in the understanding of 3-dimensional manifolds. In their seminal work on character varieties [4], Culler and Shalen provided a powerful tool to find essential surfaces in 3-manifolds. These are associated to ideal points of the character variety. However, not all essential surfaces can be detected this way: examples can be found in [14].

In this paper, we consider a very special family of essential surfaces, that is *Conway spheres*. These are spheres embedded in $\mathbf{S}^3$ which meet a link transversally in four points and whose intersections with the link exterior are essential. Here by *essential* we mean incompressible, ∂-incompressible and non boundary parallel.

In the following, given a Conway sphere C, we shall denote by C' the planar surface which is the intersection of C with the exterior of the link.

Theorem 1. *Assume L is a hyperbolic link admitting a unique Conway sphere C up to isotopy. Then C' corresponds to an ideal point of the character variety of the exterior of L.*

For hyperbolic links admitting more than one Conway sphere, the above result generalises to the following:

Theorem 2. *Let L be a hyperbolic link, and let $C_1, \ldots, C_k$ be a Bonahon-Siebenmann family of Conway spheres. Then the surface $C'_1 \cup \cdots \cup C'_k$ corresponds to an ideal point of the character variety of the exterior of L.*

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Recall that the *Bonahon-Siebenmann family of toric sub-orbifolds* [2] is the equivalent for orbifolds of the Jaco-Shalen-Johannson family for manifolds. Remark that, once the orders of ramification of the components of L are assigned, the Bonahon-Siebenmann family is unique up to orbifold isotopy, as is the Jaco-Shalen-Johannson family, but the family can change for different orders of ramification. In our situation, since the link is hyperbolic, all elements of a Bonahon-Siebenmann family are Conway spheres which meet only components with order of ramification equal to 2.

Assume the hypotheses of Theorem 1 are fulfilled. Let M_1 and M_2 be the components of $\mathbf{S}^3 \setminus C$, and M_1' respectively M_2' their intersections with the exterior of the link. The key idea in the proof of Theorem 1 is to consider the projections of the character varieties for M_1' and M_2' into the character variety for C'. Their intersection is a curve containing a distinguished point. Such point satisfies the following properties:

- It is a reducible character;
- It is a limit point of irreducible characters in the intersection of the two projections.

These conditions imply that the irreducible characters in the intersection correspond to unique characters for the link exterior. However, these characters do not have a limit in the character variety for the link exterior. This means that our distinguished point is an ideal point for the link exterior, even if it corresponds to a character of both M_1' and M_2'.

Remark that there may exist representations of the link exterior which restricted to M_i induce the same distinguished characters. For instance, this is the case if the link exterior is obtained as the double of M_1', for in this case each representation for M_1' can be extended to a representation of the link exterior, simply by doubling.

The proof of Theorem 2 relies basically on the same idea seen for Theorem 1: some extra care is however needed to deal with several components at the same time.

The paper is organised as follows: the next two sections provide background material on representation and character varieties (Section 2) and on Culler-Shalen theory (Section 3). Section 4 is devoted to the study of the character variety of the surface C' and, more precisely, of a specific subvariety we shall be working with. In Section 5 we shall discuss properties of the pieces obtained by cutting the link exterior along Conway spheres and their character varieties. In the last two sections the proofs of Theorem 1 (Section 6) and Theorem 2 (Section 7) will be given.

2 Background on varieties of representations

The variety of representations of a connected compact manifold N^n of dimension n is the set of representations of its fundamental group in $SL(2, \mathbb{C})$:

$$R(N^n) = \hom(\pi_1(N^n), SL(2, \mathbb{C})).$$

Since we assume that N^n is compact, $\pi_1(N^n)$ is finitely generated and thus $R(N^n)$ can be embedded in a product $SL(2, \mathbb{C}) \times \cdots \times SL(2, \mathbb{C})$ by mapping each representation to the image of a generating set. In this way $R(N^n)$ is an affine algebraic set, whose defining polynomials are induced by the relations of a presentation of $\pi_1(N^n)$. This structure is independent of the choice of presentation of $\pi_1(N^n)$, cf. [11].

Given a representation $\rho \in R(N^n)$, its character is the map $\chi_\rho : \pi_1(N^n) \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$, $\forall \gamma \in \pi_1(N^n)$. The set of all characters is again an affine algebraic set, denoted by $X(N^n)$ and called the *variety of characters of* N^n, even if it is often not irreducible [7]. The projection

$$\pi : R(N^n) \to X(N^n)$$

is the quotient in the sense of invariant theory of the action of $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathcal{Z}(SL(2,\mathbb{C}))$ by conjugation: namely, if $\mathbb{C}[R(N^n)]^{PSL(2,\mathbb{C})}$ denotes the $\mathbb{C}$-subalgebra of the polynomial functions on $R(N^n)$ which are $PSL(2,\mathbb{C})$-invariant, then

$$\mathbb{C}[R(N^n)]^{PSL(2,\mathbb{C})} = \pi^*(\mathbb{C}[X(N^n)]),$$

cf. [4].

When N^n is not connected, if $N^n = N_1^n \cup \cdots \cup N_k^n$ is the decomposition into connected components, one defines

$$R(N^n) = R(N_1^n) \times \cdots \times R(N_k^n),$$

and

$$X(N^n) = X(N_1^n) \times \cdots \times X(N_k^n).$$

Definition 3. *A representation $\rho \in R(N^n)$ is called* reducible *if it has an invariant line in $\mathbb{C}^2$, namely if it is conjugate to a representation whose image consists of upper triangular matrices. Otherwise a representation is called* irreducible.

Lemma 4. *[4] Given $\rho, \rho' \in R(N^n)$, if $\chi_\rho = \chi_{\rho'}$ and ρ is irreducible, then ρ and ρ' are conjugate; in particular ρ' is irreducible.*

In addition, the projection $\pi : R(N^n) \to X(N^n)$ induces a holomorphic fibre bundle of the set of irreducible representations over the set or irreducible characters.

According to Lemma 4, it makes sense to call a character *reducible* (respectively *irreducible*) if it is associated to a reducible (respectively irreducible) representation. In addition, it is proved in [4] that the set of irreducible characters is Zariski open.

Now assume that N^3 is a 3-manifold and $S \subset N^3$ is an essential connected surface, that separates N^3 into two components N_1^3 and N_2^3. Then $\pi_1(N^3)$ is an amalgamated product

$$\pi_1(N^3) = \pi_1(N_1^3) *_{\pi_1(S)} \pi_1(N_2^3).$$

Let r_i and r_i' denote the restriction maps, induced by the natural inclusions, in the following commutative diagram:

$$\begin{tikzcd} X(N^3) \arrow[r, "r_1"] \arrow[d, "r_2"'] & X(N_1^3) \arrow[d, "r_1'"] \\ X(N_2^3) \arrow[r, "r_2'"] & X(S) \end{tikzcd}$$

Lemma 5. *Let $\chi_1 \in X(N_1^3)$ and $\chi_2 \in X(N_2^3)$ be characters such that $r_1'(\chi_1) = r_2'(\chi_2)$ is irreducible. Then there exists a unique $\chi \in X(N^3)$ satisfying $r_1(\chi) = \chi_1$ and $r_2(\chi) = \chi_2$.*

