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On π-hyperbolic knots and branched coverings

Luisa Paoluzzi*

Abstract. We prove that, for any given $n > 2$, a π-hyperbolic knot is determined by its 2-fold and n-fold cyclic branched coverings. We also prove that a $2\pi/m$-hyperbolic knot which is not determined by its m-fold and n-fold cyclic branched coverings, $2 < m < n$, must have genus $(n-1)(m-1)/2$.

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1. Introduction

The aim of this work is to prove the following result:

Theorem 1. *Given $n \geq 3$, any Conway irreducible hyperbolic knot is determined by its 2-fold and n-fold cyclic branched coverings.*

A knot K is Conway irreducible if it does not admit any Conway sphere, i.e. a sphere S which meets K in four points and such that $S - \mathcal{U}(K)$ is incompressible and boundary incompressible in $\mathbf{S}^3 - \mathcal{U}(K)$, where $\mathcal{U}(K)$ is a tubular neighbourhood of K. A knot K is hyperbolic if its complement $\mathbf{S}^3 - \mathcal{U}(K)$ admits a complete hyperbolic structure of finite volume and it is $2\pi/n$-hyperbolic, $n \geq 2$, if the orbifold, whose underlying topological space is $\mathbf{S}^3$ and whose singular set of order n is K, is hyperbolic (for basic definitions about orbifolds see [12]). Equivalently, K is $2\pi/n$-hyperbolic if its n-fold cyclic branched covering is a hyperbolic manifold and the covering group acts by isometries.

We want to reduce the statement of Theorem 1 to a simpler form. To this purpose we prove:

Proposition 1. *Theorem 1 is equivalent to the following:*

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Theorem 2. *Let K and K' be two π-hyperbolic and $2\pi/n$-hyperbolic knots, $n \geq 3$. If K and K' have the same 2-fold and n-fold cyclic branched coverings then K and K' are equivalent, i.e. the pairs $(\mathbf{S}^3, K)$ and $(\mathbf{S}^3, K')$ are homeomorphic.*

Proof: According to Thurston's orbifold geometrization theorem [13], [14] (see also [3] for a proof in the case of good orbifolds of cyclic type, those we are dealing with) and to the classification of 3-dimensional Euclidean crystallographic groups (compare Dunbar's list of non-hyperbolic orbifolds with underlying space $\mathbf{S}^3$ in [5]), the class of hyperbolic knots coincides with that of $2\pi/n$-hyperbolic knots for all $n \geq 3$, with the unique exception, in the case $n = 3$, of the figure-eight knot 4_1. Observe now that a closed 3-manifold is geometric if and only if it is atoroidal. This means that the 2-fold cyclic branched covering M of a hyperbolic knot K is geometric if and only if the knot is Conway irreducible. Indeed, the only possible images of incompressible tori of M in the orbifold $(\mathbf{S}^3, K)$ are Conway spheres, since K is hyperbolic and thus atoroidal (see [4] for the characteristic toric decomposition of an orbifold). In particular, a hyperbolic knot is π-hyperbolic only if it is Conway irreducible. This proves that Theorem 1 implies Theorem 2. On the other hand, a Conway irreducible hyperbolic knot, which is not π-hyperbolic, admits a Seifert fibred 2-fold cyclic branched covering. Thurston's orbifold geometrization theorem ensures the existence of a Seifert fibration for the manifold M which is preserved by the action of the covering involution. Involutions of this type were studied by Montesinos in [7]. The possible quotient knots, which are also hyperbolic, are either 2-bridge knots or Montesinos knots with at most three tangles. Hodgson and Rubinstein proved that 2-bridge knots are determined by their 2-fold cyclic branched coverings [6]. On the contrary, the 2-fold cyclic branched covering of a Montesinos knot can be also the 2-fold cyclic branched covering of a torus knot. However, for $n \geq 3$, the n-fold cyclic branched covering of the hyperbolic Montesinos knot is hyperbolic because of Thurston's orbifold geometrization theorem, while the the n-fold cyclic branched covering of the torus knot is Seifert fibred. This shows that, for any given $n \geq 3$, a Conway irreducible hyperbolic knot which is not π-hyperbolic is determined by its 2-fold and n-fold cyclic branched coverings.

