Finite group actions on 3-manifolds and cyclic branched covers of knots

M. Boileau*, C. Franchi, M. Mecchia†, L. Paoluzzi*, and B. Zimmermann†

January 22, 2018

Abstract

As a consequence of a general result about finite group actions on 3-manifolds, we show that a hyperbolic 3-manifold can be the cyclic branched cover of at most fifteen inequivalent knots in $\mathbf{S}^3$ (in fact, a main motivation of the present paper is to establish the existence of such a universal bound). A similar, though weaker, result holds for arbitrary irreducible 3-manifolds: an irreducible 3-manifold can be a cyclic branched cover of odd prime order of at most six knots in $\mathbf{S}^3$. We note that in most other cases such a universal bound does not exist.

AMS classification: Primary 57S17; Secondary 57M40; 57M60; 57M12; 57M25; 57M50.

Keywords: Finite group actions on 3-manifolds; cyclic branched covers of knots; geometric structures on 3-manifolds; hyperbolic 3-manifolds.

1 Introduction

A classical way to construct closed, connected, orientable 3-manifolds is to consider cyclic covers of the 3-sphere branched along knots. A natural question in this setting is to understand in how many different ways a closed, connected, orientable 3-manifold can be presented as the (total space of a) cyclic branched cover of a knot. There is an extensive literature on this problem, mainly focussing on the case where the branching order is fixed. For instance, it is known that a closed hyperbolic 3-manifold is the n-fold cyclic branched cover of at most two knots in $\mathbf{S}^3$, provided $n > 2$ [Z2], and at most nine if $n = 2$ [Re]. For arbitrary closed, connected, orientable, irreducible 3-manifolds some results are known but only for prime orders of ramification. More precisely, such manifolds can be 2-fold cyclic branched covers of arbitrary many knots [Mon1, Mon2] but can cover at most two knots if the order is an odd prime [BoPa]. For branching order equal to 2, several analyses of how the different quotient knots are related can also be found (see, for instance, [MR] for hyperbolic manifolds, [Mon1, Mon2, V, MonW] for toroidal ones, [P] for Conway reducible hyperbolic knots, and [Gr] for alternating ones).

*Partially supported by ANR project 12-BS01-0003-01

†Partially supported by the FRA 2015 grant "Geometria e topologia delle varietà ed applicazioni", Università di Trieste

Possibly due to the fact that the knots that are covered by the same manifold but with different orders of branching are harder to relate, not much was known so far on the general problem, even for hyperbolic manifolds (see [RZ]). The prime motivation for this work was to understand whether it is possible to establish bounds on the number of ways a 3-manifold can be presented as a cyclic branched cover of a knot, without fixing the order of the cover. By the above discussion, no universal bound can be given in general, however the first main result of the present paper assures the existence of a universal bound for the class of closed hyperbolic 3-manifolds.

Theorem 1. *A closed hyperbolic* 3*-manifold is a cyclic branched cover of at most fifteen inequivalent knots in* $\mathbf{S}^3$.

We call two knots *equivalent* if one is mapped to the other by an orientation-preserving diffeomorphism of $\mathbf{S}^3$. In the present paper, all manifolds are closed, connected and orientable, and all maps are smooth and orientation-preserving.

The orientation-preserving isometry group of a closed hyperbolic 3-manifold M is finite, and every finite group occurs for some hyperbolic M. Suppose that M is a cyclic branched cover of a knot in $\mathbf{S}^3$; then the group of covering transformations acting on M is generated by a hyperelliptic rotation: we call a periodic diffeomorphism of a closed 3-manifold a *hyperelliptic rotation* if all of its non-trivial powers have connected, non-empty fixed-point set (a simple closed curve), and its quotient (orbit) space is $\mathbf{S}^3$. By the geometrization of 3-orbifolds, or of finite group actions on 3-manifolds ([BLP], [BoP], [DL]), the group of covering transformations is conjugate to a subgroup of the isometry group of M. Hence establishing a universal upper bound for hyperbolic 3-manifolds as in Theorem 1 is equivalent to bounding the number of conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of the isometry group of a hyperbolic 3-manifold. Now Theorem 1 is a consequence of the following more general result on finite group actions on closed 3-manifolds.

Theorem 2. *Let M be a closed* 3*-manifold not homeomorphic to* $\mathbf{S}^3$*. Let G be a finite group of orientation-preserving diffeomorphisms of M. Then G contains at most fifteen conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation (at most six for cyclic subgroups whose order is not a power of two).*

Note that the 3-sphere is the n-fold cyclic branched cover of the trivial knot for any integer $n \geq 2$ (and, by the solution to the Smith conjecture, only of the trivial knot). It is well-known that, for any branching order n, a 3-manifold can be the n-fold cyclic branched cover of an arbitrary number of non-prime knots (such a manifold is not irreducible), and that an irreducible 3-manifold can be the 2-fold branched cover of arbitrarily many prime knots (see [Mon1, Mon2, V]). Moreover, Proposition 7 in Section 6.1 shows that Seifert fibred manifolds can cover an arbitrary number of knots, all with non-prime orders. It is thus natural to restrict our attention to covers of odd prime order.

For irreducible 3-manifolds, the following holds:

Theorem 3. *Let M be a closed, irreducible* 3*-manifold. Then there are at most six inequivalent knots in* $\mathbf{S}^3$ *having M as a cyclic branched cover of odd prime order.*

The proof of Theorem 3 uses Theorem 2 in connection with the equivariant torus-decomposition of irreducible 3-manifolds into geometric pieces, see [BoP], [BLP], [CHK], and [KL]. For arbitrary 3-manifolds, as a direct consequence of Theorem 3 as well as the equivariant prime decomposition for 3-manifolds [MSY], the following remains true.

Corollary 1. *Let M be a closed 3-manifold not homeomorphic to $\mathbf{S}^3$. Then M is a p-fold cyclic branched cover of a knot in $\mathbf{S}^3$ for at most six distinct odd prime numbers p.*

Another consequence of Theorem 2 is the following characterisation of the 3-sphere which generalises the main result of [BPZ] from the case of $\mathbb{Z}$-homology 3-spheres to arbitrary closed 3-manifolds.

Corollary 2. *A closed 3-manifold M is homeomorphic to $\mathbf{S}^3$ if and only if there is a finite group G of orientation-preserving diffeomorphisms of M such that G contains sixteen conjugacy classes of subgroups generated by hyperelliptic rotations.*

A noteworthy aspect of the proof of Theorem 2 is the substantial use of finite group theory, in particular of the classification of finite simple groups. For a prime p, the p-fold cyclic branched cover of a knot in $\mathbf{S}^3$ is a rational homology 3-sphere, and we will prove in Section 7 that every finite group acts non-freely on some rational homology 3-sphere, so the use of the classification seems to be intrinsic to the proofs of our results. We note that the class of finite groups acting on a $\mathbb{Z}/2$-homology 3-sphere instead is quite restricted (see [MZ]), and in this case the much shorter Gorenstein-Harada classification of finite simple groups of sectional 2-rank at most four is sufficient for our proofs. The bounds that can be derived for $\mathbb{Z}/2$-homology 3-spheres are, however, precisely the same as those we get for arbitrary manifolds.

We have tried to separate the algebraic, purely group theoretical parts of the proof (Section 4) from the topological parts (Sections 3 and 5), so they can be read independently.

We note that, although the upper bounds in our results are quite small, at this point we do not know if they are really optimal. In the hyperbolic case, one can easily construct manifolds that are covers of orders > 2 of three distinct knots [RZ]. For hyperbolic double covers, the bound (nine) is sharp according to a result of Kawauchi [Ka], but no explicit examples are known so far. For general irreducible manifolds, Brieskorn spheres of type $\Sigma(p, q, r)$, where p, q, and r are three pairwise different odd primes, provide examples of manifolds that cover four knots: $\Sigma(p, q, r)$ is the p-fold (resp. q-fold and r-fold) cyclic cover of $\mathbf{S}^3$ branched along the torus knot $T(q, r)$ (resp. $T(p, r)$ and $T(p, q)$) as well as the double branched cover of a Montesinos knot.

The paper is organised as follows. In Section 2 we present a brief sketch of the proof of Theorem 2. Hyperelliptic rotations and their properties are considered in Section 3. Section 4 contains the main group-theoretical part of the paper, Section 5 the proof of Theorem 2, and Section 6 the proof of Theorem 3 for the irreducible case. Finally, in an Appendix we prove that every finite group acts non-freely on some rational homology sphere (adapting the result of [CL] that deals with free actions).

2 Sketch of the proof of Theorem 2

The proof of Theorem 2 is based on a series of preliminary results which are presented in Sections 3 and 4. Our choice to present the group-theoretical part of the proof in a separate section (Section 4) allows for the group-theoretical results to be read independently of the other parts of the paper. In the following, in order to make the paper more accessible, we explain the main steps of the proof.

We begin with a more detailed definition of hyperelliptic rotation. Note that throughout the paper, unless otherwise stated, 3-*manifold* will mean orientable, connected, closed 3-manifold. Also, all finite group actions by diffeomorphisms will be faithful and orientation-preserving.

Definition 1. Let $\psi : M \longrightarrow M$ be a finite order diffeomorphism of a 3-manifold M. We shall say that ψ is a *rotation* if it preserves the orientation of M, $Fix(\psi)$ is non-empty and connected, and $Fix(\psi) = Fix(\psi^k)$ for all non-trivial powers ψ^k of ψ. $Fix(\psi)$ will be referred to as the *axis of the rotation.* Note that if ψ is a periodic diffeomorphism of prime order, then ψ is a rotation if and only if $Fix(\psi) = \mathbf{S}^1$. We shall say that a rotation ψ is *hyperelliptic* if the space of orbits M/ψ of its action is $\mathbf{S}^3$, and a *hyperelliptic group* is a cyclic group generated by a hyperelliptic rotation.

We start by observing that the case of hyperelliptic rotations whose order is a power of two is already well-understood by work of Reni and Mecchia (see [Re] and [Mec1]). In particular, there are at most nine conjugacy classes of cyclic groups generated by such hyperelliptic rotations. As a consequence, from now on, we exclude this case and consider only hyperelliptic groups whose order is not a power of two.

Section 3 collects various simple facts on the geometry of hyperelliptic rotations. In particular, we prove that there are at most three hyperelliptic groups commuting pairwise. This result implies the existence of a universal bound in the solvable case. In fact, in the solvable case, the presence of Hall subgroups assures that all hyperelliptic subgroups commute, up to conjugacy, implying that there at most three conjugacy classes of such groups.

The case of non-solvable groups, where local approaches on the basis of p-groups fail, is more involved. We need a global description of the groups that may arise, which is provided in Section 4. In that section, we introduce the notion of an *algebraically hyperelliptic collection* of cyclic subgroups which have the same algebraic properties as the hyperelliptic subgroups; this allows a purely algebraic approach in Section 4. The main result there (Theorem 4) is that a non-solvable finite group generated by an algebraically hyperelliptic collection is of a very special type, in particular G has a quotient by a normal solvable subgroup which is isomorphic to the direct product of at most two simple groups.

The next step is to cover the hyperelliptic subgroups by a bounded number of conjugacy classes of solvable subgroups (i.e., to find a collection of solvable subgroups such that each element of odd prime order in a hyperelliptic subgroup has a conjugate in one of these solvable subgroups); this concept of a solvable cover of a finite group is central for the proofs in the present paper since a bound on the number of elements of such a cover implies that there is a bound on the number of conjugacy classes of hyperelliptic subgroups. Using the classification of finite simple groups, we prove, in Proposition 4, that in any finite simple group the hyperelliptic subgroups can be covered, up to conjugacy, by at most four solvable subgroups. This result, together with the characterisation of groups generated by algebraically hyperelliptic collections, gives directly the existence of a universal bound of fifty-seven, much larger than the bound of fifteen obtained in Theorem 2 by exploiting extra topological considerations. Note that the existence of a universal (although non explicit) bound is ensured by the existence of only a finite number of simple sporadic groups.

In Section 5 we conclude the proof of Theorem 2. We can suppose that G is generated by an algebraically hyperelliptic collection and that G is not solvable. Under these hypotheses, the proof is divided into two cases.

In the first case, we suppose that G contains no rotation of order two. By geometric motivations, an involution acting dihedrally by conjugation on a hyperelliptic subgroup is a rotation. The absence of this type of involutions induces further restrictions on the structure of G; in particular, up to a quotient by a solvable subgroup, G is a single simple group. By using these properties of G and the solvable covers we prove that four is the upper bound in this case.