Proof. Take representations $\rho_i \in R(N_i^3)$ such that $\chi_{\rho_i} = \chi_i$. Since $r_1'(\chi_1) = r_2'(\chi_2)$ is irreducible, after conjugation (see Lemma 4), we may assume that the restrictions of ρ_1 and ρ_2 to $\pi_1(S)$ coincide. Thus they define a representation on the amalgamated product $\pi_1(N^3) = \pi_1(N_1^3) *_{\pi_1(S)} \pi_1(N_2^3)$. In addition, this representation is unique up to conjugacy, because the restriction of ρ_i to $\pi_1(S)$ is irreducible and has no centraliser. □

Next assume that the essential surface $S \subset N^3$ is not connected. Let $S_1 \ldots, S_k$ be its components, which are non-parallel, and assume that they split N^3 into $k+1$ components $N_1^3, \ldots N_{k+1}^3$, i.e. each S_l is separating. The proof of Lemma 5 can be extended by induction to obtain the following:

Lemma 6. *Let $\chi_1 \in X(N_1^3), \ldots, \chi_{k+1} \in X(N_{k+1}^3)$ be characters whose restrictions to $N_i^3 \cap N_j^3$ coincide and are irreducible, whenever this intersection is nonempty. Then there exists a unique $\chi \in X(N^3)$ whose restriction to N_i^3 is χ_i, for $i = 1, \ldots, k+1$.*

3 Culler-Shalen theory

Culler-Shalen theory associates essential surfaces to ideal points in the projective completion of a curve of characters. Recall that a surface is called essential if it is incompressible, ∂-incompressible and not boundary parallel. The surface does not need to be connected, and from now on we assume that its components are pairwise nonparallel.

Given an algebraic curve $\mathcal{C}$ in $X(N^3)$, let $\mathcal{C}_0$ be a smooth model of the projective completion of $\mathcal{C}$. An ideal point x is a point in $\mathcal{C}_0 \setminus \mathcal{C}$, namely a point at infinity. The essential surface associated to x is constructed in essentially two main steps:

- *From ideal points to representations over a field with a discrete valuation.* The ideal point $x \in \mathcal{C}_0 \setminus \mathcal{C}$ defines a discrete valuation v_x on the function field $\mathbb{C}(\mathcal{C})$ as follows. For any rational function $f \in \mathbb{C}(\mathcal{C})$, $v_x(f) = n \geq 0$ if x is a zero of f of order n, and $v_x(f) = -n < 0$ if x is a pole of order n. Given a curve $\tilde{\mathcal{C}} \subset R(N^3)$ that projects to $\mathcal{C}$, the field $F = \mathbb{C}(\tilde{\mathcal{C}})$ is a finite extension of $\mathbb{C}(\mathcal{C})$, and (a multiple of) the valuation v_x extends to a discrete valuation $\tilde{v} : F \to \mathbb{Z}$. Construct the tautological representation $P : \pi_1(N^3) \to SL(2, F)$, by viewing the entries of a matrix in $SL(2, \mathbb{C})$ as polynomial functions on $\tilde{\mathcal{C}}$. Notice that there are elements $\gamma \in \pi_1(N^3)$ that satisfy $\tilde{v}(P(\gamma)) < 0$.

- *The action on the Bass-Serre tree.* In [15], J.-P. Serre constructed the Bass-Serre tree T associated to $SL(2, F)$, where F is a field with a discrete valuation $\tilde{v} : F \to \mathbb{Z}$. The group $SL(2, F)$ acts on this tree without reversing the orientation of its edges. This action has the property that the stabilisers of vertices are precisely the subgroups that can be conjugated into $SL(2, R_{\tilde{v}})$, where $R_{\tilde{v}} = \{f \in F \mid \tilde{v}(f) \geq 0\}$ is the ring of the valuation. Since there are elements $\gamma \in \pi_1(N^3)$ satisfying $\tilde{v}(P(\gamma)) < 0$, the induced action of $\pi_1(N^3)$ is not contained in a vertex stabiliser. Then, one considers an equivariant map from the universal covering of N^3 to the tree: to obtain an essential surface, it suffices to take the inverse image of midpoints of edges of T and to render it essential in an equivariant way. The reader is referred to [4] for details.

Notice that the essential surface constructed above is *not unique*: indeed, even the choice of the equivariant map from the universal covering of N^3 to T is not unique.

Now we want to establish a sufficient condition for a surface to be associated to an ideal point.

Let $S \subset N^3$ be an essential surface with components $S_1, \ldots, S_k$, each one separating, so that $N^3_1, \ldots, N^3_{k+1}$ are the complementary components of S. Let $\mathcal{C} \subset X(N^3)$ be an algebraic curve and $\{\chi_n\}$ be a sequence of characters in $\mathcal{C}$. Assume that the sequence of the restrictions to N^3_i, $\{\chi^i_n\}$, converges to the irreducible character χ^i_∞. Let $\rho_i \in R(N^3_i)$ be the representation whose character is $\chi_{\rho_i} = \chi^i_\infty$, for $i = 1, \ldots, k+1$.

Lemma 7. *Assume that whenever $S_l = N^3_i \cap N^3_j$, the restriction of ρ_i to S_l is* not conjugate *to the restriction of ρ_j to S_l. Then a subsequence of $\{\chi_n\}$ converges to an ideal point χ_∞ of $\mathcal{C}$, and S is a surface associated to this ideal point.*

Proof. Assume first that $k = 1$, i.e. $S = S_1$ is connected. Up to a subsequence, $\{\chi_n\}$ converges to either a point in $\mathcal{C}$ or to an ideal point. We want to see that it converges to an ideal point: seeking a contradiction, assume that it converges to $\chi_\infty \in \mathcal{C} \subset X(N^3)$. In particular $\chi_\infty = \chi_{\rho_\infty}$ for some $\rho_\infty \in R(N^3)$ and since χ_∞ restricted to N^3_i is the irreducible character χ^i_∞, then ρ_∞ restricted to N^3_i is conjugate to ρ_i (see Lemma 4). In particular the restrictions of ρ_1 and ρ_2 to S_1 are conjugate, leading to a contradiction. Thus $\chi_n \to x$, an ideal point. Now consider the Bass-Serre tree T associated to this ideal point. By construction, since the valuation restricted to N^3_i is non-negative, $P(\pi_1(N^3_i))$ is contained in the stabiliser of a vertex v_i of T (where P is the tautological representation described above). Notice that $v_1 \neq v_2$, otherwise the point x would not be ideal.

Consider the graph of groups G, with two vertices, with vertex groups $\pi_1(N^3_1)$ and $\pi_1(N^3_2)$ respectively, and with one edge between them, with edge group $\pi_1(S)$. This graph is dual to the decomposition along S. Now construct an equivariant map from $\tilde{G}$ to T, by mapping the i-th vertex to v_i, the edge of G to the unique path in T joining them (note that this path can consists of more than one edge). Compose it with the obvious equivariant map from $\tilde{N}^3$ to $\tilde{G}$:

$$\tilde{N}^3 \to \tilde{G} \to T.$$

Taking into account this very action, and given the fact that S is essential, it is clear that the surface obtained by taking the inverse image of midpoints of edges and eliminating parallel copies is precisely S.