Remark that the proof of Theorem 2 does not require Thurston's orbifold geometrization theorem any longer. Under the extra assumption n even, Theorem 2 was proved by Zimmermann. In fact, Zimmermann proved more generally that a $2\pi/n$- and $2\pi/m$-hyperbolic knot is determined by its n-fold and m-fold cyclic branched coverings provided that n and m are not coprime [17]. We are then left to consider only the case n odd of the Theorem.

The proof of the Theorem relies on the following facts:

i) The existence of a minimal genus H-equivariant Seifert surface for a knot K and any given finite group H of symmetries of the knot [15].

Remark. Throughout the paper the expression *symmetry of a knot* K will stand for finite order diffeomorphism of the pair $(\mathbf{S}^3, K)$ preserving the orientation of $\mathbf{S}^3$.

ii) The positive solution of the Smith conjecture; i.e. if a diffeomorphism of $\mathbf{S}^3$ of finite order fixes pointwise a link L, then L is the trivial knot [8].

iii) A $2\pi/n$-hyperbolic knot K which is not determined by its n-fold cyclic branched covering M, $n \geq 3$, admits a symmetry of order n with non-empty fixed point set A such that the quotient of K under the action of the symmetry is the trivial knot. Let $\bar{K}$ and $\bar{A}$ be the images of K and A in this quotient: $\bar{K} \cup \bar{A}$ is a two trivial component link. If n is not a power of 2 there are at most two $2\pi/n$-hyperbolic knots K and K' with the same n-fold cyclic branched covering M. K' is the preimage of $\bar{A}$ in the cyclic covering of $\mathbf{S}^3$ branched along $\bar{K}$. These results are due to Zimmermann [17] and their proof is based on certain considerations on the Sylow subgroups of $Iso_+(M)$ and on the Smith conjecture.

Notice that the fact that a π-hyperbolic knot is determined by its 2-fold and n-fold cyclic branched coverings for n odd is quite peculiar. Indeed, it was proved by Zimmermann [16] that, for any two coprime integers $n > m > 2$, there exist arbitrarily many pairs of $2\pi/m$- and $2\pi/n$-hyperbolic knots with the same m-fold and n-fold cyclic branched coverings. As a corollary of the proof of the Theorem (see Lemma in Section 2) we obtain that all the $2\pi/m$- and $2\pi/n$-hyperbolic knots which are not determined by both their m-fold and n-fold cyclic branched coverings have the same genus, which is equal to $(m-1)(n-1)/2$. Finally, observe that there are infinitely many sets of four (resp. three) different π-hyperbolic knots with the same 2-fold cyclic branched covering (see [9], [16]), so that the 2-fold cyclic branched covering alone is not sufficient to determine a π-hyperbolic knot.

To conclude this Section, we want to remark that the problem solved in this paper was originally suggested by the following question put by Boileau and Flapan in [2]:

> *"Is there an integer $n \geq 3$ such that any two prime knots having the same m-fold cyclic branched coverings for $2 \leq m \leq n$ are necessarily equivalent?"*

Notice that, if we restrict our attention to the class of Conway irreducible hyperbolic knots, the answer to this question is positive and one can choose $n = 3$ which is obviously the best possible. However, we can also consider the class of all hyperbolic knots and in this case the answer is still positive. It is easy to see that $n = 5$ is sufficient to determine a hyperbolic knot although perhaps not best possible. Indeed, suppose that a hyperbolic knot K is not determined by its 2-fold, 3-fold, 4-fold and 5-fold cyclic branched coverings. In particular K is not determined by its 4-fold and 5-fold cyclic branched coverings and K is $\pi/2$- and $2\pi/5$-hyperbolic because of Thurston's orbifold geometrization theorem. Because of the above remark on the genus of a $2\pi/m$- and $2\pi/n$-hyperbolic knot which is not determined by both their m-fold and n-fold cyclic branched coverings, K has genus 6. This means that K is not the figure-eight knot and thus it is $2\pi/3$-hyperbolic. Again, since it is not determined by its 3-fold and 4-fold cyclic branched coverings its genus must be 3 which is absurd.