In the second case, G contains a rotation of order two. The groups that may act on 3-manifolds containing such an involution are listed in [Mec2]. We combine this with our result about groups generated by an algebraically hyperelliptic collection and we obtain that a quotient of G by a solvable subgroup must be isomorphic to one of $\mathbb{A}_8$, $PSL_2(q)$ or $PSL_2(q) \times PSL_2(q')$. For these groups we explicitly find a solvable cover with a bounded number of elements and hence a universal bound as in Theorem 2.

3 Rotations and their properties

In this section we shall establish some properties of rotations in general and hyperelliptic ones in particular.

Remark 1. Assume that ψ is a hyperelliptic rotation acting on a 3-manifold M. The natural projection from M to the space of orbits M/ψ of ψ is a cyclic cover of $\mathbf{S}^3$ branched along a knot $K = Fix(\psi)/\psi$. The converse is also true, that is any deck transformation generating the automorphism group of a cyclic covering of $\mathbf{S}^3$ branched along a knot is a hyperelliptic rotation.

We observe that cyclic branched covers of prime order are closely related to $\mathbb{Q}$-homology 3-spheres.

Remark 2.

1. If the order of ψ is a prime p, then M is a $\mathbb{Z}/p$-homology sphere [Go].
2. By Smith theory, if f is a periodic diffeomorphism of order p, a prime number, acting on a $\mathbb{Z}/p$-homology sphere, then f either acts freely or is a rotation.

We start with a somehow elementary remark which is however central to determine constraints on finite groups acting on 3-manifolds.

Remark 3. Let $G \subset Diff^+(M)$ be a finite group of diffeomorphisms acting on a 3-manifold M. One can choose a Riemannian metric on M which is invariant by G and with respect to which G acts by isometries. Let now $\psi \in G$ be a rotation. Since the normaliser $\mathcal{N}_G(\langle\psi\rangle)$ of ψ in G consists precisely of those diffeomorphisms that leave the circle $Fix(\psi)$ invariant, we deduce that $\mathcal{N}_G(\langle\psi\rangle)$ is a finite subgroup of $\mathbb{Z}/2 \ltimes (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$, where the nontrivial element in $\mathbb{Z}/2$ acts by conjugation sending each element of $\mathbb{Q}/\mathbb{Z}\oplus\mathbb{Q}/\mathbb{Z}$ to its inverse. Note that the elements of $\mathcal{N}_G(\langle\psi\rangle)$ are precisely those that rotate about $Fix(\psi)$, translate along $Fix(\psi)$, or invert the orientation of $Fix(\psi)$; in the last case the elements have order 2 and non-empty fixed-point set meeting $Fix(\psi)$ in two points.

Remark 4. Let us consider the 3-sphere $\mathbf{S}^3$. According to Smith's theory, an orientation-preserving finite-order diffeomorphism of $\mathbf{S}^3$ is a rotation if and only if its fixed-point set is non-empty. Because of the positive solution to the Smith conjecture the fixed-point set of a rotation of $\mathbf{S}^3$ is the trivial knot. Morover, any group of symmetries of a non-trivial knot K (that is, any finite group of orientation-preserving diffeomorphisms of $\mathbf{S}^3$ acting on the pair $(\mathbf{S}^3, K)$) is either cyclic or dihedral.

Lemma 1. *Let φ and ψ be two rotations contained in a finite group of orientation-preserving diffeomorphisms of a 3-manifold M.*

1. *A non-trivial power of ψ of order different from 2 commutes with a non-trivial power of φ, if and only if φ and ψ commute.*
2. *Assume $M \neq \mathbf{S}^3$. If φ and ψ are hyperelliptic and $Fix(\varphi) = Fix(\psi)$, then $\langle\varphi\rangle = \langle\psi\rangle$ (in particular they have the same order).*
3. *Assume $M \neq \mathbf{S}^3$. If φ and ψ are hyperelliptic, then $\langle\varphi\rangle$ and $\langle\psi\rangle$ are conjugate if and only if some non-trivial power of φ is conjugate to some non-trivial power of ψ.*

Proof.
Part 1 The sufficiency of the condition being obvious, we only need to prove the necessity. Remark that we can assume that both rotations act as isometries for some fixed Riemannian metric on the manifold. Denote by f and g the non trivial powers of φ and ψ, respectively. Note that, by definition, $Fix(\psi) = Fix(g)$ and $Fix(\varphi) = Fix(f)$. Since g and f commute, g leaves invariant $Fix(\varphi) = Fix(f)$ and thus normalises every rotation about $Fix(\varphi)$. Moreover g and φ commute, for the order of g is not 2 (see Remark 3). In particular, φ leaves $Fix(\psi) = Fix(g)$ invariant and normalises every rotation about $Fix(\psi)$. The conclusion follows.
Part 2 Reasoning as in Part 1, one sees that the two rotations commute. Assume, by contradiction, that the subgroups they generate are different. Under this assumption, at least one of the two subgroups is not contained in the other. Without loss of generality we can assume that $\langle\varphi\rangle \not\subset \langle\psi\rangle$. Take the quotient of M by the action of ψ. The second rotation φ induces a non-trivial rotation of $\mathbf{S}^3$ which leaves the quotient knot $K = Fix(\psi)/\psi \subset \mathbf{S}^3$ invariant. Moreover, this induced rotation fixes pointwise the knot K. The positive solution to the Smith conjecture implies now that K is the trivial knot and thus $M = \mathbf{S}^3$, against the hypothesis.
Part 3 follows from 2 since the conjugate of a hyperelliptic rotation is again a hyperelliptic rotation. □

Corollary 3. *Let G be a finite group of orientation-preserving diffeomorphisms acting on a 3-manifold $M \neq \mathbf{S}^3$. Let ψ be a rotation and let $f \in G$ be an element* *of odd prime order which is a power of ψ. Then we have $\mathcal{C}_G(\langle f\rangle) = \mathcal{C}_G(\langle\psi\rangle)$ and $\mathcal{N}_G(\langle f\rangle) = \mathcal{N}_G(\langle\psi\rangle)$.*

There is a natural bound on the number of hyperelliptic subgroups of order not a power of two which commute pairwise; we begin analysing the situation of the symmetry group of a knot.

Definition 2. A *rotation of a knot K* in $\mathbf{S}^3$ is a rotation ψ of $\mathbf{S}^3$ such that $\psi(K) = K$ and $K \cap Fix(\psi) = \emptyset$. We shall say that ψ is a *full rotation* if K/ψ in $\mathbf{S}^3 = \mathbf{S}^3/\psi$ is the trivial knot.

Remark 5. Let ψ and φ be two commuting rotations acting on some manifold M and with orders not both equal to 2. Assume that ψ is hyperelliptic and φ is not a power of ψ. According to Remark 3 we have two situations: Either $Fix(\psi) \cap Fix(\varphi) = \emptyset$ and φ induces a rotation ϕ of $K = Fix(\psi)/\psi$, or $Fix(\psi) = Fix(\varphi)$. In the former situation we have that φ is hyperelliptic if and only if ϕ is a full rotation. This can be shown by considering the quotient of M by the action of the group generated by ψ and φ. This quotient is $\mathbf{S}^3$ and the projection onto it factors through M/φ, which can be seen as a cyclic cover of $\mathbf{S}^3$ branched along K/ϕ. By the positive solution to the Smith conjecture, M/φ is $\mathbf{S}^3$ if and only if K/ϕ is the trivial knot. In the latter situation, again because of the positive solution to Smith's conjecture, we have $M = \mathbf{S}^3$ (compare also Part 2 of Lemma 1).

The following finiteness result about commuting rotations of a non-trivial knot in $\mathbf{S}^3$ is one of the main ingredients in the proof of Theorem 2 (see [BoPa, Proposition 2], and [BoPa, Theorem 2] for a stronger result where commutativity is not required).

Proposition 1. *Let K be a non-trivial knot in $\mathbf{S}^3$. Let us consider a set of pairwise commuting full rotations in $Diff^+(\mathbf{S}^3, K)$. The elements of the set generate at most two pairwise distinct cyclic subgroups.*

Proof. Note, first of all, that, according to Remark 4 the finite subgroup of $Diff^+(\mathbf{S}^3, K)$ generated by a finite set of pairwise commuting full rotations is cyclic.

Assume now, by contradiction, that there are three pairwise distinct cyclic subgroups generated by commuting full rotations of K, φ, ψ and ρ respectively. Note that such cyclic subgroups have distinct orders. Assume that two of them -say φ, ψ- have the same axis. Fix the one with smaller order -say ψ-: since ψ is a full rotation, the quotient K/ψ is the trivial knot, and φ induces a rotation of K/ψ which is non-trivial since φ commutes with ψ and its order is larger than that of ψ. The axis A of this induced symmetry is the image of $Fix(\psi)$ in the quotient $\mathbf{S}^3/\psi$ by the action of ψ. In particular K/ψ and A form a Hopf link and K is the trivial knot: this follows from the equivariant Dehn lemma, see [Hil].

We can thus assume that the axes are pairwise disjoint. Indeed, this follows from Remark 3 taking into account that the rotations commute pairwise and at most one of them has order 2. Therefore we would have that the axis of ρ, which is a trivial knot, admits two commuting rotations, φ and ψ, with distinct axes, which is impossible: this follows, for instance, from the fact (see [EL, Thm 5.2]) that one can find a fibration of the complement of the trivial knot which is equivariant with respect to the two symmetries. $\square$

Observe that the proof of the proposition shows that two commuting full rotations of a non-trivial knot either generate the same cyclic subgroup or have disjoint axes.

We stress that if a knot $K \subset \mathbf{S}^3$ admits a full rotation, then it is a prime knot, see [BoPa, Lemma 2].

Lemma 2. *Let G be a finite group of diffeomorphisms acting on a 3-manifold $M \neq \mathbf{S}^3$ and $\{H_1, \ldots, H_m\}$ be a set of hyperelliptic subgroups of G of order not a power of two. Suppose that there exists an abelian subgroup of G, containing at least an element of odd prime order of each H_i, then $m \leq 3$. Moreover either the orders of the H_i are pairwise coprime or $m \leq 2$.*

Proof.

By Lemma 1 we obtain that the subgroups H_i commute and have trivial intersection pairwise. Consider the cyclic branched covering $M \to M/H_1 \cong \mathbf{S}^3$ over the knot $Fix(H_1)/H_1$. By projecting H_i with $i \geq 2$ to M/H_1 we obtain full rotations of $Fix(H_1)/H_1$. Note that if two subgroups H_i have the same order that is moreover a divisor of the order of H_1, a priori they might map to the same subgroup in $\langle H_1, \ldots, H_m \rangle / H_1$. We claim, however, that the induced full rotations are distinct so there are $m-1$ of them. This follows from the fact that, for different indices i and j, $Fix(H_i)$ and $Fix(H_j)$ are disjoint according to Part 2 of Lemma 1. Now, since the subgroups commute, for all $i = 1, \ldots, m$ we have $H_1(Fix(H_i)) = Fix(H_i)$, so that the fixed-point sets of the induced full rotations are disjoint, too, and the full rotations are pairwise distinct. By Proposition 1 we obtain $m - 1 \leq 2$. Note that, by Remark 4 a non-trivial knot cannot admit two distinct and commuting cyclic groups of symmetries of the same order. This proves the latter part of the lemma. $\square$

The above lemma implies directly the following corollary.

Corollary 4. *Let p be an odd prime and assume that $H \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$ acts on a 3-manifold $M \neq \mathbf{S}^3$, the group H contains at most two distinct cyclic subgroups generated by non-trivial powers of hyperelliptic rotations.*

We now collect several facts relative to hyperelliptic rotations that can be deduced from the discussion in this section.

Remark 6. Let G be a finite group of orientation-preserving diffeomorphisms of a 3-manifold $M \neq \mathbf{S}^3$. Let $\psi \in G$ be a hyperelliptic rotation of order not a power of 2. Recall that the structures of the centraliser and of the normaliser of $\langle \psi \rangle$ are described in Remark 3. Let N denote the normaliser $N_G(\langle \psi \rangle)$.

1. The centraliser $\mathcal{C}_G(\langle \psi \rangle)$ of ψ in G satisfies $1 \longrightarrow \langle \psi \rangle \longrightarrow \mathcal{C}_G(\langle \psi \rangle) \longrightarrow H \longrightarrow 1$, where H is cyclic, possibly trivial. This follows from Remark 4.

2. Since the symmetries of a knot not acting freely have connected fixed-point set (see again Remark 4), the possible elements of $N \setminus \mathcal{C}_G(\langle \psi \rangle)$ are rotations of order two.

3. According to last part of Lemma 2, the centraliser $\mathcal{C}_G(\langle \psi \rangle)$ contains at most one more cyclic subgroup of the same order as $\langle \psi \rangle$ and generated by a hyperelliptic rotation.