The general case follows easily, since each component S_l separates. ◻

Remark 8. *In the hypothesis of Lemma 7 the representations ρ_i and ρ_j restricted to S_l are not conjugate, but they have the same character: $\chi_i|_{S_l} = \chi_j|_{S_l}$. This follows from the fact that this is the case for the restrictions of χ_n to N^3_i and N^3_j and that the corresponding sequences converge.*

This situation can occur when the restrictions of ρ_i and ρ_j to S_l are reducible, and this is precisely what happens in our applications.

4 The character variety for C'

Recall that, for a given Conway sphere C, C' denotes the four-holed sphere which is the intersection of C with the link exterior. The fundamental group of C' is a free group of rank 3. For our purposes, we choose the presentation

$$\pi_1(C') = \langle \mu_1, \mu_2, \mu_3, \mu_4 \mid \mu_1\mu_2\mu_3\mu_4 \rangle$$

where the μ_i's correspond to the four peripheral elements.

According to [7] the $SL(2,\mathbb{C})$-character variety for a free group of rank 3 is a hypersurface in $\mathbb{C}^7$, whose coordinates correspond to the traces of the images of μ_i, $i = 1, 2, 3$, $\mu_i\mu_j$, $1 \le i < j \le 3$ and $\mu_1\mu_2\mu_3 = \mu_4^{-1}$.

We will only be interested in characters coming from representations induced by those of M_1' and M_2' and that potentially extend to the link exterior. In particular, when the link is a knot, we have that the traces of the μ_i's must be the same. We shall then only need to consider the intersection of the character variety with a 4-plane. Even in the case where the link has more than one component, we shall only consider this subvariety, which is again a hypersurface but in $\mathbb{C}^4$.

The hypersurface obtained by imposing all traces of meridians to be equal is

$$\mathcal{Y} = \{(x, y, z, t) \in \mathbb{C}^4 \mid (t^2 - (x + y + z - 2))^2 = (2 - x)(2 - y)(2 - z)\}, \quad (1)$$

where t represents the trace of the image of μ_i, $i = 1, 2, 3$, and $\mu_1\mu_2\mu_3 = \mu_4^{-1}$, while x, y and z those of $\mu_1\mu_2$, $\mu_1\mu_3$ and $\mu_2\mu_3$ respectively.

One can prove that $\mathcal{Y}$ is an irreducible hypersurface whose singular set consists of the point $(-2, -2, -2, 0)$ together with three one-dimensional components. The three singular curves meet at the character corresponding to the trivial representation and contain the points $(-2, 2, 2, 0)$, $(2, -2, 2, 0)$ and $(2, 2, -2, 0)$ respectively.

Note that, since $\pi_1(C')$ is a free group, the projection from $X(C', SL(2,\mathbb{C}))$ to $X(C', PSL(2,\mathbb{C}))$ is a surjection. The points of $\mathcal{Y}$ we are interested in correspond to characters of lifts of parabolic representations in $PSL(2,\mathbb{C})$, where the meridians are rotations of angle π. In particular their holonomies are conjugate to the matrix

$$\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

According to the possible lifts (i.e. choices of signs), the possible values for (x, y, z, t) are $(-2, -2, -2, 0)$, $(-2, 2, 2, 0)$, $(2, -2, 2, 0)$ or $(2, 2, -2, 0)$. In fact, we shall show in Lemma 10 that the point $(x, y, z, t) = (-2, -2, -2, 0)$ does not occur. We shall need to work in neighbourhoods of these points, thus considering $\mathbb{C}$-analytic varieties. We follow the usual notation and call germ at a point the analytic variety defined around the point, without specifying the neighbourhood.

Lemma 9. *The analytic germ of the hypersurface $\mathcal{Y}$ in $\mathbb{C}^4$ of equation (1) at the points $(x, y, z, t) = (-2, 2, 2, 0)$, $(2, -2, 2, 0)$ and $(2, 2, -2, 0)$ is irreducible.*

Proof. We shall analyse the germ at the point $(-2,2,2,0)$: for symmetry reasons, it is sufficient to consider this case. We apply the following algebraic change of variables: $\xi = t^2 - x - 2$, $\upsilon = 2 - y$, $\zeta = 2 - z$ and $\tau = t$ so that the coordinates of our point become $(0,0,0,0)$ and the equation

$$\mathcal{Y} = \{(\xi, \upsilon, \zeta, \tau) \in \mathbb{C}^4 \mid (\xi - \upsilon - \zeta)^2 = (\xi - \tau^2 + 4)\upsilon\zeta\}. \tag{2}$$

We now consider the algebraic hypersurface in $\mathbb{C}^4$ defined by the equation

$$\mathcal{H} = \{(w, u, v, t) \in \mathbb{C}^4 \mid u^2 + v^2 + uvw - w^2 - t^2 + 4 = 0\}. \tag{3}$$

We can define a regular morphism $\mathcal{H} \longrightarrow \mathcal{Y}$ as follows:

$$(w, u, v, t) \mapsto (\xi = u^2 + v^2 + uvw, \upsilon = u^2, \zeta = v^2, \tau = t).$$

Note that we have $\xi - \upsilon - \zeta = uvw$ and $\xi - \tau^2 + 4 = u^2 + v^2 + uvw - t^2 + 4 = w^2$. It is easy to check that this morphism is finite to one, hence proper, and surjective. The preimage in $\mathcal{H}$ of $(0,0,0,0)$ consists of two points: $(\pm 2, 0, 0, 0)$. It is trivial to see that these are smooth points of $\mathcal{H}$, for the derivative with respect to w is non zero. It is now clear that the analytic germ of $\mathcal{Y}$ at $(-2,2,2,0)$ has at most two components. Irreducibility follows from the fact that the involution $(w, u, v, t) \mapsto (-w, u, -v, t)$ acting on $\mathcal{H}$ exchanges the two points $(\pm 2, 0, 0, 0)$ without changing the regular morphism.

□

5 Cutting off along Conway spheres

Cut off the exterior of the link along a family of pairwise non-parallel Conway spheres $C_1, \ldots, C_k$, obtaining components $M_1, \ldots, M_{k+1}$. Let $\mathcal{O}_i$ denote the orbifold with underlying space M_i, branching locus $L \cap M_i$. We require that the ramification indices are equal to 2 along the components that meet some Conway sphere.

Assumption. *The orbifolds $\mathcal{O}_1, \ldots, \mathcal{O}_{k+1}$ are geometric, either hyperbolic with finite volume or Seifert fibred with hyperbolic base.*

We shall show in Section 6 that under the hypotheses of Theorem 1 this assumption is satisfied for an appropriate choice of the ramification. Note that, under the hypotheses of Theorem 2, all orbifolds $\mathcal{O}_i$ (where the orders of ramification are those that determine the Bonahon-Siebenmann family) are geometric by definition, but the Seifert fibred ones do not have necessarily a hyperbolic base. We shall assume that this does not happen, and we shall see later how to avoid this particular situation. Note that if the link is a knot, all bases of Seifert fibred components must be hyperbolic.