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2. Proof of Theorem 2

Throughout the paper we shall use the following conventions:
p_* denotes the covering projection induced by the covering transformation $*$ (i.e. $*$ generates the group of covering transformations);
$Fix(*)$ denotes the fixed-point set of the map $*$.
Let K be a knot and h a symmetry of K with $Fix(h) \neq \varnothing$; in particular, because of Smith conjecture, $Fix(h)$ is the trivial knot. If $Fix(h) \cap K = \varnothing$ and the order of h is n we say that h is an *n-periodic symmetry*. If $Fix(h) \cap K \neq \varnothing$, then it consists of two points, the order of h is 2 and we say that h is a *strong inversion*. A knot is *strongly invertible* if it admits a strong inversion. For other basic definitions about knots, the reader is referred to [10].

Assume that K and K' are two π- and $2\pi/n$-hyperbolic knots with the same 2-fold and n-fold cyclic branched coverings for some n odd. Denote by M the hyperbolic manifold which is the common 2-fold cyclic branched covering of K and K'. The orbifold hyperbolic structure of $(\mathbf{S}^3, K)$ and the one of $(\mathbf{S}^3, K')$ induce hyperbolic structures on the manifold M. By Mostow's rigidity theorem, these two structures on M coincide (for the Mostow's rigidity theorem and other basic facts in hyperbolic geometry see [1]). This means that the we can choose two elements τ and $\tau' \in Iso_+(M)$ such that the quotient of M with respect to the action of τ (resp. τ') is $\mathbf{S}^3$ branched along K (resp. K'). It is easy to see that, if K and K' are distinct, τ and τ' cannot be conjugate. Since $Iso_+(M)$ has finite order, τ and τ' generate a dihedral group where the element $(\tau\tau')$ has even order, say, $2d$; else τ and τ' would be conjugate. Define $r := (\tau\tau')^d$. By [17, Corollary 1] both K and K' admit n-periodic symmetries $\bar{h}$ and $\bar{h}'$ respectively whose actions on K and K' give the trivial knot. Let h and h' be lifts of these symmetries in $Iso_+(M)$ and note that τ and h (resp. τ' and h') commute. Note that $p_{\bar{h}}(K \cup Fix(\bar{h})) = p_{\bar{h}'}(Fix(\bar{h}') \cup K')$ is a two trivial non exchangeable component link [17, Theorem 1]. Indeed if the components were exchangeable K and K' would coincide.

Remark now that $p_h(M) = \mathbf{S}^3$ since it is the 2-fold cyclic covering of $\mathbf{S}^3$ branched along one of the two trivial components of $p_{\bar{h}}(K \cup Fix(\bar{h}))$. In particular

M is the n-fold cyclic covering of $\mathbf{S}^3$ branched along L which is either a knot, if the linking number of the two components of $p_{\bar{h}}(K \cup Fix(\bar{h}))$ is odd, or else a two component link.

Repeating the same reasoning for h' we obtain another link L' with the same number of components of L. It is clear that L and L' are $2\pi/n$-hyperbolic.

The idea is now to study these n-fold cyclic branched coverings which are better understood and easier to handle than the 2-fold cyclic branched coverings (compare Zimmermann's result stated in iii) Section 1). We distinguish two cases according to if the groups generated by h and h' are conjugate or not. We shall see that these two different algebraic situations geometrically stand for the cases when L and L' are links or knots respectively.

Case A: *The groups generated by h and h' are not conjugate in $Iso(M)$.*

Let q be any maximal prime power divisor of n and consider the cyclic groups of order q $\langle h^{n/q}\rangle$ and $\langle h'^{n/q}\rangle$. These groups cannot be conjugate in $Iso(M)$ else the element conjugating the first to the second would map $Fix(h) = Fix(h^{n/q})$ to $Fix(h') = Fix(h'^{n/q})$ and conjugate the group $\langle h\rangle$ to the group $\langle h'\rangle$. In particular, for each prime p dividing n, the p-Sylow subgroup of $\langle h\rangle$ is a proper subgroup of a p-Sylow subgroup of $Iso(M)$.