4. Observe that we have $\mathcal{N}_G(N) = \mathcal{N}_G(\mathcal{C}_G(\langle \psi \rangle))$.

5. Of course, the conjugate of a hyperelliptic rotation is again a hyperelliptic rotation. Consider an element $g \in \mathcal{N}_G(N)$. Then either g conjugates $\langle\psi\rangle$ to itself and so g belongs to N, or g does not normalise $\langle\psi\rangle$. In this case, $\mathcal{C}_G(\langle\psi\rangle)$ contains precisely two cyclic subgroups generated by hyperelliptic rotations of the same order as ψ ($\langle\psi\rangle$ and another one) and g exchanges them. It follows that $\mathcal{C}_G(\langle\psi\rangle)$ has index a divisor of 4 in $\mathcal{N}_G(N)$.

6. Let $f \in G$ be an element of odd prime order which is a power of ψ. Then we have $\mathcal{C}_G(\langle f\rangle) = \mathcal{C}_G(\langle\psi\rangle)$ and $\mathcal{N}_G(\langle f\rangle) = N$. This is just Corollary 3.

The following general observation will be useful in the sequel.

Remark 7. With the notation used in the previous remark, assume that there are elements in $\mathcal{N}_G(N)$ which do not act in the same way on $\mathcal{C}_G(\langle\psi\rangle)$, then necessarily the index of N in $\mathcal{N}_G(N)$ is two and $\mathcal{C}_G(\langle\psi\rangle)$ contains two distinct subgroups conjugate to $\langle\psi\rangle$.

We end this section by providing a dictionary translating between algebraic properties of the structure of $\mathcal{N}_G(N)$ and the symmetries of the knot $K = Fix(\psi)/\psi$. This will not be needed in the proofs of our results but will provide a geometric interpretation of the different situations that occur in the proof of Theorem 2 (see Remark 9). We start with the following definition.

Definition 3. A 2-component link is called *exchangeable* if there exists an orientation-preserving diffeomorphism of $\mathbf{S}^3$ which exchanges the two components of the link.

Let K be a knot and ρ a rotation of K of order n and with axis A. Consider the 2-component link $\overline{K} \cup \overline{A}$ consisting of the images of the knot K and of the axis A in the quotient $\mathbf{S}^3/\rho$ of the 3-sphere by the action of ρ. Note that at least one component of this link (i.e. $\overline{A}$) is trivial. We call K *n-self-symmetric* if $\overline{K} \cup \overline{A}$ is exchangeable. In this case ρ is a full rotation of K.

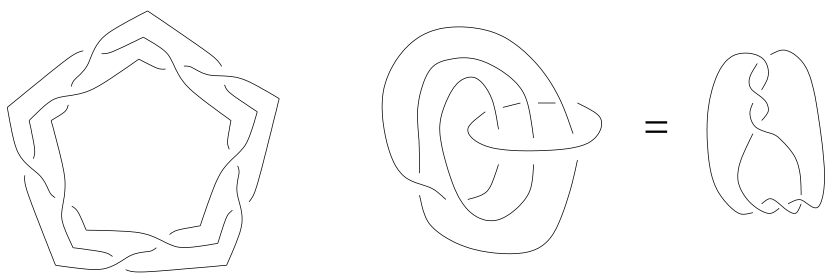

Figure 1: A 5-self-symmetric knot on the left, and its exchangeable quotient link on the right.

Since the structure of the normaliser of $\langle\psi\rangle$ and of its centraliser, only depend on the symmetries of K that lift to G, we introduce the following definitions:

Definition 4. Let G be a finite group of orientation preserving diffeomorphisms of a closed connected 3-manifold M. Let ψ be a hyperelliptic rotation contained in G with quotient knot K. We say that K is *strongly invertible with respect* *to G* if K admits a strong inversion that lifts to G. Similarly we say that K is *self-symmetric with respect to G* if G contains an element ψ' conjugate to ψ such that the subgroup $\langle \psi, \psi' \rangle$ is abelian of rank 2, i.e. not cyclic. Remark that in the latter case K is n-self-symmetric, where n is the order of ψ.

The proof of the following facts is elementary and left to the reader.

Proposition 2. *Let K, ψ, N, and G be as above.*

- *The centraliser of $\langle \psi \rangle$ in G is contained with index 2 in N if and only if K is strongly invertible with respect to G.*
- *N is contained with index 2 in $\mathcal{N}_G(N)$ if and only if K is self-symmetric with respect to G.*

Moreover, if M is hyperbolic and $G = Iso^+(M)$, K is strongly invertible if and only if it is strongly invertible with respect to G, and it is self-symmetric with respect to G if and only if it is n-self-symmetric, where n is the order of the hyperelliptic rotation ψ.

4 Group theoretical results

This section is devoted to the proofs of the group-theoretical results that are used to obtain the bound provided by Theorem 2.

The first result, Theorem 4, is probably the more interesting of the two from a group-theoretical point of view. It describes the finite groups G that may be generated by hyperelliptic rotations. In this settings, a hyperelliptic rotation is simply an element of G of order not a power of two, satisfying a purely algebraic condition (see Definition 5, below) on the structure of the normaliser of (a power of odd prime order of) the element. More precisely, Theorem 4 states that either G is solvable, or it admits a quotient by a normal solvable subgroup so that the quotient is either the product of a simple group times a cyclic group of odd order, or the product of two simple groups. We see that the fact of being generated by hyperelliptic rotations puts very strong constraints on the structure of G.

The second result, Proposition 4, although possibly not as striking as the previous one, is however key to be able to carry out the strategy of bounding the number of hyperelliptic subgroups by covering them with solvable subgroups (see Definition 6). Indeed, in Proposition 4 we establish the fact that four conjugacy classes of solvable subgroups are sufficient to contain, up to conjugacy, all hyperelliptic subgroups of orders not a power of two that may sit inside a finite simple group.

Note that these two main results imply that 4×4 solvable subgroups suffice to cover all hyperelliptic groups in G, and together with Lemma 3, and Lemma 2 show that a finite group G can contain at most forty-eight conjugacy classes of hyperelliptic subgroups of orders that are not powers of 2.

In the following G will denote a finite group. We start with some preliminary observations and definitions.

Definition 5. A collection $\{C_i\}$ of subgroups of G, each of odd prime order p_i, is said to be *algebraically hyperelliptic* if, for each i, the following conditions are satisfied:

1. the centraliser of C_i in G is abelian of rank at most two and has index at most two in the normaliser of C_i in G;

2. each element belonging to the normaliser of C_i but not to the centraliser inverts by conjugation each element in the centraliser;

3. if C_i is contained in a Sylow p_i-subgroup S_i of G, then S_i contains at most two distinct conjugates of C_i.

We remark that in this definition the primes p_i are not necessarily pairwise distinct.

Proposition 3. *Let S be a Sylow p_i-subgroup where p_i is the order of a group C_i belonging to an algebraically hyperelliptic collection.*

1. *S is either cyclic or the product of two cyclic groups, and*

2. *$\mathcal{N}_G(S)$ contains with index at most 2 the normaliser of a conjugate of C_i, and contains an abelian subgroup of rank at most 2 with index a divisor of 4. In particular $\mathcal{N}_G(S)$ is solvable.*

Proof.
Up to conjugation we can suppose that S contains C_i. By Properties 1 and 2 in Definition 5 and the fact that p_i is odd, the normaliser $\mathcal{N}_S(C_i)$ is abelian of rank at most two. Property 3 in Definition 5 implies that $\mathcal{N}_S(\mathcal{N}_S(C_i)) = \mathcal{N}_S(C_i)$. Since S is a p_i-subgroup we obtain that $S = \mathcal{N}_S(C_i)$ and we get the first part of the thesis.

Since the subgroup $\mathcal{N}_G(S)$ normalises the maximal elementary abelian subgroup of S, we obtain also the second part of the thesis. $\square$

Lemma 3. *Let G be a solvable group containing an algebraically hyperelliptic collection $\{C_1, \cdots, C_m\}$ of subgroups of odd prime order. Then there exists an abelian subgroup of G containing a conjugate of C_i, for each $i = 1, \ldots, m$. In particular the subgroups C_i commute pairwise up to conjugacy.*

Proof. Let π be the set of the orders of the C_i and let B be a Hall π-subgroup of G. Each C_i is conjugate to a subgroup of B. Since π contains only odd primes, Definition 5 yields that centraliser and normaliser of every Sylow p-subgroup of B coincide. By Burnside's p-complement theorem (see [Su, Theorem 2.10 page 144]), every Sylow p-subgroup of B has a normal complement, and hence B is abelian. $\square$

Remark 8. Suppose that N is a normal subgroup of G and H is a p-subgroup of G. If the order of N is coprime with p, then the normaliser of the projection of H to G/N is the projection of the normaliser of H in G, that is

$$\mathcal{N}_{G/N}(HN/N) = \mathcal{N}_G(H)N/N.$$

The inclusion $\supseteq$ holds trivially. We prove briefly the other inclusion. Let fN be an element of $\mathcal{N}_{G/N}(HN/N)$, then $H^f \subseteq HN$. Both H^f and H are Sylow

p-subgroups of HN and by the second Sylow theorem they are conjugate by an element $hn \in HN$. We obtain that $H^{fhn} = H$, and hence $f \in \mathcal{N}_G(H)N$.

Analogously if f is an element of prime order coprime with the order of N, then $\mathcal{C}_{G/N}(fN) = \mathcal{C}_G(f)N/N$.

Recall that a finite group Q is *quasisimple* if it is perfect (the abelianised group is trivial) and the factor group $Q/\mathcal{Z}$ of Q by its centre $\mathcal{Z}$ is a nonabelian simple group (see [Su, chapter 6.6]). A group E is *semisimple* if it is perfect and the factor group $E/\mathcal{Z}(E)$ is a direct product of nonabelian simple groups. A semisimple group E is a central product of quasisimple groups which are uniquely determined. Any finite group G has a unique maximal semisimple normal subgroup $E(G)$ (maybe trivial), which is characteristic in G. The subgroup $E(G)$ is called the *layer* of G and the quasisimple factors of $E(G)$ are called the *components* of G.

The maximal normal nilpotent subgroup of a finite group G is called the *Fitting subgroup* and is usually denoted by $F(G)$. The Fitting subgroup commutes elementwise with the layer of G. The normal subgroup generated by $E(G)$ and by $F(G)$ is called the *generalised Fitting subgroup* and is usually denoted by $F^*(G)$. The generalised Fitting subgroup has the important property to contain its centraliser in G, which thus coincides with the centre of $F^*(G)$. For further properties of the generalised Fitting subgroup see [Su, Section 6.6.].

Theorem 4. *Suppose that G is generated by the algebraically hyperelliptic collection $\mathcal{H} := \{C_1, \cdots, C_m\}$. Denote by p_i the order of C_i and by A the maximal normal solvable subgroup of order coprime with every p_i.*

If G is non-solvable, then the following properties hold:

1. *every p_i divides the order of any component of G/A;*

2. *either G/A is the direct product of a cyclic group of odd order and a simple group or it is the direct product of two simple groups;*

3. *if in addition G does not contain any involution acting dihedrally on any C_i, then $E(G/A)$ is simple, every p_i divides the order of $F(G/A)$ and a Sylow p_i-subgroup of G contains exactly two conjugates of C_i.*

Proof. Let π be the set of the primes p_i.

By Remark 8 we can suppose that A is trivial and $F(G)$ is a π-group.

Claim 1. *$F(G)$ is cyclic and $E(G)$ is not trivial.*

Suppose first that $F(G)$ contains an abelian p_i-subgroup S of rank two. Then S, being the maximal elementary abelian p_i-subgroup contained in $F(G)$, is normal in G and contains C_i. This implies that G is solvable and we get a contradiction. Hence $F(G)$ is cyclic.

If $E(G)$ is trivial, then $F(G) = F^*(G)$ and $G/F(G)$ is isomorphic to a subgroup of $\mathrm{Aut}F(G)$. Since $F(G)$ is cyclic $\mathrm{Aut}F(G)$ is solvable, and we get again a contradiction.

Claim 2. *Each p_i divides the order of any component of G. Moreover the components of G are simple groups and are at most two.*

Since the Sylow p_i-subgroups are abelian and A is trivial, by [Su, Exercise 1, page 161] the components of G have trivial centre.

Now we prove that each component of G is normalised by any C_i. Let f_i be a generator of C_i and Q a component of G. Suppose by contradiction that Q is not normalised by f_i. We define the following subgroup:

$$Q_c = \{x f_i x f_i^{-1} \dots f_i^{p_i-1} x f_i^{p_i-1} \mid x \in Q\}.$$

Since the components of G commute elementwise, Q_c is a subgroup of G isomorphic to Q. Moreover, each element of Q_c commutes with f_i and this gives a contradiction.