Let C_l be a Conway sphere and M_i a component adjacent to it. Let us denote, as usual, C'_l and M'_i their respective intersections with the exterior of the link. The holonomy of the hyperbolic structure of $\mathcal{O}_i$, or of its hyperbolic base, if $\mathcal{O}_i$ is Seifert fibred, restricts to a $PSL(2, \mathbb{C})$-representation of C'_l that, up to conjugacy, maps each μ_i to an element of the form

$$\pm \begin{pmatrix} i & * \\ 0 & -i \end{pmatrix}. \tag{4}$$

The character of its lift to $SL(2,\mathbb{C})$ will be denoted by χ_0 and has coordinates $(x,y,z,t) = (-2,-2,-2,0)$, $(-2,2,2,0)$, $(2,-2,2,0)$ or $(2,2,-2,0)$. In fact the case $(-2,-2,-2,0)$ will be ruled out in Lemma 10. Notice that χ_0 is *reducible*, in particular there might be several non conjugate representations of C_l' whose characters are χ_0.

Let $\chi_i \in X(M_i')$ be the character of a lift of the representation induced by the holonomy of the complete hyperbolic structure on $\mathcal{O}_i$, or the hyperbolic structure of its base when it is Seifert fibred. The existence of the lift is due to [3]. Recall that, even if χ_0 is reducible, χ_i is *irreducible*.

When C_l is a component of ∂M_i, let

$$r_{il} : X(M_i') \to X(C_l')$$

denote the restriction map.

Lemma 10. *The lifts $\chi_i \in X(M_i')$, for $i = 1, \ldots, k+1$ can be chosen so that*

$$r_{il}(\chi_i) = r_{jl}(\chi_j),$$

whenever $M_i \cap M_j = C_l$.

In addition, the coordinates of $r_{il}(\chi_i) = r_{jl}(\chi_j)$ are $(x,y,z,t) = (-2,2,2,0)$, $(2,-2,2,0)$ or $(2,2,-2,0)$ (i.e. the case $(-2,-2,-2,0)$ does not occur).

Remark 11. *We will show in Remark 19 that the restricted representations are not conjugate, even if they have the same character, cf. Remark 8. This will be used to apply Lemma 7.*

Proof of Lemma 10. Two different lifts may differ by a change of sign at the meridians of the arcs $L \cap M_i$. To make a consistent choice, we fix an *orientation* of the components of L.

For each meridian $\mu \in \pi_1(M_i')$, choose an isometry $a \in PSL(2,\mathbb{C})$ that maps the oriented line from 0 to ∞ to the end-points of the oriented axis of $\rho_i(\mu)$, in the upper half space model for $\mathbf{H}^3$. In particular, a lift ρ_i of the holonomy must satisfy:

$$a^{-1}\rho_i(\mu)a = \epsilon \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \text{for some } \epsilon = \pm 1. \tag{5}$$

We fix the following convention: we choose $\epsilon = 1$ if the orientations of L and μ induce the standard orientation of $\mathbf{S}^3$ and $\epsilon = -1$ otherwise, cf. Figure 1. A different choice of isometry $a \in PSL(2,\mathbb{C})$ differs by an isometry that preserves the oriented line from 0 to ∞, hence it commutes with the isometry in (5). Note also that the inverse in $SL(2,\mathbb{C})$ of the matrix in the above identity coincides with its opposite, which is in accordance with the chosen convention.

We need to check that this choice is consistent. We identify M_i' with the exterior of an embedded graph in $\mathbf{S}^3$ so that its vertices are 4-valent, and they correspond precisely to the Conway spheres in ∂M_i. In this way, one can compute the fundamental group with the Wirtinger method, and we obtain that $\pi_1(M_i')$ is generated by meridians of the arcs, and there are two kinds of relations:

$r_{crossings}$ The usual conjugacy relations corresponding to transverse crossings of a projection, as for links.

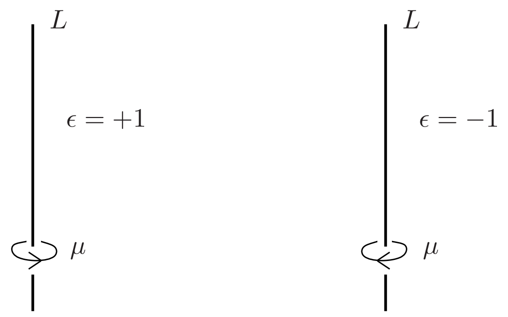

Figure 1: The choice of $\epsilon = \pm 1$ according to the orientation.

$r_{vertices}$ Relations corresponding to the vertices, i.e. to Conway spheres. Using the presentation at the beginning of Section 4, for $\mu_1, \mu_2, \mu_3, \mu_4 \in \pi_1(C_l')$, the relation is $\mu_1\mu_2\mu_3\mu_4 = 1$.

As for links, this family of relations is not minimal, but they generate all relations. Let us check compatibility. First we deal with $r_{crossings}$, which are relations of conjugations between meridians. If $\mu, \mu' \in \pi_1(M_i')$ are two meridians for the same component of $L \cap M_i$ with the same orientation, then $\mu = \gamma^{-1}\mu'\gamma$ for some $\gamma \in \pi_1(M_i')$. In particular, for $a \in PSL(2, \mathbb{C})$ as in Equation 5,

$$a^{-1}\rho_i(\mu)a = (\rho(\gamma)a)^{-1}\rho_i(\mu')\rho(\gamma)a.$$

This shows compatibility for relations $r_{crossings}$.

For $r_{vertices}$, look at the restriction of ρ_i to C_l'. Up to conjugation, we may assume that the point in $\partial_\infty(\mathbf{H}^3)$ fixed by $\pi_1(C_l')$ is ∞, hence

$$\rho(\mu_i) = \epsilon_i \begin{pmatrix} i & * \\ 0 & -i \end{pmatrix}, \qquad \epsilon_i = \pm 1. \tag{6}$$

Since the intersection number between L and C_l is zero, the set $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ contains precisely twice $+1$ and twice -1. Consequently

$$\rho(\mu_1)\rho(\mu_2)\rho(\mu_3)\rho(\mu_4) = \begin{pmatrix} i^4 & 0 \\ 0 & (-i)^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the lift is coherent with all the relations of the Wirtinger representation (i.e. the lifted representation maps the relators to the identity matrix, instead of minus the identity).

We look again at the restriction of ρ_i to C_l'. Since x is the trace of $\mu_1\mu_2$, from our convention and formula (6), it follows easily that $x = -2$ if the arcs of μ_1 and μ_2 cross C_l in the same direction, and $x = 2$ if they do it in opposite directions. Of course the same holds true for y and z with the respective oriented arcs. Since the values of x, y, z and $t = 0$ determine the character, this proves the compatibility.

To prove that the case $(-2, -2, -2, 0)$ does not occur, notice that it requires that L crosses at least three times C_l in the same direction; this is impossible, since the intersection number between L and C_l is zero. $\square$

As in previous section, $\mathcal{Y}$ denotes the subset of $X(C')$ obtained by requiring that the traces of all meridians are the same. We use the notation $\mathcal{Y}_l \subset X(C'_l)$ to refer to the l-th component C_l. Recall that $\chi_0 \in \mathcal{Y}_l$ denotes the character of $X(C'_l)$ such that $r_{il}(\chi_i) = \chi_0$.