Claim 1. *L and L' are knots.*

Suppose, on the contrary, that L and L' are two component links. We need the following result whose proof can be found in [11, Chapter 2, 1.5].

Proposition 2. *Let H be a subgroup of a finite p-group S; then either H is normal in S or a conjugate subgroup sHs^{-1} of H, different from H, is contained in the normalizer $N_S(H)$ of H in S.*

Let p be a fixed prime divisor of n and q its maximal power dividing n. By the above discussion, Proposition 2 applies to $H := \langle h^{n/q}\rangle$ and S a p-Sylow subgroup of $Iso(M)$ containing $\langle h^{n/q}\rangle$. We can assume, up to conjugation, that $h'^{n/q}$ belongs to such p-Sylow subgroup. According to Proposition 2, either $h'^{n/q}$ or $sh^{n/q}s^{-1}$ normalizes $h^{n/q}$ and thus normalizes h. It is worth remarking that the groups generated by $h^{n/q}$ and by $sh^{n/q}s^{-1}$ have trivial intersection for, otherwise, their fixed-point set would coincide implying that the two groups would coincide. Remark moreover that $Fix(h'^{n/q})$ and $Fix(sh^{n/q}s^{-1})$ are both non-empty with exactly two components. This means that there is an element $\eta \in Iso(M)$ of order q which normalizes $\langle h\rangle$ and induces a q-periodic symmetry $\bar{\eta}$ of L; this symmetry cannot be a strong inversion because q is odd. $Fix(\bar{\eta})$ is a trivial knot by Smith conjecture and $p_h^{-1}(Fix(\bar{\eta})) = \cup_{i=1}^{n} h^i(Fix(\eta))$. Clearly the number of components of $p_h^{-1}(Fix(\bar{\eta}))$ divides n. Now we want to prove that $h^i(Fix(\eta))$ and $h^j(Fix(\eta))$ either coincide or are disjoint. Since the number of components of $h^i(Fix(\eta))$ is two for all i while n is odd, we reach a contradiction. Indeed, assume that $h^i(Fix(\eta))$ and $h^j(Fix(\eta))$ have a common component. Then both $h^i\eta h^{-i}$ and $h^j\eta h^{-j}$ would act locally as rotations along such component (and project on $\bar{\eta}$). Since $h^i\eta h^{-i}$ and $h^j\eta h^{-j}$ are isometries, the groups they generate would coincide and their fixed-point sets as well.

Claim 2. *K admits two n-periodic symmetries induced by h and h'.*

Since the groups generated by h and h' are not conjugate, by [17, pages 668-669] we have that, up to conjugation, h and h' commute. We also know that h and τ commute thus both h' and τ preserve $Fix(h)$ and generate a group of isometries, which does not contain h, isomorphic to $\mathbb{Z}_{2n}$, for τ is not an inversion of $Fix(h)$. We have that h' and τ commute and h and h' project to distinct n-periodic symmetries of K.

Claim 3. *The two n-periodic symmetries induced by h and h' must coincide.*

Let $\hat{h}'$ the isometry induced by h' on the orbifold $(\mathbf{S}^3, K)$. Obviously $\hat{h}'$ must preserve K. By Smith conjecture $\hat{h}'$ cannot fix K pointwise so it must act as an n-periodic symmetry of K. Since $\bar{h}$ and $\hat{h}'$ have the same order and act by rotating the knot K, there exist a number t, prime with n, such that $\bar{h}^t\hat{h}'$ fixes K pointwise. By Smith conjecture we obtain that the group generated by $\bar{h}$ and that generated by $\hat{h}'$ coincide.

This contradiction shows that Case A cannot occur.

Case B: *The groups generated by h and h' are conjugate in $Iso(M)$.*

In this case we can assume $h = h'$ (up to a conjugation and perhaps a change of generator in one of the two groups) and we have the group $\langle \tau, \tau', h\rangle \cong \mathbf{D}_{2d} \oplus \mathbb{Z}_n$. Since τ and r commute, r must project to a symmetry $\bar{r}$ of K.