We have that C_i normalises Q but cannot centralise it, so the action by conjugation of f_i on Q is not trivial.

Assume that Q is either sporadic or alternating. Since the order of the outer automorphism group of any such simple group is a (possibly trivial) power of 2 (see [GLS, Section 5.2 and 5.3]), we conclude that f must induce an inner automorphism of Q. In particular p_i divides the order of Q.

We can thus assume that Q is a simple group of Lie type.

Recall that, by [GLS, Theorem 2.5.12], $Aut(Q)$ is the semidirect product of a normal subgroup $Inndiag(Q)$, containing the subgroup $Inn(Q)$ of inner automorphisms, and a group $\Phi\Gamma$, where, roughly speaking, Φ is the group of automorphisms of Q induced by the automorphisms of the defining field and Γ is the group of automorphisms of Q induced by the symmetries of the Dynkin diagram associated to Q (see [GLS] for the exact definition). By [GLS, Theorem 2.5.12.(c)], every prime divisor of $|Inndiag(Q)|$ divides $|Q|$. Thus we can assume that the automorphism induced by f_i on Q is not contained in $Inndiag(Q)$ and its projection θ on $Aut(Q)/Inndiag(Q) \cong \Phi\Gamma$ has order p_i. We will find a contradiction showing that in this case the centraliser of f_i in Q is not abelian.

Write $\theta = \phi\gamma$, with $\phi \in \Phi$ and $\gamma \in \Gamma$. If $\phi = 1$, then γ is nontrivial and f_i induces a graph automorphism according to [GLS, Definition 2.5.13]. Since p_i is odd, the only possibility is that Q is $D_4(q)$ and $p_i = 3$ (see [GLS, Theorem 2.5.12 (e)]). The centraliser of f_i in Q is nonabelian by [GLS, Table 4.7.3 and Proposition 4.9.2.]. If $\phi \neq 1$ and Q is not isomorphic to the group ${}^3D_4(q)$, then the structure of the centraliser of f_i in Q is described by [GLS, Theorem 4.9.1], and it is nonabelian. Finally, if $\phi \neq 1$ and $Q \cong {}^3D_4(q)$, the structure of the non abelian centraliser of f_i in Q follows from [GLS, Proposition 4.2.4]. We proved that the automorphism induced by f_i is contained in $Inndiag(Q)$ and p_i divides $|Q|$.

Since p_i divides the order of any component, G has at most two components.

Claim 3. $G = E(G)F(G)$.

We prove first that $G = E(G)\mathcal{C}_G(E(G))$. Let us assume by contradiction that there exists C_i with trivial intersection with $E(G)\mathcal{C}_G(E(G))$ and denote by f a generator of C_i. Since p_i divides the order of every component of G and the Sylow p_i-subgroup has rank at most 2, we get that $E(G)$ has only one component which we denote by Q. The Sylow p_i-subgroups of Q are cyclic. Moreover, by the first part of the proof, the automorphism induced by f on Q is inner-diagonal. If it is inner, we obtain f as a product of an element that centralises Q and an element in Q, a contradiction to our assumption; otherwise, we get again contradiction, since, by [GLS, Theorem 2.5.12] and [A, (33.14)], a group of Lie type with cyclic Sylow p_i-subgroup cannot have a diagonal automorphism of order p_i.

Hence, all the subgroups C_i are contained in $E(G)\mathcal{C}_G(E(G))$ and, since they generate G, we obtain that $G = E(G)\mathcal{C}_G(E(G))$.

Now if $F(G) = 1$, then $F^*(G) = E(G)$ and hence $\mathcal{C}_G(E(G)) = \mathcal{C}_G(F^*(G)) = \mathcal{Z}(E(G)) = 1$ and the claim is proved. So suppose that $F(G) \neq 1$. Then, since $F(G)$ is a π-group, there is at least one subgroup C_i that is contained in $E(G)F(G)$. Hence there is a subgroup T_1 of $E(G)$ with order p_i and a subgroup T_2 of $F(G)$ with order p_i such that $C_i \leq T_1T_2$. Since $F(G)$ is cyclic, T_2 is normal in G and so $\mathcal{C}_G(E(G)) \leq \mathcal{N}_G(T_1T_2)$. But $\mathcal{N}_G(T_1T_2)$ has an abelian normal subgroup of index a divisor of 4, so $\mathcal{C}_G(E(G))$ is solvable. This shows that every Sylow p-subgroup of $\mathcal{C}_G(E(G))$ for p odd is contained in $F(G)$, whence it follows that $G = E(G)F(G)$.

Claim 4. *If no involution of G acts dihedrally on any C_i, then $E(G)$ is simple, and, for every i, p_i divides the order of $F(G)$ and a Sylow p_i-subgroup of G contains exactly two conjugates of C_i.*

Suppose no involution of G acts dihedrally on any C_i. By Definition 5, it follows that $\mathcal{N}_G(C_i) = \mathcal{C}_G(C_i)$ for every $i = 1, \dots, m$. Let Q be a component of $E(G)$ and suppose by contradiction that Q contains a Sylow p_i-subgroup S of rank two, for some i. Up to conjugation we can suppose that S contains C_i. By Definition 5, $\mathcal{N}_Q(S)$ contains with index at most two the abelian group $\mathcal{C}_Q(C_i)$. Since Q is simple, Burnside's p-complement theorem (see [Su, Theorem 2.10 page 144]) yields that $\mathcal{N}_Q(S)$ is not abelian. Therefore, $\mathcal{N}_Q(S)$ contains with index two $\mathcal{N}_Q(C_i)$ and the elements of $\mathcal{N}_Q(S) \setminus \mathcal{N}_Q(C_i)$ conjugate C_i to a cyclic subgroup distinct from C_i. By using [Su, Exercise 1, page 161] and the fact that Q is perfect we get again a contradiction. Hence, for every $i \in \{1, \dots, m\}$, the Sylow p_i-subgroups of Q are cyclic. Now, as above for every Sylow subgroup S of Q we have $\mathcal{N}_Q(S) \neq \mathcal{C}_Q(S)$. Since $\mathcal{N}_G(C_i) = \mathcal{C}_G(C_i)$ for every $i = 1, \dots, m$, that implies that the Sylow p_i-subgroups of G are not cyclic and hence p_i divides the order of $F(G)$ for every i. $\square$

To bound above the number of conjugacy classes of hyperelliptic rotations, our strategy will consist of conjugating hyperelliptic rotations in solvable subgroups of G, where they are forced to commute, hence, in analogy with the standard definition of normal covers, we introduce the notion of solvable normal π-cover and we prove the following lemma.

Definition 6. Let G be a finite group. Let π be a set of primes dividing $|G|$. We will call a collection $\mathcal{C}$ of subgroups of G a *solvable normal π-cover* of G if every element of G of prime order p belonging to π is contained in an element of $\mathcal{C}$ and for every $g \in G$, $H \in \mathcal{C}$ we have that $H^g \in \mathcal{C}$. We denote by $\gamma^s_\pi(G)$ the smallest number of conjugacy classes of subgroups in a solvable normal π-cover of G. Here, the letter "s" stands for "solvable", and is used to distinguish this number from $\gamma(G)$, that is the standard notation in the case of covers by subgroups that are not requested to be solvable. Note that, since Sylow subgroups are clearly solvable, $\gamma^s_\pi(G) \leq |\pi|$.

Proposition 4. *Let G be a finite nonabelian simple group. If π is the set of odd primes p such that G has cyclic Sylow p-subgroup, the centraliser of $C_G(g)$ is* *abelian for every element $g \in G$ of order p and the normaliser of any subgroup of order p contains with index at most two its centraliser, then $\gamma^s_\pi(G) \leq 4$.*

Proof.

If G is a sporadic group, the primes dividing the order of the group do not satisfy the condition on the normaliser.

If G is isomorphic to the alternating group $\mathbb{A}_n$ and $p \in \pi$, then the condition on the centraliser of the elements of order p implies that $p > n-4$ and $\gamma^s_\pi(G) \leq |\pi| \leq 2$.

The only remaining case is that of groups of Lie type.

Let $G \cong \Sigma_n(q)$ or $G \cong {}^d\Sigma_n(q)$, where q is a power of a prime t. Here we use the same notation as in [GLS]: the symbol $\Sigma(q)$ (resp. ${}^d\Sigma(q)$) may refer to finite groups in different isomorphism classes, each of them is an untwisted (resp. twisted) finite group of Lie type with root system Σ (see [GLS, Remark 2.2.5]). Any finite group of Lie type is quasisimple with the exception of the following groups: $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$, ${}^2B_2(2)$, $B_2(2)$, $G_2(2)$, ${}^2F_4(2)$ and ${}^2G_2(3)$ (see [GLS, Theorem 2.2.7]).

If $t \in \pi$, then by [GLS, Theorem 3.3.3], either $t = 3$ and $G \cong ({}^2G_2(3))'$ or $G \cong A_1(s)$. In the former case the order of ${}^2G_2(3)'$ is divided only by two odd primes, thus $\gamma^s_\pi(G) \leq 2$; in the latter case we have $\gamma^s_\pi(G) \leq 2$ (see for example [H]).

Assume now that $t \notin \pi$. By [GLS, Paragraph 4.10], since the Sylow subgroups are cyclic, every element of order $p \in \pi$ is contained in a maximal torus of G, and clearly a maximal torus is abelian.

Therefore, we need only to bound the number of conjugacy classes of cyclic maximal tori in G with abelian centraliser. Note that the number of conjugacy classes of maximal tori in G is bounded by the number of different cyclotomic polynomials evaluated in q appearing as factors of $|G|$. Moreover the power of a cyclotomic polynomial in the order of G gives the rank of the corresponding maximal torus (except possibly when the prime divides the order of the centre but in this case the Sylow subgroup is not cyclic, see [A, (33.14)])

Recall Σ is the root system associated to G as in [GLS, 2.3.1]; let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a fundamental system for Σ as in Table 1.8 in [GLS], α_* be the lowest root relative to Π as defined in [GLS, Paragraph 1.8] and set $\Pi^* = \Pi \cup \{\alpha_*\}$. We recall that $|G|$ can be deduced from [GLS, Table 2.2] and the Dynkin diagrams can be found in [GLS, Table 1.8]. Observe that, by [GLS, Proposition 2.6.2], if Σ_0 is a root subsystem of Σ, then G contains a subsystem subgroup H, which is a central product of groups of Lie type corresponding to the irreducible constituents of Σ_0. In order to prove the lemma, we shall show that for every group G and for every element g of order a prime r lying in a maximal torus belonging to any but four conjugacy classes of maximal tori, either the Sylow r-subgroup is not cyclic or we find a subsystem subgroup H that is a central product of two groups H_1 and H_2 such that H_1 contains g and H_2 is not abelian. Note that for every prime power q, $A_1(q)$ is a non-abelian group (see [GLS, Theorem 2.2.7]).

We treat the case $G \cong A_n(q)$ in detail as an example. All other cases can be dealt similarly. Assume $G \cong A_n(q)$. Let m be the minimum index i such that r divides $q^{i+1}-1$ and let Σ_0 be generated by $\Pi^* \backslash \{\alpha_1, \alpha_n\}$. Then the corresponding subsystem subgroup is $H = H_1 \cdot H_2$, where $H_1 \cong A_{n-2}(q)$ and $H_2 \cong A_1(q)$. Thus if $m \leq n-1$, then H_1 contains an element g of order r and $C_G(g)$ contains

H_2 which is not abelian. Therefore, since g has an abelian centraliser, r may divide only $(q^n-1)(q^{n+1}-1)$, that is r divides $\Phi_n(q)\Phi_{n+1}(q)$. Hence we have at most two conjugacy classes of maximal tori with abelian centraliser. $\square$

5 Proof of Theorem 2

Let G be a finite group of orientation preserving diffeomorphisms of M, a closed orientable connected 3-manifold which is not homeomorphicto $\mathbf{S}^3$.

As noted in Section 2, there at most nine conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of order 2^n, so we concentrate on hyperelliptic subgroups whose order is not a power of two and we will prove that there are at most six conjugacy classes of such subgroups.