Lemma 12. *Let $\{\chi_s \in \mathcal{Y}_l\}_{s\in[0,\varepsilon)}$ be a continuous deformation of χ_0 for which $t(s) \neq 0$ for all $s \in (0,\varepsilon)$. Assume that $\chi_s = r_{il}(\chi_{i,s})$ for all $s \in [0,\varepsilon)$, where $\{\chi_{i,s} \in X(M'_i)\}_{s\in[0,\varepsilon)}$ is a continuous deformation of χ_i. Then there exists an $s \in (0,\varepsilon)$ such that χ_s is irreducible.*

Proof. Assume by contradiction that all χ_s are reducible. Since χ_i is irreducible, and the set of irreducible characters is Zariski open, $\chi_{i,s}$ is irreducible $\forall s \in [0,\varepsilon)$. Thus, by Lemma 4, the deformation $\{\chi_{i,s} \in X(M'_i)\}_{s\in[0,\varepsilon)}$ lifts to a continuous deformation of representations in $R(M'_i)$. By considering its restriction to $\pi_1(C'_l)$, we get, for each s, a (reducible) representation $\rho_{0,s} \in R(C'_l)$ whose character is χ_s, and which depends continuously on s.

Up to symmetry, we may assume that the coordinates of χ_0 are $(x,y,z,t) = (-2,2,2,0)$. We have

$$\rho_{0,s}(\mu_i) = \begin{pmatrix} \lambda_i(s) & * \\ 0 & \lambda_i(s)^{-1} \end{pmatrix}$$

and $\lambda_1(0) = \lambda_2(0) = \frac{1}{\lambda_3(0)} = \frac{1}{\lambda_4(0)} = \pm i$. Recall that we are working in $\mathcal{Y}$, where the traces of the $\rho_{0,s}(\mu_i)$ are all the same. As a consequence, $\lambda_i(s) + \lambda_i(s)^{-1} = \lambda_j(s) + \lambda_j(s)^{-1}$ for all i and j. This means that $\lambda_i(s) = \lambda_j(s)^{\pm 1}$ for all i, j and $s \in (0,\varepsilon)$. For small $\varepsilon > 0$, by continuity we must have

$$\lambda_1(s) = \lambda_2(s) = \frac{1}{\lambda_3(s)} = \frac{1}{\lambda_4(s)},$$

for s in a neighbourhood of 0. By the relation between the $\lambda_i(s)$'s, we get $y^2(\chi_s) = z^2(\chi_s) = 4$ and $x^2(\chi_s)$ is non-constant, as t is non-constant either.

We project the characters $\chi_{i,s} \in X(M'_i)$ to $PSL(2,\mathbb{C})$ and we lift them again to $SL(2,\mathbb{C})$, continuously on s. All possible different lifts can be realised by maps from $\pi_1(M'_i)$ to $\mathbb{Z}/2$, by composing one given representation with an abelian representation into

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

One can easily prove the existence of maps $\pi_1(M'_i) \to \mathbb{Z}/2$ that are non trivial on two meridians of C'_l and are trivial on the other two meridians of C'_l. This will change the lift to one with coordinates $(2,-2,2,0)$, $(2,2,-2,0)$ or $(-2,-2,-2,0)$ (notice that here we only extend the character to M'_i not to the whole $\mathbf{S}^3 \setminus L$, and therefore the case$(-2,-2,-2,0)$ can occur). Assume first that the coordinates are $(x,y,z,t) = (2,-2,2,0)$. The previous discussion implies that $x^2(\chi_s) = z^2(\chi_s) = 4$ and $y^2(\chi_s)$ is non-constant, hence a contradiction, because x^2, y^2 and z^2 are functions that do not depend on the lift from $PSL(2,\mathbb{C})$ to $SL(2,\mathbb{C})$.

The case $(x,y,z,t) = (2,2,-2,0)$ being analogous, we next assume that $(x,y,z,t) = (-2,-2,-2,0)$. We repeat the construction of the representation of $\rho_{0,s}$ as above, with the difference that now

$$\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = \lambda_4(s),$$

which implies that $x^2(\chi_s) = y^2(\chi_s) = z^2(\chi_s)$, leading again to a contradiction. $\square$

6 A unique Conway sphere

The aim of this section is to study the case when the link admits a unique Conway sphere $C = C_1$. With the same notation as in the previous section, we shall denote by $\mathcal{O}_i$, $i = 1, 2$, the two orbifolds of the decomposition. Since the Bonahon-Siebenmann family must be contained in $\{C\}$, we see that the two orbifolds $\mathcal{O}_i$ are geometric. Note, moreover, that if both orbifolds are Seifert, their fibrations cannot match, for else the base of the global fibration would be large and the link would then admit some other Conway sphere. In particular C is precisely the only element of the Bonahon-Siebenmann family.

Lemma 13. *Assume $\mathcal{O}_i$ is a Seifert fibred orbifold, with Euclidean base. Then the associated tangle consists of two straight vertical arcs plus an unknotted circle isotopic to the equator.*

Proof. The base of the fibration is a 2-dimensional Euclidean orbifold. These are completely classified (see [17]), and we can check that there are only two planar orbifolds with a unique boundary component (non planar bases cannot correspond to fibrations of the ball). The first one is obtained in the following way: quotient a "pillowcase" by the reflection in a great circle containing its four cone points and cut in half the resulting orbifold. This way one obtains a rectangle with three mirrored sides and one boundary component, and two corner points (with angle $\pi/2$ and of dihedral type). The second one is obtained by quotienting a Möbius band by a reflection in a line orthogonal to its core. The result is a bigon with one mirrored side and one cone point of order 2 in its interior (note that the reflection fixes an extra point on the central curve of the Möbius band). Both base orbifolds can also be seen as the quotient by two different reflections of a disc with two cone points of order 2: the axis of the first reflection contains both cone points, while the second reflection exchanges them. It is not hard to see that these bases correspond to the two different fibrations of the tangle described in the statement of the lemma, which are pictured in Figure 2. Note that the extra closed component of the tangle is a fibre of the second fibration, corresponding to the cone point of the base orbifold. $\square$

By putting a branching index 3 on the closed component of the tangle of Lemma 13 we get the following:

Corollary 14. *We can modify the branching indices of the orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ so that:*

- *The singular components that meet the Conway sphere have branching index 2;*
- *The two resulting orbifolds, which will again be denoted by $\mathcal{O}_1$ and $\mathcal{O}_2$, are either hyperbolic or Seifert fibred with hyperbolic base.*

Remark 15. *From now on $\mathcal{O}_i$, $i = 1, 2$, will denote the orbifold with underlying topological space M_i and singular set $M_i \cap L$ with branching order 2 on the components meeting the boundary and 3 on the closed components. Remark*

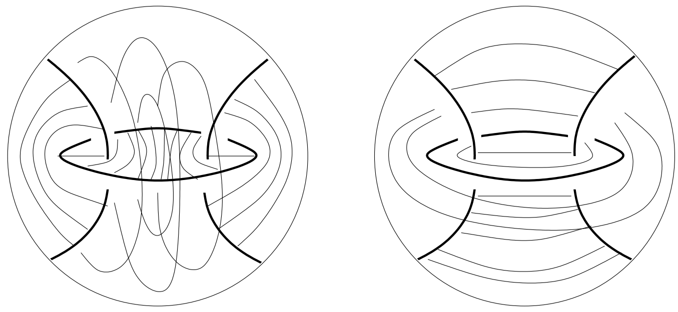

Figure 2: Two fibrations of the tangle with Euclidean base.

that now $\mathcal{O}_i$, $i = 1, 2$, satisfies the conclusions of the above corollary. We shall denote by $\mathcal{O}_i'$, $i = 1, 2$, the compact orbifold obtained from $\mathcal{O}_i$ by drilling out open regular neighbourhoods of the two singular components of branching index 2, i.e. those that meet the boundary.