Claim 4. *$\bar{r}$ is a 2-periodic symmetry and $p_{\bar{r}}(K)$ is the trivial knot.*

Indeed, remark that $\bar{r}$ lifts to r and $r\tau$ in $Iso_+(M)$ and that $r\tau$ is conjugate either to τ (if d is even) or to τ' (if d is odd). Thus $r\tau$ and consequently $\bar{r}$ have non-empty fixed point set. Since $\bar{r}$ and $\bar{h}$ commute, $\bar{r}$ cannot be a strong inversion. Now τ and $r\tau$ commute and so τ induces a 2-periodic symmetry $\bar{\tau}$ of $p_{r\tau}(M)$ (this orbifold is either $(\mathbf{S}^3, K)$ if d is even or $(\mathbf{S}^3, K')$ if d is odd). Clearly $Fix(\bar{\tau})$ is the trivial knot because of Smith conjecture. By commutativity $p_{\bar{r}}(K) = p_{\bar{r}}p_\tau(Fix(\tau))$ and $p_{\bar{\tau}}(Fix(\bar{\tau})) = p_{\bar{\tau}}p_{r\tau}(Fix(\tau))$ coincide thus proving that $p_{\bar{r}}(K)$ is the trivial knot.

We have thus seen that K admits a 2-periodic symmetry $\bar{r}$ and an n-periodic symmetry $\bar{h}$ such that both $p_{\bar{r}}(K)$ and $p_{\bar{h}}(K)$ are the trivial knot. Note that $\bar{r}$ and $\bar{h}$ generate a cyclic group of order $2n$. We can then apply the following Lemma to prove that the genus of K must be $(n-1)/2$ and L is a two component link.

Lemma. *Let H be a cyclic group acting smoothly and orientation preservingly on $\mathbf{S}^3$ and leaving a non trivial knot K invariant. Assume that two elements $h, h' \in H$ are respectively an n-periodic symmetry and an m-periodic symmetry of K, with $1 < m < n$. If both $p_h(K)$ and $p_{h'}(K)$ are the trivial knot, then the genus g of K satisfies $g = (n-1)(m-1)/2$ and m and n are coprime.*

Proof: Since H is finite we can assume that it acts as a group of isometries for some Riemannian metric on $\mathbf{S}^3$.

Claim 5. *There exists F a Seifert surface of minimal genus g for K which is invariant by the action of h and h'.*

The proof of the Claim follows easily from the existence of a minimal genus Seifert surface which is H-equivariant [15]; one must only observe that $Fix(h)$ and $Fix(h')$ are non empty and must intersect F. Notice, that since F has minimal genus and since $p_h(K)$ and $p_{h'}(K)$ are trivial knots, $p_h(F)$ and $p_{h'}(F)$ have genus 0. In particular if $Fix(h)$ and $Fix(h')$ did not intersect F, F would have genus 0, which is impossible.

We distinguish two cases:

a) $Fix(h) = Fix(h')$.

Let k be the number of points in the intersection $Fix(h) \cap F$. The Riemann-Hurwitz formula yields:

$$(n-1)(k-1) = 2g = (m-1)(k-1).$$

If $k = 1$ the genus of the knot is 0 and the knot is trivial, against the hypothesis. Then it must be $n = m$ and this is again against the hypothesis. So this case cannot happen.

b) $Fix(h) \cap Fix(h') = \varnothing$*; in particular n and m are coprime.*

Let k be the number of points in the intersection $Fix(h) \cap F$ and k' the number of points in the intersection $Fix(h') \cap F$. Applying as above the Riemann-Hurwitz formula we have:

$$(n-1)(k-1) = 2g = (m-1)(k'-1).$$

Note that h' (resp. h) acts freely on the k (resp. k') points of intersection of F with $Fix(h)$ (resp. $Fix(h')$), so that $k = mv$ and $k' = nv'$. Notice moreover that h (resp. h') induces an n-periodic (resp. m-periodic) symmetry on the quotient of $(\mathbf{S}^3, K)$ by the action of h' (resp. h) by commutativity. We can now apply once more the Riemann-Hurwitz formula obtaining

$$(n-1)(v-1) = 0 = (m-1)(v'-1)$$

from which one deduces $k = m$, $k' = n$ and $2g = (n-1)(m-1)$.