Let $\mathcal{S} = \{C_1, \ldots, C_m\}$ be the set of cyclic subgroups of odd prime orders that are generated by powers of the hyperelliptic rotations of G. We recall that the conjugate of a power of a hyperelliptic rotation is the power of a hyperelliptic rotation too. By Remark 3 and Corollary 4, $\mathcal{S}$ is an algebraically hyperelliptic collection, and actually the definition of an algebraically hyperelliptic collection was chosen precisely to capture the behaviour of cyclic subgroups of odd prime orders generated by powers of the hyperelliptic rotations. For $i \in \{1, \ldots, m\}$, let p_i be the prime order of C_i and π be the set including every p_i. We denote by G_0 the subgroup generated by the subgroups C_i. Let A be the maximal normal solvable subgroup of G_0 of order coprime with all p_i. We denote by $\overline{G_0}$ the quotient group G_0/A and by $\overline{C}_i$ the projection of C_i to $\overline{G_0}$.

Case 1. If G_0 is solvable, we are in the hypotheses of Lemma 3 which affirms that up to conjugacy the hyperelliptic rotations of G commute. Since they commute we can apply Lemma 2 and conclude that there are at most three conjugacy classes of subgroups generated by hyperelliptic rotations of order not a power of two.

Even if its proof is easier than that of the remaining situations, the solvable case is interesting in its own right, in particular for it plays an important role also in the proof of Theorem 3. We then summarise the conclusions in the following proposition.

Proposition 5. *Let M be a closed 3-manifold not homeomorphic to $\mathbf{S}^3$. Let G be a finite group of orientation preserving diffeomorphisms of M. If G is solvable, it contains at most three conjugacy classes of hyperelliptic subgroups of order not a power of two, and any two such subgroups commute up to conjugacy. Moreover, if there are three conjugacy classes, then their orders must be pairwise coprime.*

Case 2. Suppose G_0 is not solvable and it has no rotation of order 2 outside A. Then, because of the structure of the normaliser of a hyperelliptic rotation as described in Remark 3, G_0 has no involution acting dihedrally on any C_i and by Part 3 of Theorem 4 describing the structure of G_0, $E(\overline{G_0})$ is simple and, for each p_i, any Sylow p_i-subgroup contains exactly two distinct conjugates of $\bar{C}_i$. By Remark 3 every hyperelliptic rotation commuting with one of these two subgroups of order p_i commutes also with the other one. From this fact, Lemma 3, and Lemma 2, it follows that $\gamma^s_\pi(G_0)$ bounds from above the number of conjugacy classes of hyperelliptic subgroups of order not a power of two (see also Proposition 5). It is easy to see that $\gamma_\pi^s(G_0) \leq \gamma_\pi^s(E(\overline{G_0}))$. By Proposition 4 we have $\gamma_\pi^s(E(\overline{G_0})) \leq 4$ and so we get the thesis in this case.

Case 3. Suppose G_0 is not solvable and it has a rotation of order 2 not contained in A. The groups containing a rotation of order two are studied in [Mec2] where the following result was proved.

Theorem 5. *[Mec2] Let D be a finite group of orientation-preserving diffeomorphisms of a closed orientable 3-manifold. Let O be the maximal normal subgroup of odd order and $E(\tilde{D})$ be the layer of $\tilde{D} = D/O$. Suppose that D contains an involution which is a rotation.*

1. *If $E(\tilde{D})$ is trivial, there exists a normal subgroup H of D such that H is solvable and D/H is isomorphic to a subgroup of $\mathbb{A}_8$, the alternating group on 8 letters.*

2. *If the semisimple group $E(\tilde{D})$ is not trivial, it has at most two components and the factor group of $\tilde{D}/E(\tilde{D})$ is solvable.*

Moreover if D contains a rotation of order 2 such that its projection is contained in $E(\tilde{D})$, then $E(\tilde{D})$ is isomorphic to one of the following groups:

$$PSL_2(q), \;\; SL_2(q) \times_{\mathbb{Z}/2} SL_2(q')$$

where q and q' are odd prime powers greater than 4.

Applying Theorem 5 and Theorem 4 to G_0, we get that $E(\overline{G_0})$ is isomorphic either to a subgroup of $\mathbb{A}_8$, or to $PSL_2(q)$, or to $PSL_2(q) \times PSL_2(q')$. In the first case, there are at most three odd primes dividing the order of $E(\overline{G_0})$ and the thesis follows again from Lemma 2 and Lemma 3.

In the remaining cases, we will use a solvable normal π-cover to bound the number of conjugacy classes. We have that $\gamma_\pi^s(PSL_2(q)) \leq 2$. In fact the upper triangular matrices form a solvable subgroup of $SL_2(q)$ of order $(q-1)q$, moreover $SL_2(q)$ contains a cyclic subgroup of order $q+1$ (see [H]). The conjugates of the projections of these two subgroups to $PSL_2(q)$ give a solvable normal π-cover of $PSL_2(q)$. It is again easy to see that, if $E(\overline{G_0})$ is isomorphic to $PSL_2(q)$, then $\gamma_\pi^s(G) \leq 2$. As above, by Lemma 3 and Lemma 2, we get a bound of six in this case.

Finally, if $E(\overline{G})$ is isomorphic to $PSL_2(q) \times PSL_2(q')$, then it follows from the discussion above that $\gamma_\pi^s(E(\overline{G_0})) \leq 4$, and hence $\gamma_\pi^s(G_0) \leq 4$. Now, by Part 1 of Theorem 4, for each $p_i \in \pi$, p_i divides the order of both $PSL_2(q)$ and $PSL_2(q')$, so that the Sylow p_i-subgroup of $\overline{G_0}$ has rank 2. For each Sylow p_i-subgroup S_i, one can find in $E(\overline{G_0})$ elements of the group which normalise S_i but do not act in the same way on all of its elements. Indeed, let H be a cyclic subgroup of order p_i in $PSL_2(q)$. If p_i does not divide q, then there is an element of order 2 in $PSL_2(q)$ which acts dihedrally on H but commutes with all elements of order p in $PSL_2(q')$. If p_i divides q, then $q = p_i$ since the Sylow p_i-subgroup of the component must be cyclic. Since $q > 3$, we have again elements in $PSL_2(q)$ that normalise the Sylow p-subgroup but do not centralise it (note that in this case the structure of the normaliser is compatible with the description given in Remark 3 only if $p = q = 5$). We now deduce from Remark 7 and Corollary 3 that S_i contains two subgroups conjugate to C_i.

Reasoning as in Case 2 we get that $\gamma_\pi^s(G_0)$ bounds from above the number of conjugacy classes of hyperelliptic subgroups of order not a power of two. This concludes the proof. $\square$

Remark 9. We wish to stress that the three cases that appear in the proof of Theorem 2 correspond to different types of symmetries possessed by the knots that are branched covered by the manifold. Let M be a closed, connected, oriented 3-manifold and let G a finite group of diffeomorphisms of M generated by a set $\{\psi_1, \ldots, \psi_k\}$ of hyperelliptic rotations. Denote by K_i the quotient knot $Fix(\psi_i)/\psi_i$, $i = 1, \ldots, k$. If there exists a K_i which is neither strongly invertible nor self-symmetric with respect to G, then we are in Case 1, that is, the hyperelliptic rotations commute up to conugacy (see [Su, Thm 2.10, page 144]). If there is a K_i which is strongly invertible with respect to G, then G contains a rotation of order 2 and we are in Case 3. Otherwise, every K_i is self-symmetric with respect to G but not strongly invertible with respect to G, and we are in Case 2.

The proof of Theorem 2 shows that topology and geometry can impose extra constraints on the conditions that can be derived by the algebra alone. In this spirit, note that if M is a (closed, connected, oriented) reducible 3-manifold and G a finite group of diffeomorphisms of M generated by hyperelliptic rotations, then, by the equivariant sphere theorem, G is isomorphic to a finite subgroup of $SO(3)$, that is cyclic, dihedral or a spherical triangular group. It follows readily, that, up to conjugacy, at most three cyclic hyperelliptic subgroups can be contained in any such G. We know, however, that the group of diffeomorphisms of a reducible 3-manifolds can admit arbitrarily many conjugacy classes of cyclic hyperelliptic subgroups.

The main point here is that, generically, one expects that two hyperelliptic rotations in the group $Diff^+(M)$ generate an infinite subgroup. As a consequence, Theorem 2 cannot be directly exploited to obtain bounds for non-hyperbolic manifolds and new strategies must be developped, as we will see in the next section.

6 Proof of Theorem 3

The statement of Theorem 3 is equivalent to the following:

Theorem 6. *Let M be a closed, orientable, connected, irreducible 3-manifold which is not homeomorphic to $\mathbf{S}^3$, then the group $Diff^+(M)$ of orientation preserving diffeomorphisms of M contains at most six conjugacy classes of hyperelliptic subgroups of odd prime order.*

6.1 Proof of Theorem 6 for Seifert manifolds

In this section we prove Proposition 6 which implies Theorem 6 for closed Seifert fibred 3-manifolds. We also show that the assumption that the hyperelliptic rotations have odd prime orders cannot be avoided in general by exhibiting examples of closed Seifert fibred 3-manifolds M such that $Diff^+(M)$ contains an arbitrarily large number of conjugacy classes of hyperelliptic subgroups of odd, but not prime, orders.

Proposition 6. *Let M be a closed Seifert fibred 3-manifold which is not homeomorphic to $\mathbf{S}^3$. Then the group $Diff^+(M)$ of orientation preserving diffeomorphisms of M contains at most one conjugacy class of hyperelliptic subgroups of odd prime order except if M is a Brieskorn integral homology sphere with 3 exceptional fibres. In this latter case $Diff^+(M)$ contains at most three non conjugate hyperelliptic subgroups of odd prime orders.*

Proof.

By hypothesis M is a cyclic cover of $\mathbf{S}^3$ branched over a knot, so it is orientable and a rational homology sphere by Remark 2. Notably, M cannot be $\mathbf{S}^1 \times \mathbf{S}^2$ nor a Euclidean manifold, except for the Hantzsche-Wendt manifold, see [Or, Chap. 8.2]. In particular, since M is prime it is also irreducible.

Consider a hyperelliptic rotation ψ on M of odd prime order p and let K be the image of $Fix(\psi)$ in the quotient $\mathbf{S}^3 = M/\psi$ by the action of ψ. The knot K must be hyperbolic or a torus knot, otherwise its exterior would be toroidal and have a non-trivial JSJ-collection of essential tori which would lift to a non-trivial JSJ-collection of tori for M, since the order of ψ is $p > 2$ (see [JS, J] and [BS]). By the orbifold theorem (see [BoP], [CHK]), the cyclic branched cover with order $p \geq 3$ of a hyperbolic knot is hyperbolic, with a single exception for $p = 3$ when K is the the figure-eight knot and M is the Hantzsche-Wendt Euclidean manifold. But then, by the orbifold theorem and the classification of 3-dimensional christallographic groups, ψ generates the unique, up to conjugacy, Euclidean hyperelliptic cyclic subgroup of $Diff^+(M)$, see for example [Dun], [Z1].

So we can assume that M is the p-fold cyclic cover of $\mathbf{S}^3$ branched along a non-trivial torus knot K of type (a, b), where $a > 1$ and $b > 1$ are coprime integers. Then M is a Brieskorn-Pham manifold $M = V(p, a, b) = \{z^p + x^a + y^b = 0 \text{ with } (z, x, y) \in \mathbb{C}^3 \text{ and } |z|^2 + |x|^2 + |y|^2 = 1\}$. A simple computation shows that M admits a Seifert fibration with 3, p or $p+1$ exceptional fibres and base space $\mathbf{S}^2$, see [Ko, Lem. 2], or [BoPa, Lemma 6 and proof of Lemma 7]. In particular M has a unique Seifert fibration, up to isotopy: by [Wa], [Sco] and [BOt] the only possible exception with base $\mathbf{S}^2$ and at least 3 exceptional fibres is the double of a twisted I-bundle, which is not a rational homology sphere, since it fibers over the circle. We distinguish now two cases:

Case 1: *The integers a and b are coprime with p, and there are three singular fibres of pairwise relatively prime orders a, b and p.* By the orbifold theorem any hyperelliptic rotation of M of order > 2 is conjugate into the circle action $\mathbf{S}^1 \subset Diff^+(M)$ inducing the Seifert fibration, hence the uniqueness of the Seifert fibration, up to isotopy, implies that M admits at most 3 non conjugate hyperelliptic groups of odd prime orders belonging to $\{a, b, p\}$. Indeed M is a Brieskorn integral homology sphere, see [BPZ].