Let $\chi_i \in X(\mathcal{O}_i')$ be a lift of the character induced by the holonomy of the geometric structure of $\mathcal{O}_i$, chosen so that it satisfies the compatibility condition established in Lemma 10. Notice that Culler's theorem [3] allows to lift the representation of $\pi_1(M_i')$, and since M_i' is obtained from $\mathcal{O}_i'$ by removing all components of order 3, the lift extends from M_i' to $\mathcal{O}_i'$, by changing the signs of the corresponding meridians as in the proof of Lemma 10, if needed.

Lemma 16. *The character χ_i is a smooth point in $X(\mathcal{O}_i')$. The traces of the two meridians and of a peripheral element of $\mathcal{O}_i$ define local coordinates in a neighbourhood of χ_i.*

Proof. For a manifold admitting an irreducible representation, Theorem 5.6 in Thurston's notes [16] states that the complex dimension of the character variety is at least $-3e/2 + \theta$ where e is the Euler characteristic of the boundary of the manifold and θ the number of tori in the boundary. If we fill some toric components with singular solid tori, in such a way that the resulting orbifold still admits an irreducible representation, then the same dimension bound still holds. This means that we can apply Thurston's result to conclude that the variety of characters of $\mathcal{O}_i'$ has dimension at least three. This gives a lower bound on the dimension.

To obtain an upper bound, we must work in the Zariski tangent space, i.e. the first cohomology group with coefficients in the Lie algebra twisted by the adjoint representation, that we denote by $H^1(\mathcal{O}_i'; \mathfrak{sl}(2,\mathbb{C}))$.

We first show that the variety of $PSL(2,\mathbb{C})$-characters $X(\mathcal{O}_i, PSL(2,\mathbb{C}))$ is one-dimensional, locally parametrised by deformations of its boundary $\partial\mathcal{O}_i$ (since $\pi_1(\mathcal{O}_i)$ has 2-torsion we must work in $PSL(2,\mathbb{C})$ instead of $SL_2(\mathbb{C})$). Moreover the Zariski tangent space at the holonomy character is also one dimensional, namely for the holonomy representation we have: $H^1(\mathcal{O}_i, \mathfrak{sl}(2,\mathbb{C})) \cong \mathbb{C}$. In the hyperbolic case, this follows from the proof of Thurston's hyperbolic Dehn filling theorem [16, 10] (see also [1] for the precise statement for orbifolds.) In the Seifert fibred case with hyperbolic base, since all irreducible representations factor through the base, it suffices to determine the variety of characters of the base (cf. [12, Lemma 3.4]). This is well-known, using the fact that the base is a small orbifold, cf. [6].

After lifting, the $PSL(2,\mathbb{C})$-character variety of $\mathcal{O}_i$ can be identified to the the subvariety of $X(\mathcal{O}_i')$ obtained by imposing that the traces of the meridians are zero, that is by intersecting the latter variety with two hyperplanes. Since $X(\mathcal{O}_i)$ is (non empty) of dimension 1, standard results on the dimension of intersections of algebraic varieties imply the dimension of $X(\mathcal{O}_i')$ is at most three. Similarly, since the tangent space to $X(\mathcal{O}_i, PSL(2,\mathbb{C}))$ at χ_i is one-dimensional, we see that the Zariski tangent space of $X(\mathcal{O}_i')$ at χ_i is again of dimension at most 3.

As a consequence, χ_i is a smooth point and the lemma follows. □

We consider the restriction maps

$$r_i\colon X(\mathcal{O}_i') \to X(C')$$

for $i = 1, 2$.

Lemma 17. *For a sufficiently small neighbourhood U_i of χ_i in $X(\mathcal{O}_i')$, $r_i(U_i) \cap \mathcal{Y}$ is a $\mathbb{C}$-analytic surface. Moreover the Zariski closure of $\mathrm{Im}(r_i) \cap \mathcal{Y}$ is two-dimensional.*

Proof. This follows from the description of the local coordinates around χ_i given in Lemma 16.

□

For hyperbolic orbifolds, Lemma 16 can also be understood geometrically, by applying the deformation theory of hyperbolic manifolds due to Weiss and Hodgson-Kerckhoff [9, 18].

Lemma 18. *The analytic germs $r_1(U_1) \cap \{t = 0\}$ and $r_2(U_2) \cap \{t = 0\}$ are curves. In addition χ_0 is an isolated point of their intersection.*

Proof. To prove that they are curves, notice that imposing $t = 0$ on a representation of $\mathcal{O}_i'$ implies that it factors through a representation of the orbifold $\mathcal{O}_i$. Thus $r_i(U_i) \cap \{t = 0\}$ is a curve, by the discussion on the variety of representations of $\mathcal{O}_i$ in the proof of Lemma 16 (for its restriction to $\pi_1(\partial\mathcal{O}_i)$ is non-trivial).

To prove why χ_0 is an isolated point, we need to understand how the deformations of representations of $\mathcal{O}_i$ are seen in $\partial\mathcal{O}_i$, i.e. the orbifold with underlying space C, branching locus $C \cap L$ and branching indices 2. We shall consider the induced representations on the index 2 subgroup of $\pi_1(\partial\mathcal{O}_i)$ consisting of all elements of infinite order. It is a characteristic subgroup corresponding to the cyclic branched covering of the torus onto the pillowcase. In particular, it makes sense to talk about slopes in $\partial\mathcal{O}_i$.

When $\mathcal{O}_i$ is Seifert fibred, its deformations are realised by perturbing the base, while keeping the fibre trivial. Thus the slope of $\partial\mathcal{O}_i$ corresponding to the fibre remains constant for every perturbation. This proves the lemma when both $\mathcal{O}_1$ and $\mathcal{O}_2$ are Seifert fibred, because their fibrations do not match, and two different slopes on the torus generate its fundamental group.

When $\mathcal{O}_i$ is hyperbolic, the tangent space to the curve of deformations depends on the cusp shape, as observed in the proof of Thurston's hyperbolic

Dehn filling. If $\langle m, l \mid [m,l] \rangle < \pi_1(\partial \mathcal{O}_i)$ is a presentation of the characteristic torsion-free subgroup of index two, then the cusp shape is the complex number τ such that

$$\rho(l) = \pm \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

where ρ is the holonomy representation for which

$$\rho(m) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this case, the deformation satisfies [5, Lemma 3.11]

$$\frac{d\,\mathrm{trace}(l)}{d\,\mathrm{trace}(m)} = \tau^2.$$

In particular this proves the lemma for the case of a Seifert fibred orbifold glued to a hyperbolic one, because no cusp shape τ can be zero.