Assume now that the two fixed-point sets intersect but do not coincide. Since H is a group of isometries, the intersection has exactly two points and one of the two symmetries must be a strong inversion for the fixed-point set of the other. In this case, however the group would be dihedral and not cyclic. Since the above are the only possible cases, this proves the Lemma.

Let us go back to the proof of the Theorem. We have the following commutative diagram of coverings:

$$\begin{array}{ccccc} (\mathbf{S}^3, K) & \stackrel{p_\tau}{\longleftarrow} & M & \stackrel{p_{\tau'}}{\longrightarrow} & (\mathbf{S}^3, K') \\ \downarrow p_{\bar{h}} & & \downarrow p_h & & \downarrow p_{\bar{h'}} \\ (\mathbf{S}^3, p_{\bar{h}}(K \cup Fix(\bar{h}))) & \stackrel{p_{\bar\tau}}{\longleftarrow} & (\mathbf{S}^3, L) & \stackrel{p_{\bar{\tau'}}}{\longrightarrow} & (\mathbf{S}^3, p_{\bar{h'}}(K' \cup Fix(\bar{h'}))) \end{array}$$

where $\bar\tau$ and $\bar{\tau'}$ denote the projections of τ and τ' on the orbifold $p_h(M)$, $\bar{h}$ and $\bar{h}'$ the projections of h on the orbifolds $(\mathbf{S}^3, K)$ and $(\mathbf{S}^3, K')$ respectively and $p_{\bar{h}}(K \cup Fix(\bar{h})) = p_{\bar{h'}}(Fix(\bar{h}') \cup K')$ are the same link (where components are taken in the order).

Claim 6. *There exists a minimal genus Seifert surface F for K which is preserved by the action of $\bar{h}$ such that the number of points of intersection of F and $Fix(\bar{h})$ is* 2.

This is a consequence of the second part of the proof of the Lemma. The linking number of the two components of $p_{\bar{h}}(K \cup Fix(\bar{h}))$ is congruent modulo 2 to the intersection number of F and $Fix(\bar{h})$, which is even. This implies in particular that L is a two component $2\pi/n$-hyperbolic link. One can reach the same conclusion also by considering the group $\mathbf{D}_{2d} \oplus \mathbb{Z}_n$, generated by τ, τ' and h, which preserves $Fix(h)$ but cannot be a group of symmetries of a geodesic.

Claim 7. *The link L has genus* 0.

Consider $p_{\bar{h}}(F)$: this is a disk with boundary $p_{\bar{h}}(K)$ intersecting $p_{\bar{h}}(Fix(\bar{h}))$ in two points. Since the two links $p_{\bar{h}}(K \cup Fix(\bar{h}))$ and $p_{\bar{h'}}(Fix(\bar{h}') \cup K')$ are the same, $p_{\bar{h}}(F)$ is also a disk with boundary $p_{\bar{h'}}(Fix(\bar{h}'))$ intersecting $p_{\bar{h'}}(K')$ in two points. The lift of this disk to the orbifold $(\mathbf{S}^3, L)$, i.e. $p_{\bar{\tau'}}^{-1} p_{\bar{h}}(F)$, is a Seifert surface for the link L. Again an easy computation with the Riemann-Hurwitz formula yields that its genus is 0. Clearly L must have genus 0.

A hyperbolic ($2\pi/n$-hyperbolic) two component link cannot have genus 0 because it is anannular. This final contradiction proves Theorem 2.

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Luisa Paoluzzi
Laboratoire de Mathématiques E. Picard
Université Paul Sabatier
118, route de Narbonne
F-31062 Toulouse – cédex 4
France
e-mail: paoluzzi@picard.ups-tlse.fr

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