Case 2: *Either $a = p$ and M has p singular fibres of order b, or $a = a'p$ with $a' > 1$, and M has p singular fibres of order b and one extra singular fibre of order a'.* In both situations, there are $p \geq 3$ exceptional fibres of order b which are cyclically permuted by the hyperelliptic rotation ψ. As before, M has a unique Seifert fibration, up to isotopy. Therefore, up to conjugacy, ψ is the only hyperelliptic rotation of order p on M, and by the discussion above M cannot admit a hyperelliptic rotation of odd prime order $q \neq p$. □

Remark 10. The requirement that the rotations are hyperelliptic is essential in the proof of Proposition 6. The Brieskorn homology sphere $\Sigma(p_1,\dots,p_n)$, $n \geq 4$, with $n \geq 4$ exceptional fibres admits n rotations of pairwise distinct prime orders but which are not hyperelliptic.

The hypothesis that the orders of the hyperelliptic rotations are $\neq 2$ cannot be avoided either.

Indeed, Montesinos's construction of fibre preserving hyperelliptic involutions on Seifert fibered rational homology spheres [Mon1], [Mon2], (see also [BS, Appendix A], [BZH, Chapter 12]), shows that for any given integer n there are infinitely many closed orientable Seifert fibred 3-manifolds with at least n conjugacy classes of hyperelliptic rotations of order 2.

On the other hand, the hypothesis that the orders are odd primes is sufficient but not necessary: A careful analysis of the Seifert invariants shows that if $M \neq \mathbf{S}^3$ is a Seifert rational homology sphere, then M can be the cyclic branched cover of a knot in $\mathbf{S}^3$ of order > 2 in at most three ways.

The hypotheses of Proposition 6 cannot be relaxed further, though: Proposition 7 below shows that there exist closed 3-dimensional circle bundles with arbitrarily many conjugacy classes of hyperelliptic rotations of odd, but not prime, orders.

Proposition 7. *Let N be an odd prime integer. For any integer $1 \leq q < \frac{N}{2}$ the Brieskorn-Pham manifold $M = V((2^q+1)(2^{(N-q)}+1), 2^q+1, 2^{(N-q)}+1)$ is a circle bundle over a closed surface of genus $g = 2^{N-1}$ with Euler class ± 1. Hence, up to homeomorphism (possibly reversing the orientation), M depends only on the integer N and admits at least $\frac{N-1}{2}$ conjugacy classes of hyperelliptic groups of odd orders.*

Proof. We remark first that the integers q and $N-q$ are relatively prime, because N is prime. If k is a common prime divisor of 2^q+1 and $2^{(N-q)}+1$, by the Bezout identity we have $2^1 = 2^{aq+b(N-q)} \equiv (-1)^{a+b} \mod k$ which implies that $k=3$. But then $(-1)^q \equiv (-1)^{(N-q)} \equiv -1 \mod 3$ and thus $(-1)^N \equiv 1 \mod 3$ which is impossible since N is odd. Hence 2^q+1 and $2^{(N-q)}+1$ are relatively prime.

So the Brieskorn-Pham manifold M is the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover of $\mathbf{S}^3$ branched over the torus knot $K_q = T(2^q+1, 2^{(N-q)}+1)$. It is obtained by Dehn filling the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover of the exterior of the torus knot K_q along the lift of its meridian. The $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover of the exterior of K_q is a trivial circle bundle over a once punctured surface of genus $g = 2^N - 1$. On the boundary of the torus-knot exterior the algebraic intersection between a meridian and a fibre of the Seifert fibration of the exterior is ± 1 (the sign depends on a choice of orientation, see for example [BZH, Chapter 3]). So on the torus boundary of the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover the algebraic intersection between the lift of a meridian of the torus knot and an $\mathbf{S}^1$-fiber is again ± 1. Hence the circle bundle structure of the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover of the exterior of the torus knot K_q can be extended with Euler class ± 1 to the Dehn filling along the lift of the meridian. So M is a circle bundle over a closed surface of genus $g = 2^{N-1}$ with Euler class ± 1.

Since the torus knots $K_q = T(2^q+1, 2^{(N-q)}+1)$ are pairwise inequivalent for $1 \leq q \leq \frac{N-1}{2}$, the hyperelliptic subgroups corresponding to the $(2^q+1)(2^{(N-q)}+$

1)-fold cyclic branched covers of the knots K_q are pairwise not conjugate in $Diff^+(M)$. $\square$

Remark 11. Note that the Seifert manifolds M and their hyperelliptic rotations constructed in Proposition 7 enjoy the following properties: If $N > 8$, then no hyperelliptic rotation can commute up to conjugacy with all the remaining ones (see Proposition 5 and [BoPa, Theorem 2]). If $N > 14$ no finite subgroup of $Diff^+(M)$ can contain up to conjugacy all hyperelliptic rotations of M, according to Theorem 2.

6.2 Reduction to the finite group action case

The fact that Theorem 6 implies Corollary 1 follows from the existence of a decomposition of a closed, orientable 3-manifold M as a connected sum of prime manifolds and the observation that a hyperelliptic rotation on M induces a hyperelliptic rotation on each of its prime summands. A 3-manifold admitting a hypereliptic rotation must be a rational homology sphere, and so M cannot have $\mathbf{S}^2 \times \mathbf{S}^1$ summands. Hence all prime summands are irreducible and at least one is not homeomorphic to $\mathbf{S}^3$, since M itself is not homeomorphic to $\mathbf{S}^3$. This is enough to conclude.

The remaining of this section is devoted to the proof that Theorem 2 implies Theorem 6.

We prove the following proposition:

Proposition 8. *If M is a closed, orientable, irreducible 3-manifold such that there are $k \geq 7$ conjugacy classes of hyperelliptic subgroups of $Diff^+(M)$ whose order is an odd prime, then M is homeomorphic to $\mathbf{S}^3$.*

Proof.

Let M be a closed, orientable, irreducible 3-manifold such that $Diff^+(M)$ contains $k \geq 7$ conjugacy classes of hyperelliptic subgroups of odd prime orders.

According to the orbifold theorem (see [BoP], [BMP], [CHK]), a closed orientable irreducible manifold M admitting a rotation has geometric decomposition. This means that M can be split along a (possibly empty) finite collection of π_1-injective embedded tori into submanifolds carrying either a hyperbolic or a Seifert fibered structure. This splitting along tori is unique up to isotopy and is called the JSJ-decomposition of M, see for example [NS], [BMP, chapter 3]. In particular, if its JSJ-decomposition is trivial, M admits either a hyperbolic or a Seifert fibred structure.

First we see that M cannot be hyperbolic. Indeed, if the manifold M is hyperbolic then, by the orbifold theorem, any hyperelliptic rotation is conjugate into the finite group $Isom^+(M)$ of orientation preserving isometries of M. Hence, applying Theorem 2 to $G = Isom^+(M)$, we see that $k \leq 6$ against the hypothesis.

If the manifold M is Seifert fibred, it follows readily from Proposition 6 of the previous section that $M = \mathbf{S}^3$. So we are left to exclude the case where the JSJ-decomposition of M is not empty.

Consider the JSJ-decomposition of M: each geometric piece admits either a complete hyperbolic structure with finite volume or a Seifert fibred product structure with orientable base. Moreover, the geometry of each piece is unique, up to isotopy.

Let $\Psi = \{\psi_1, \dots, \psi_k,\ k \geq 7\}$ be the set of hyperelliptic rotations which generate non conjugate cyclic subgroups in $Diff^+(M)$. By the orbifold theorem [BoP], [BMP], [CHK], after conjugacy, one can assume that each hyperelliptic rotation preserves the JSJ-decomposition, acts isometrically on the hyperbolic pieces, and respects the product structure on the Seifert pieces. We say that they are *geometric*.

Let Γ be the dual graph of the JSJ-decomposition: it is a tree, for M is a rational homology sphere (in fact, the dual graph of the JSJ-decomposition for a manifold which is the cyclic branched cover of a knot is always a tree, regardless of the order of the covering). Let $H \subset Diff^+(M)$ be the group of diffeomorphisms of M generated by the set Ψ of geometric hyperelliptic rotations. By [BoPa, Thm 1], there is a subset $\Psi_0 \subset \Psi$ of $k_0 \geq 4$ hyperelliptic rotations with pairwise distinct odd prime orders, say $\Psi_0 = \{\psi_i,\ i = 1, \dots, k_0\}$.

Let H_Γ denote the image of the induced representation of H in $Aut(\Gamma)$. Since rotations of finite odd order cannot induce an inversion on any edge of Γ, the finite group H_Γ must fix pointwise a non-empty subtree Γ_f of Γ.

The idea of the proof is now analogous to the ones in [BoPa] and [BPZ]: we start by showing that, up to conjugacy, the $k_0 \geq 4$ hyperelliptic rotations with pairwise distinct odd prime orders can be chosen to commute on the submanifold M_f of M corresponding to the subtree Γ_f. We consider then the maximal subtree corresponding to a submanifold of M on which these hyperelliptic rotations commute up to conjugacy and prove that such subtree is in fact Γ. Then the conclusion follows by applying Lemma 2.

The first step of the proof is achieved by the following proposition:

Proposition 9. *The hyperelliptic rotations in Ψ_0 commute, up to conjugacy in $Diff^+(M)$, on the submanifold M_f of M corresponding to the subtree Γ_f.*

Proof.

Since the hyperelliptic rotations in Φ have odd orders, either Γ_f contains an edge, or it consists of a single vertex. We shall analyse these two cases.

Case 1: M_f contains an edge.

Claim 5. *Assume that Γ_f contains an edge and let T denote the corresponding torus. The hyperelliptic rotations in Ψ_0 commute, up to conjugacy in $Diff^+(M)$, on the geometric pieces of M adjacent to T.*

Proof.

The geometric pieces adjacent to T are left invariant by the hyperelliptic rotations in Ψ_0, since their orders are odd. Let V denote one of the two adjacent geometric pieces: each hyperelliptic rotation acts non-trivially on V with odd prime order. We distinguish two cases according to the geometry of V.

V is hyperbolic. In this case all rotations act as isometries and leave a cusp invariant. Since their order is odd, the rotations must act as translations along horospheres, and thus commute.

Note that, even in the case where a rotation has order 3, its axis cannot meet a torus of the JSJ-decomposition of M for each such torus is separating and cannot meet the axis in an odd number of points.

V **is Seifert fibred.** In this case we can assume that the hyperelliptic rotations in Ψ preserve the Seifert fibration with orientable base. Since their orders are odd and prime, each one preserves the orientation of the fibres and of the base. The conjugacy class of a fiber-preserving rotation of V with odd prime order depends only on its combinatorial behaviour, i.e. its translation action along the fibre and the induced permutation on cone points and boundary components of the base. In particular, two geometric rotations with odd prime order having the same combinatorial data are conjugate via a diffeomorphism isotopic to the identity.

Since the hyperelliptic rotations in Ψ_0 have pairwise distinct odd prime orders, an analysis of the different cases described in Lemma 4 below shows that at most one among these hyperelliptic rotations can induce a non-trivial action on the base of the fibration, and thus the remaining ones act by translation along the fibres and induce the identity on the base. Since the translation along the fibres commutes with every fiber-preserving diffeomorphism of V, the hyperelliptic rotations in Ψ_0 commute on V.

Lemma 4 describes the Seifert fibred pieces of a manifold admitting a hyperelliptic rotation of odd prime order, as well as the action of the rotation on the pieces. Its proof can be found in [BoPa, Lemma 6 and proof of Lemma 7], see also [Ko, Lem. 2].

Lemma 4. *Let M be an irreducible 3-manifold admitting a non-trivial JSJ-decomposition. Assume that M admits a hyperelliptic rotation of prime odd order p. Let V be a Seifert piece of the JSJ-decomposition for M. Then the base B of V can be:*

1. *A disc with 2 cone points. In this case either the rotation freely permutes p copies of V or leaves V invariant and acts by translating along the fibres.*
2. *A disc with p cone points. In this case the rotation leaves V invariant and cyclically permutes the singular fibres, while leaving a regular one invariant.*
3. *A disc with $p+1$ cone points. In this case the rotation leaves V and a singular fibre invariant, while cyclically permuting the remaining p singular fibres.*
4. *An annulus with 1 cone point. In this case either the rotation freely permutes p copies of V or leaves V invariant and acts by translating along the fibres.*
5. *An annulus with p cone points. In this case the rotation leaves V invariant and cyclically permutes the p singular fibres.*
6. *A disc with $p-1$ holes and 1 cone point. In this case the rotation leaves V invariant and cyclically permutes all p boundary components, while leaving invariant the only singular fibre and a regular one.*
7. *A disc with k holes, $k \geq 2$. In this case either the rotation freely permutes p copies of V or leaves V invariant. In this latter case either the rotation acts by translating along the fibres, or $k = p-1$ and the rotation permutes all the boundary components (while leaving invariant two fibres), or $k = p$*

and the rotation permutes p boundary components, while leaving invariant the remaining one and a regular fibre.