Finally, assume that both $\mathcal{O}_i$'s are hyperbolic. When replacing l by $l\,m$, the previous formula becomes:

$$\frac{d\,\mathrm{trace}(l\,m)}{d\,\mathrm{trace}(m)} = (\tau+1)^2.$$

So the cusp shape τ determines the tangent direction of the curve of deformations, and different values of τ correspond to different directions of deformation. Hence, it suffices to show that the cusp shapes are different, for in that case the curves are transversal. In fact, the cusp shapes must be different because the induced orientations on the boundary are opposite, hence one of the cusp shapes has positive imaginary part, while the other one has negative imaginary part. □

Remark 19. *The proof of the previous lemma shows that the holonomy of $\mathcal{O}_1$ restricted to its boundary is not conjugate to the restriction to the boundary of the holonomy of $\mathcal{O}_2$. In particular, the induced representations on C' are not conjugate even if they have the same character.*

Proposition 20. *The intersection $r_1(U_1) \cap r_2(U_2) \cap \mathcal{Y}$ is a germ of an analytic curve on which t is not constant.*

Proof. Since the germ of $\mathcal{Y}$ at χ_0 is irreducible by Lemma 9, this is just a dimension count, using Lemmata 17 and 18. In addition, t is nonconstant, again by Lemma 18. □

Let $\mathcal{C}$ be the algebraic curve which is the Zariski closure of the intersection $r_1(U_1) \cap r_2(U_2) \cap \mathcal{Y}$. Note that it contains χ_0.

Lemma 21. *There exists an algebraic curve $\mathcal{D} \subset X(S^3 \setminus L)$ such that the restriction from $\pi_1(S^3 \setminus L)$ to $\pi_1(C')$ induces a non-trivial regular map $\mathcal{D} \longrightarrow \mathcal{C}$.*

Proof. Let $\mathcal{C}^{irr}$ denote the set of irreducible characters of $\mathcal{C}$, which is Zariski open. By Lemma 12, $\mathcal{C}^{irr} \neq \emptyset$. For $i = 1, 2$, let $\mathcal{C}_i \subset r_i^{-1}(\mathcal{C}) \subset X(\mathcal{O}_i')$ be an irreducible curve which contains the holonomy character χ_i of $\mathcal{O}_i$. If such curve is chosen generically then we can assume that $r_i \colon \mathcal{C}_i \to \mathcal{C}$ in non constant.

Let $p_i : X(S^3 \setminus L) \to X(M_i')$ be the morphism induced by the inclusion of fundamental groups. Notice that $X(M_i') \supset X(\mathcal{O}_i')$. We consider the algebraic set:

$$\mathcal{D} = \{\chi \in p_1^{-1}(\mathcal{C}_1) \cap p_2^{-1}(\mathcal{C}_2) \mid r_1(p_1(\chi)) = r_2(p_2(\chi))\}.$$

The intersection $Z = r_1(\mathcal{C}_1) \cap r_2(\mathcal{C}_2) \cap \mathcal{C}^{irr}$ is a nonempty (see Lemma 12) Zariski open subset of $\mathcal{C}$. In addition, every point of Z is the restriction of a *unique* character in $\mathcal{D}$, by Lemma 5. So if we choose an irreducible component of $\mathcal{D}$ containing a point with image in Z, then the restriction to this component of $r_1 \circ p_1 = r_2 \circ p_2$ is non-trivial and the component is a curve. □

Proof of Theorem 1. With the notation of the previous lemma, choose a sequence of points in $r_1(\mathcal{C}_1 \cap U_1) \cap r_2(\mathcal{C}_2 \cap U_2) \subset \mathcal{C}$ which converges to χ_0. The sequence lifts to sequences in $\mathcal{C}_i$ which converge to χ_i, $i = 1, 2$. Up to passing to a subsequence, all these sequences are induced by a sequence in $\mathcal{D}$. We claim that such sequence has no accumulation point in $\mathcal{D}$. This follows from the fact that the representations inducing the characters χ_i do not match (Remark 19). On the other hand, the restrictions of these characters to the M_i's are induced by representations that converge to the holonomy representations of $\mathcal{O}_i$. Now apply Lemma 7. □

7 Bonahon-Siebenmann families

Let L be a hyperbolic link in the 3-sphere and consider the orbifold whose underling topological space is $\mathbf{S}^3$ and whose singular set is L with branching indices of order 2 or 3. Assume that its Bonahon-Siebenmann decomposition is non trivial. Note that if we replace the ramification indices of order 3 by higher orders of ramification, the decomposition will still be the same, and even the type of geometric structure will not change. The Conway spheres of the decomposition cut the orbifold into pieces which are either hyperbolic or Seifert fibred. Note that we can increase the order of singularity of the components of L which do not meet the toric family without changing the decomposition (although the geometries involved may change) and so that no Seifert piece has a Euclidean base. This follows from Lemma 13 which holds for decompositions with an arbitrary number of pieces. From now on we shall thus assume, without loss of generality, that the order of singularity of the components which do not meet the Bonahon-Siebenmann family is 3, while the order of singularity of the remaining ones is 2.

Assume a Bonahon-Siebenmann family as above is given. Let $C_1, \ldots, C_k$ be the Conway spheres of the family. We denote by C_l', $l = 1, \ldots k$, the intersection of C_l with the exterior of the link. Let $M_1, \ldots, M_{k+1}$ be the $k + 1$ connected components of $S^3 \setminus (C_1 \cup \cdots \cup C_k)$ and $M_1', \ldots, M_{k+1}'$ their respective intersections with the exterior of the link. Finally let $\mathcal{O}_1, \ldots, \mathcal{O}_{k+1}$ denote the orbifolds with the chosen orders of ramification, corresponding to $M_1, \ldots, M_{k+1}$, and $\mathcal{O}_1', \ldots, \mathcal{O}_{k+1}'$ the orbifolds obtained by removing the branching components of order 2, i.e. those that meet $C_1 \cup \cdots \cup C_k$.

In the same spirit of what was done in the single Conway sphere case, we denote by $\mathcal{Y}_l$, $l = 1, \ldots, k$, the subvariety of $X(\mathcal{C}_l')$ obtained by imposing that all meridians have the same trace. It has the same equation as in (1).

We now need to prove analogues of Lemmata 16, 17 and 21 in this new setting.

For $i = 1, \dots, k+1$, let b_i the number of boundary components of M_i. Note that the number of arcs with ramification of order 2 in $\mathcal{O}_i$ is $2b_i$.

Lemmata 16 and 17 will be replaced by the following remark in the hyperbolic case and by the subsequent lemma which deals with the Seifert fibred case.

Remark 22. *Assume $\mathcal{O}_i$ is hyperbolic and let χ_i be a character in $X(\mathcal{O}'_i)$ that is a lift of the holonomy representation for $\mathcal{O}_i$. The very same proof of Lemma 16 shows that χ_i is a smooth point and the traces of the $2b_i$ meridians and b_i peripheral elements of $\mathcal{O}_i$, one for each boundary component, constitute a local coordinate system in a neighbourhood of χ_i. Lemma 17 (with dimension $b_i + 1$ instead of 2) follows at once.*

Assume now that $\mathcal{O}_i$ is Seifert fibred. The hyperbolic structure of the base may be not unique if the base is large for, in this case, the Teichmüller space has positive dimension. In particular, the choice for the holonomy character is not unique. However, it is easy to prove that there is a preferred structure. The base is a polygonal orbifold with mirror boundary, some of whose corners are cusps, and with at most one cone point in the interior. The preferred structure corresponds to the situation in which the base polygonal orbifold admits an inscribed circle tangent to every side. The character $\chi_i \in X(\mathcal{O}'_i)$ will denote a lift of the holonomy of this preferred structure for $\mathcal{O}_i$. The interest of this specific χ_i comes from the fact that it admits deformations which correspond to hyperbolic cone structure as proved in [12]. This means that χ_i is contained in an irreducible component W_i of the variety of $X(\mathcal{O}'_i)$ which contains points representing hyperbolic cone structures. Lemma 23 investigates some properties of the subvariety V_i of W_i, obtained by imposing that all meridians have the same trace.