□

We conclude that the rotations in Ψ_0 can be chosen to commute on the submanifold M_f of M corresponding to Γ_f by using inductively at each edge of Γ_f the gluing lemma below (see [Lemma 6][BPZ]). We give the proof for the sake of completeness.

Lemma 5. *If the rotations preserve a JSJ-torus T then they commute on the union of the two geometric pieces adjacent to T.*

Proof.
Let V and W be the two geometric pieces adjacent to T. By Claim 5, after conjugacy in $Diff^+(M)$, the rotations in Ψ_0 commute on V and W. Since they have pairwise distinct odd prime orders, their restrictions on V and W generate two cyclic groups of the same finite odd order. Let g_V and g_W be generators of these two cyclic groups. Since they have odd order, they both act by translation on T. We need the following result about the slope of translation for such periodic transformation of the torus:

Claim 6. *Let ψ be a periodic diffeomorphism of the product $T^2 \times [0,1]$ which is isotopic to the identity and whose restriction to each boundary torus $T \times \{i\}$, $i = 0, 1$, is a translation with rational slopes α_0 and α_1 in $H_1(T^2;\mathbb{Z})$. Then $\alpha_0 = \alpha_1$.*

Proof.
By Meeks and Scott [MS, Thm 8.1], see also [BS, Prop. 12], there is a Euclidean product structure on $T^2 \times [0,1]$ preserved by ψ such that ψ acts by translation on each fiber $T \times \{t\}$ with rational slope α_t. By continuity, the rational slopes α_t are constant. □

Now the the following claim shows that the actions of g_W and g_V can be glued on T.

Claim 7. *The translations $g_V|_T$ and $g_W|_T$ have the same slope in $H_1(T^2;\mathbb{Z})$.*

Proof.
Let $\Psi_0 = \{\psi_i, i = 1, \ldots, k_0\}$. Let p_i the order of ψ_i and $q_i = \Pi_{j \neq i} p_j$. Then the slopes α_V and α_W of $g_V|_T$ and $g_W|_T$ verify: $q_i \alpha_V = q_i \alpha_W$ for $i = 1, ..., k_0$, by applying Claim 6 to each ψ_i. Since the GCD of the q_i is 1, it follows that $\alpha_V = \alpha_W$. □

This finishes the proof of Lemma 5 and of Proposition 9 when M_f contains an edge. □

To complete the proof of Proposition 9 it remains to consider the case where Γ_f is a single vertex.

Case 2: M_f is a vertex.

Claim 8. *Assume that Γ_f consists of a single vertex and let V denote the corresponding geometric piece. Then the hyperelliptic rotations in Ψ_0 commute on V, up to conjugacy in $Diff^+(M)$.*

Proof.

We consider again two cases according to the geometry of V.

The case where V is Seifert fibred follows once more from Lemma 4.

We consider now the case where V is hyperbolic.

In this case, the hyperelliptic rotations in Ψ act non-trivially on V by isometries of odd prime orders. The restriction $H_{|V} \subset Isom^+(V)$ of the action of the subgroup H that they generate in $Diff^+(M)$ is finite.

If the action on V of the cyclic subgroups generated by two of the hyperelliptic rotations in Ψ are conjugate in $H_{|V}$, one can conjugate the actions in $Diff^+(M)$ to coincide on V, since any diffeomorphism in $H_{|V}$ extends to M. Then by [BoPa, Lemma 10] these actions must coincide on M, contradicting the hypothesis that the conjugacy classes of cyclic subgroups generated by the hyperelliptic rotations in Ψ are pairwise distinct in $Diff^+(M)$. Hence, the cyclic subgroup generated by the $k \geq 7$ hyperelliptic rotations in Ψ are pairwise not conjugate in the finite group $H_{|V} \subset Isom^+(V)$.

Since the dual graph of the JSJ-decomposition of M is a tree, a boundary torus $T \subset \partial V$ is separating and bounds a component U_T of $M \setminus int(V)$. Since, by hypothesis, Γ_f consists of a single vertex, no boundary component T is setwise fixed by the finite group $H_{|V}$. This means that there is a hyperelliptic rotation $\psi_i \in \Psi$ of odd prime order p_i such that the orbit of U_T under ψ_i is the disjoint union of p_i copies of U_T. In particular U_T projects homeomorphically onto a knot exterior in the quotient $\mathbf{S}^3 = M/\psi_i$. Therefore on each boundary torus $T = \partial U_T \subset \partial V$, there is a simple closed curve λ_T, unique up to isotopy, that bounds a properly embedded incompressible and ∂-incompressible Seifert surface S_T in the knot exterior U_T.

By pinching the surface S_T onto a disc $\mathbf{D}^2$, in each component U_T of $M \setminus int(V)$, one defines a degree-one map $\pi : M \longrightarrow M'$, where M' is the rational homology sphere obtained by Dehn filling each boundary torus $T \subset V$ along the curve λ_T.

For each hyperelliptic rotation ψ_i in Ψ, of odd prime order p_i, the ψ_i-orbit of each component U_T of $M \setminus int(V)$ consists of either one or p_i elements. As a consequence, by [Sa], ψ_i acts equivariantly on the set of isotopy classes of curves $\lambda_T \subset \partial V$. Hence, each ψ_i extends to periodic diffeomorphism ψ_i' of order p_i on M'. Moreover, M' is a $\mathbb{Z}/p_i$-homology sphere, since so is M and $\pi : M \longrightarrow M'$ is a degree-one map. According to Smith theory, if $Fix(\psi')$ is non-empty on M', then ψ_i' is a rotation on M'. To see that $Fix(\psi') \neq \emptyset$ on M' it suffices to observe that either $Fix(\psi) \subset V$ or ψ_i is a rotation of some U_T; in this latter case, ψ_i' must have a fixed point on the disc $\mathbf{D}^2$ onto which the surface S_T is pinched. To show that ψ_i' is hyperelliptic it remains to show that the quotient M'/ψ_i' is homeomorphic to $\mathbf{S}^3$.

Since ψ_i acts equivariantly on the components U_T of $M \setminus int(V)$ and on the set of isotopy classes of curves $\lambda_T \subset \partial V$, the quotient $\mathbf{S}^3 = M/\psi_i$ is obtained from the compact 3-manifold V/ψ_i by gluing knot exteriors (maybe solid tori) to its boundary components, in such a way that the boundaries of the Seifert surfaces of the knot exteriors are glued to the curves $\lambda_T/\psi_i \subset \partial V/\psi_i$.

In the same way, the rotation ψ_i' acts equivariantly on the components $M' \setminus int(V)$ and on the set of isotopy classes of curves $\lambda_T \subset \partial V$. By construction, these components are solid tori, and either the axis of the rotation is contained in V or there exists a unique torus $T \in \partial V$ such that the solid torus glued to T to obtain M' contains the axis. In the latter case, by [EL, Cor. 2.2], the rotation ψ'_i preserves a meridian disc of this solid torus and its axis is a core of W_T. It follows that the images in the quotient M'/ψ'_i of the the solid tori glued to ∂V are again solid tori. Hence M'/ψ'_i is obtained from $\mathbf{S}^3$ by replacing each components of $\mathbf{S}^3 \setminus V/\psi_i$ by a solid torus, in such way that boundaries of meridian discs of the solid tori are glued to the curves $\lambda_T/\psi'_i \subset \partial V/\psi'_i$. It follows that M'/ψ'_i is again $\mathbf{S}^3$.

So far we have constructed a closed orientable 3-manifold M' with a finite subgroup of orientation preserving diffeomorphisms H_V that contains at least seven conjugacy classes of hyperelliptic subgroups of odd prime orders. Theorem 2 implies that M' must be $\mathbf{S}^3$, and thus by the orbifold theorem [BLP] H_V is conjugate to a finite subgroup of $SO(4)$. In particular the subgroup $H_0 \subset H_V$ generated by the subset Ψ_0 of at least 4 hyperelliptic rotations with pairwise distinct odd prime orders must be solvable. Therefore, by Proposition 5 the induced rotations commute on M' and, by restriction, the hyperelliptic rotations in Ψ_0 commute on V. □

In the final step of the proof we extend the commutativity on M_f to the whole manifold M. The proof of this step is analogous to the one given in [BPZ], since the proof there was not using the homology assumption. We give the argument for the sake of completeness.

Proposition 10. *The $k_0 \geq 4$ hyperelliptic rotations in Ψ_0 commute, up to conjugacy in $Diff^+(M)$, on M.*

Proof.

Let Γ_c be the largest subtree of Γ containing Γ_f, such that, up to conjugacy in $Diff^+(M)$, the rotations in Ψ_0 commute on the corresponding invariant submanifold M_c of M. We shall show that $\Gamma_c = \Gamma$. If this is not the case, we can choose an edge contained in Γ corresponding to a boundary torus T of M_c. Denote by U_T the submanifold of M adjacent to T but not containing M_c and by $V_T \subset U_T$ the geometric piece adjacent to T.

Let $H_0 \subset Diff^+(M)$ be the group of diffeomorphisms of M generated by the set of geometric hyperelliptic rotations $\Psi_0 = \{\psi_i,\, i = 1, \ldots, k_0\}$. Since the rotations in Ψ_0 commute on M_c and have pairwise distinct odd prime orders, the restriction of H_0 on on M_c is a cyclic group with order the product of the orders of the rotations. Since $\Gamma_f \subset \Gamma_c$, the H_0-orbit of T cannot be reduced to only one element. Moreover each rotation $\psi \in \Psi_0$ either fixes T or acts freely on the orbit of T since its order is prime.

If no rotation in Ψ_0 leaves T invariant, the H_0-orbit of T contains as many elements as the product of the orders of the rotations, for they commute on M_c. In particular, only the identity (which extends to U) stabilises a torus in the H_0-orbit of T. Note that all components of ∂M_c that are in the H_0-orbit of T bound a manifold homeomorphic to U_T.

Since the rotation ψ_i acts freely on the H_0-orbit of U_T, U_T is a knot exterior in the quotient $M/\psi_i = \mathbf{S}^3$. Hence there is a well defined meridian-longitude system on $T = \partial U_T$ and also on each torus of the H_0-orbit of T. This set of meridian-longitude systems is cyclically permuted by each ψ_i and thus equivariant under the action of H_0.

Let M_c/H_0 be the quotient of M_c by the induced cyclic action of H_0 on M_c. Then there is a unique boundary component $T' \subset \partial(M_c/H_0)$ which is the image of the H_0-orbit of T. We can glue a copy of the knot exterior U_T to M_c/H_0 along

T' by identifying the image of the meridian-longitude system on ∂U_T with the projection on T' of the equivariant meridian-longitude systems on the H_0-orbit of T. Denote by N the resulting manifold. For all $i = 1, \dots, k_0$, consider the cyclic (possibly branched) cover of N of order $q_i = \prod_{j \neq i} p_j$ which is induced by the cover $\pi_i : M_c/\psi_i \longrightarrow M_c/H_0$. This makes sense because $\pi_1(T') \subset \pi_{i*}(\pi_1(M_c/\psi_i))$. Call $\tilde{N}_i$ the total space of such covering. By construction it follows that $\tilde{N}_i$ is the quotient $(M_c \cup H_0 \cdot U_T)/\psi_i$. This implies that the ψ_i's commute on $M_c \cup H_0 \cdot U_T$ contradicting the maximality of Γ_c.

We can thus assume that some rotations fix T and some do not. Since all rotations commute on M_c and have pairwise distinct odd prime orders, we see that the orbit of T consists of as many elements as the product of the orders of the rotations which do not fix T and each element of the orbit is fixed by the rotations which leave T invariant. The rotations which fix T commute on the orbit of V_T according to Claim 5 and Lemma 5, and form a cyclic group generated by, say, γ. The argument for the previous case shows that the rotations acting freely on the orbit of T commute on the orbit of U_T and thus on the orbit of V_T, and form again a cyclic group generated by, say, η. To reach a contradiction to the maximality of M_c, we shall show that γ, after perhaps some conjugacy, commutes with η on the H_0-orbit of V_T, in other words that γ and $\eta\gamma\eta^{-1}$ coincide on $H_0 \cdot V_T$. Since η acts freely and transitively on the H_0-orbit of V_T there is a natural and well-defined way to identify each element of the orbit $H_0 \cdot V_T$ to V_T itself. Note that this is easily seen to be the case if V_T is hyperbolic: this follows from Claim 5 and Claim 6. We now consider the case when V_T is Seifert fibred.