Let $C_{l_1}, \dots, C_{l_{b_i}}$ be the boundary components of M_i. We shall follow the usual convention that the character variety of a disjoint union is the product of the character varieties of the components. Let

$$r_{\partial_i} : X(\mathcal{O}'_i) \longrightarrow X(C'_{l_1} \cup \dots \cup C'_{l_{b_i}}) = X(C'_{l_1}) \times \dots \times X(C'_{l_{b_i}})$$

be the projection induced by restriction.

Lemma 23. *For a sufficiently small neighbourhood U_i of χ_i in V_i, $r_{\partial_i}(U_i)$ is a $\mathbb{C}$-analytic (b_i+1)-variety and its Zariski closure is again (b_i+1)-dimensional.*

Remark 24. *When the base of the fibration of $\mathcal{O}_i$ is large, namely when the Teichmüller space of this base is non-trivial, the character χ_i is contained in another component of $X(\mathcal{O}'_i)$. This other component can be identified to the character variety of $\mathcal{O}_i$, whose points correspond to deformations of the base of the fibration, and it is locally the complexification of its Teichmüller space. In this other component t is constant, i.e. $t = 0$.*

Proof of Lemma 23. Consider the map $\pi : U_i \subset V_i \to \mathbb{C}^{b_i+1}$, whose components are t and the trace of peripheral elements corresponding to the boundary components of the base of the fibration, one for each boundary component of M_i. By the analogue of [12, Claim 6.6] (and with exactly the same proof),

$\pi^{-1}(\pi(\chi_i))) = \{\chi_i\}$. Since π factors through r_{∂_i}, by Remmert's proper map theorem $r_{\partial_i}(U_i)$ is a $\mathbb{C}$-analytic subvariety [13] (cf. [19, Thm. V.4.A] or [8, Thm. V.C.5]). In addition, by the openness principle, $\pi(U_i)$ is a neigbourhood of $\pi(\chi_i)$ in $\mathbb{C}^{b_i+1}$. This proves that $r_{\partial_i}(U_i)$ has dimension $b_i + 1$. □

Given $\chi_i \in X(\mathcal{O}'_i)$ a lift of the holonomy as above, we take V_i to be the irreducible subvariety defined as follows.

- When $\mathcal{O}_i$ is hyperbolic, V_i is defined to be the unique irreducible component of the subvariety of $X(\mathcal{O}'_i)$ obtained by imposing that the traces of all meridians are the same, and containing χ_i. This is well defined by Remark 22.
- When $\mathcal{O}_i$ is Seifert fibred, then V_i is as in Lemma 23.

By Remark 22 and Lemma 23, for every $i = 1, \dots, k+1$ there exists an open neighborhood $\chi_i \in U_i \subset V_i$ such that $r_{\partial_i}(U_i)$ is a $\mathbb{C}$-analytic $(b_i + 1)$-variety.

Now group the components $M_1, \dots, M_{k+1}$ in two families, so that if M_i and M_j are in the same family and $i \neq j$, then $M_i \cap M_j = \emptyset$. This is possible because the dual graph to the decomposition along Conway spheres is a tree. Up to permuting the indices, we may assume that the first family is $M_1, \dots, M_{k_0}$ and the second one $M_{k_0+1}, \dots, M_{k+1}$. We denote:

$$M_+ = M_1 \cup \cdots \cup M_{k_0} \quad \text{and} \quad M_- = M_{k_0+1} \cup \cdots \cup M_{k+1}.$$

Similarly one defines $\mathcal{O}_\pm$ and $\mathcal{O}'_\pm$. Notice that, since $\mathcal{O}'_\pm$ is not connected,

$$X(\mathcal{O}'_+) = X(\mathcal{O}'_1) \times \cdots \times X(\mathcal{O}'_{k_0}),$$

and analogously for $X(\mathcal{O}'_-)$. Let

$$\chi_+ = (\chi_1, \dots, \chi_{k_0}) \in X(\mathcal{O}'_+) \quad \text{and} \quad \chi_- = (\chi_{k_0+1}, \dots, \chi_{k+1}) \in X(\mathcal{O}'_-).$$

Remember that M_i has b_i boundary components, then

$$b_1 + \cdots + b_{k_0} = b_{k_0+1} + \cdots + b_{k+1} = k$$

Consider the product

$$\mathcal{Y} = \mathcal{Y}_1 \times \cdots \mathcal{Y}_k,$$

and the restriction maps whose components are the r_{∂_i}

$$r_\pm : X(\mathcal{O}'_\pm) \to \mathcal{Y}.$$

Take $V_+ = V_1 \times \cdots \times V_{k_0}$ and $V_- = V_{k_0+1} \times \cdots \times V_{k+1}$. By Remark 22 and Lemma 23, there exist $U_\pm \subset V_\pm$ neighbourhoods of $\chi_\pm$ such that $r_\pm(U_\pm)$ is an analytic variety of dimension $b_\pm$, where

$$b_+ = \sum_{i=1}^{k_0} (b_i + 1) = k + k_0 \quad \text{and} \quad b_- = \sum_{i=k_0+1}^{k+1} (b_i + 1) = 2k + 1 - k_0.$$

Notice that $b_+ + b_- = 3k + 1 = \dim \mathcal{Y} + 1$. Now the analogue of Lemma 18, that uses the same arguments and the results of Remark 22 and Lemma 23, is the following lemma.

Lemma 25. *The analytic varieties $r_+(U_+) \cap \{t = 0\}$ and $r_-(U_-) \cap \{t = 0\}$ have both dimension k. In addition, χ_0 is an isolated point of the intersection $r_+(U_+) \cap r_-(U_-) \cap \{t = 0\}$.*

Notice that the traces of meridians of $\mathcal{Y}_l$ and $\mathcal{Y}_{l'}$ are different variables, for $l \neq l'$. In $\mathcal{Y} \cap \{t = 0\}$ we require that they are all the same and equal to zero. In particular $\dim(\mathcal{Y} \cap \{t = 0\}) = 2k$. On the other hand, by construction, in $r_+(U_+) \cap r_-(U_-)$ the traces of the meridians of $\mathcal{Y}_l$ are the same.

As a corollary of Lemma 25, we obtain the analogue of Proposition 20, using Lemma 9.

Proposition 26. *The intersection $r_+(U_+) \cap r_-(U_-) \cap \mathcal{Y}$ is a curve on which t is not constant.*

Exactly the same proof as that of Lemma 21 gives:

Lemma 27. *There exists an algebraic curve $\mathcal{D} \subset X(S^3 \setminus L)$ such that the restriction from $\pi_1(S^3 \setminus L)$ to $\pi_1(C'_1), \ldots, \pi_1(C'_k)$ induces a non-trivial regular map $\mathcal{D} \longrightarrow \mathcal{C}$.*

Finally, the proof of Theorem 2 is the same as the proof of Theorem 1.

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