Claim 9. *Assume that V_T is Seifert fibred and that the restriction of γ induces a non-trivial action on the base of V_T. Then γ induces a non-trivial action on the base of each component of the H_0-orbit of V_T. Moreover, up to conjugacy on $H_0 \cdot V_T \setminus V_T$ by diffeomorphisms which are the identity on $H_0 \cdot T$ and extend to M , we can assume that the restrictions of γ to these components induce the same permutation of their boundary components and the same action on their bases.*

Proof.

By hypothesis γ and $\eta\gamma\eta^{-1}$ coincide on ∂M_c. The action of γ on the base of V_T is non-trivial if and only if its restriction to the boundary circle of the base corresponding to the fibres of the torus T is non-trivial. Therefore the action of γ is non-trivial on the base of each component of $H_0 \cdot V_T$.

By Lemma 4 and taking into account the fact that V_T is a geometric piece in the JSJ-decomposition of the knot exterior U_T, the only situation in which the action of γ on the base of V_T is non-trivial is when the base of V_T consists of a disc with p holes, where p is the order of one of the rotations that generate γ. Moreover, the restriction of γ to the elements of $H_0 \cdot V_T$ cyclically permutes their boundary components which are not adjacent to M_c. Up to performing Dehn twists, along vertical tori inside the components of $H_0 \cdot V_T \setminus V_T$, which permute the boundary components, we can assume that the restriction of γ induces the same cyclic permutations on the boundary components of each element of the orbit $H_0 \cdot V_T$. We only need to check that Dehn twists permuting two boundary components extend to the whole manifold M. This follows from the fact that the manifolds adjacent to these components are all homeomorphic and that Dehn twist act trivially on the homology of the boundary.

Since the actions of the restrictions of γ on the bases of the elements of $H_0 \cdot V_T$ are combinatorially equivalent, after perhaps a further conjugacy by an isotopy, the different restrictions can be chosen to coincide on the bases. $\square$

By Claim 5 and Claim 9 we can now deduce that the restrictions of γ and $\eta\gamma\eta^{-1}$ to the H_0-orbit of V_T commute, up to conjugacy of γ which is the identity on the H_0-orbit of T. Since γ and $\eta\gamma\eta^{-1}$ coincide on this H_0-orbit of T, we can conclude that they coincide on the H_0-orbit of V_T. This finishes the proof of Proposition 10. $\square$

Since there are at least four hyperelliptic rotations with paiwise distinct odd prime orders in Ψ_0, Proposition 8 is consequence of Proposition 10 and Proposition 1, like in the solvable case. $\square$

Remark 12. As we have seen, the strategy to prove that an irreducible manifold M with non-trivial JSJ-decomposition cannot admit more than six conjugacy classes of hyperelliptic subgroups of odd prime order inside $Diff^+(M)$ consists in modifying by conjugacy any given set of hyperelliptic rotations so that the new hyperelliptic rotations commute pairwise. Note that this strategy cannot be carried out in general if the orders are not pairwise coprime (see, for instance, [BoPa, Section 4.1] where the case of two hyperelliptic rotations of the same odd prime order, generating non conjugate subgroups, is considered). Similarly, for hyperelliptic rotations of arbitrary orders > 2 lack of commutativity might arise on the Seifert fibred pieces of the decomposition, as it does for the Seifert fibred manifolds constructed in Proposition 7.

7 Appendix: non-free finite group actions on rational homolgy spheres

In this section we will show that every finite group G admits a faithful action by orientation preserving diffeomorphisms on some rational homology sphere so that some elements of G have non-empty fixed-point sets.

Cooper and Long's construction of G-actions on rational homolgy spheres in [CL] consists in starting with a G-action on some 3-manifold and then modifying the original manifold, notably by Dehn surgery, so that the new manifold inherits a G-action but has "smaller" rational homology. Since their construction does not require that the G-action is free, it can be used to prove the existence of non-free G-actions. We will thus start by exhibiting non-free G-actions on some 3-manifold before pointing out what need to be taken into account in this setting in order for Cooper and Long's construction to work.

Since every (non-trivial) cyclic group acts as a group of rotations of $\mathbf{S}^3$, for simplicity we will assume that G is a finite non-cyclic group.

Claim 10. *Let G be a finite non-cyclic group. There is a closed, connected, orientable 3-manifold M on which G acts faitfully, by orientation preserving diffeomorphisms so that there are $g \in G \setminus \{1\}$ with the property that $Fix(g) \neq \emptyset$.*

Proof.

Let $k \geq 2$ and let $\{g_i\}_{1 \leq i \leq k+1}$ be a system of generators for G satisfying the following requirements:

- for all $1 \leq i \leq k+1$, the order of g_i is $n_i \geq 2$;

- $g_{k+1} = g_1 g_2 \cdots g_k$.

Since G is not cyclic these conditions are not difficult to fulfill, and actually they can be fulfilled even in the case when G is cyclic for an appropriate choice of the set $\{g_i\}_{1\leq i\leq k+1}$.

Consider the free group of rank k that we wish to see as the fundamental group of a $(k+1)$-punctured 2-sphere: each generator x_i corresponds to a loop around a puncture of the sphere so that a loop around the $k+1$st puncture corresponds to the element $x_1 x_2 \dots x_k$. Build an orbifold $\mathcal{O}$ by compactifying the punctured-sphere with cone points so that the ith puncture becomes a cone point of order n_i. The resulting orbifold has (orbifold) fundamental group with the following presentation:

$$\langle x_1, x_2, \dots, x_k, x_{k+1} \mid \{x_i^{n_i}\}_{1\leq i\leq k+1},\, x_1 \dots x_k x_{k+1}^{-1}\rangle.$$

Clearly this group surjects onto G. Such surjection gives rise to an orbifold covering $\Sigma \longrightarrow \mathcal{O}$, where Σ is an orientable surface on which G acts in such a way that each element g_i has non-empty fixed-point set. One can consider the 3-manifold $\Sigma \times \mathbf{S}^1$: the action of G on Σ extends to a product action of G on $\Sigma \times \mathbf{S}^1$ which is trivial on the $\mathbf{S}^1$ factor. $\square$

To be able to repeat Cooper and Long's construction it is now easy to observe that it is always possible to choose G-equivariant families of simple closed curves in M so that they miss the fixed-point sets of elements of G and either their homology classes generate $H_1(M;\mathbb{Q})$ (so that the hypothesis of [CL, Lemma 2.3] are fulfilled when we choose X to be the exterior of such families) or the family is the G-orbit of a representative of some prescribed homology class (as in the proof of [CL, Proposition 2.5]).

Acknowledgement The authors are indebted to M. Broué for valuable discussions on the topics of the paper.

References

[A] M. Aschbacher, *Finite groups theory*, Cambridge studies in advanced mathematics **10**, Cambridge University Press, 1986.

[BMP] M. Boileau, S. Maillot, and J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses **15**, Société Mathématique de France, Paris, 2003.

[BLP] M. Boileau, B. Leeb and J. Porti, *Geometrization of 3-dimensional orbifolds*, Annals of Math. **162**, (2005), 195-290.

[BOt] M. Boileau and J.P. Otal. *Scindements de Heegaard et groupe des homéotopies des petites variétés de Seifert*, Invent. Math. **106**, (1991), 85-107.

[BoPa] M. Boileau and L. Paoluzzi, *On cyclic branched coverings of prime knots*, J. Topol. **1**, (2008), 557-583.

[BPZ] M. Boileau, L. Paoluzzi and B. Zimmermann, *A characterisation of* $\mathbf{S}^3$ *among homology spheres*, The Zieschang Gedenkschrift. Geom. Topol. Monogr. **14**, Geom. Topol. Publ., Coventry, (2008), 83-103.

[BoP] M. Boileau and J. Porti, *Geometrization of 3-orbifolds of cyclic type*, Astérisque Monograph, **272**, 2001.

[BS] F. Bonahon and L. Siebenmann, *The characteristic splitting of irreducible compact 3-orbifolds*, Math. Math. **278**, (1987), 441-479.

[BZH] G. Burde and H. Zieschang, M. Heusener, *Knots*, Third edition, De Gruyter Studies in Mathematics, 5. De Gruyter, Berlin, 2014.

[CL] D. Cooper and D. D. Long, *Free actions of finite groups on rational homology 3-spheres*, Topology Appl. **101**, (2000), 143148.

[CHK] D. Cooper, C. Hodgson, and S. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds*, MSJ Memoirs **5**, 2000.

[DL] J. Dinkelbach and B. Leeb, *Equivariant Ricci flow with symmetry and applications to finite group actions on 3-manifolds*, Geom. Topol. **13**, (2009), 11291173.

[Dun] W. D. Dunbar, *Geometric Orbifolds*, Rev. Mat. Univ. Comp. Madrid, **1**, 1998, 67-99.

[EL] A. L. Edmonds and C. Livingston, *Group actions on fibered three-manifolds*, Comm. Math. Helv. **58**, (1983), 529-542.

[Gr] J. E. Greene, *Lattices, graphs, and Conway mutation.* Invent. Math. **192** (2013), no. 3, 717750.

[Go] C. McA. Gordon, *Some aspects of classical knot theory*, Knot Theory, Proceedings, Plans-sur-Bex, Switzerland (J.C. Hausmann ed.), Lect. Notes Math. **685**, (1977), Springer-Verlag, 1-60.

[GLS] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups, Number 3*, Math. Surveys Monogr. **40.3**, Amer. Math. Soc., Providence, RI, 1998.

[Hil] J. Hillman, *Links with infinitely many semifree periods are trivial*, Arch. Math. **42**, (1984), 568-572.

[H] B. Huppert, *Endlichen Gruppen I*, Springer-Verlag, New York, 1968.

[KL] B. Kleiner and J. Lott. *Local Collapsing, Orbifolds, and Geometrization*, Astérisque Monograph **365**, 2014.

[JS] W. H. Jaco and P. B. Shalen, *Seifert fibred spaces in 3-manifolds*, Mem. Amer. Math. Soc. **220**, 1979.

[J] K. Johannson, *Homotopy equivalence of 3-manifolds with boundary*, Lecture Notes in Math. **761**, Springer-Verlag, Berlin, 1979.

[Ka] A. Kawauchi, *Topological imitations and Reni-Mecchia-Zimmermann's conjecture*, Kyungpook Math. J. **46** (2006), 19.

[Ko] S. Kojima, *Determining knots by branched covers*, in Low Dimensional Topology and Kleinian groups, London Math. Soc. Lecture Note Ser. **112**, Cambridge Univ. Press (1986), 193-207.

[Mec1] M. Mecchia, *How hyperbolic knots with homeomorphic cyclic branched coverings are related*, Topology Appl. **121**, (2002), 521-533.

[Mec2] M. Mecchia, *Finite groups acting on 3-manifolds and cyclic branched coverings of knots*, The Zieschang Gedenkschrift. Geom. Topol. Monogr. **14**, Geom. Topol. Publ., Coventry, (2008), 393-416.

[MR] M. Mecchia and M. Reni, *Hyperbolic 2-fold branched coverings of links and their quotients*, Pacific J. Math. **202**, (2002), 429447.

[MZ] M. Mecchia and B. Zimmermann, *On finite groups acting on $\mathbb{Z}_2$-homology 3-spheres*, Math. Z. **248**, (2004), 675-693.

[MSY] W. H. Meeks, L. Simon, and S.-T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. **116** (1983), 621659.

[MS] W. H. Meeks and P. Scott, *Finite group actions on 3-manifolds*, Invent. Math. **86**, (1986), 287-346.

[Mon1] J. M. Montesinos, *Variedades de Seifert que son recubridores ciclicos ramificados de dos hojas*, Bol. Soc. Mat. Mexicana **18**, (1973), 1-32.

[Mon2] J. M. Montesinos, *Revêtements ramifiés de noeuds, espaces fibrés de Seifert et scindements de Heegaard*, Publicaciones del Seminario Mathematico Garcia de Galdeano, Serie II, Seccion 3, (1984).

[MonW] J. M. Montesinos and W. Whitten *Constructions of two-fold branched covering spaces.* Pac. J. Math. **125**, 415446 (1986)

[NS] W. Neumann and G. Swarup, *Canonical decompositions of 3-manifolds*, Geom. Topol. **1**, (1998), 21-40.

[Or] P. Orlik, *Seifert manifolds*, Lecture Notes in Mathematics **291**, Springer, 1972.

[P] L. Paoluzzi, *On hyperbolic type involutions*, Rend. Istit. Mat. Univ. Trieste **32** suppl. 1, (2001), 257-288.

[Re] M. Reni, *On π-hyperbolic knots with the same 2-fold branched coverings*, Math. Ann. **316**, (2000), no. 4, 681697

[RZ] M. Reni and B. Zimmermann, *Hyperbolic 3-manifolds as cyclic branched coverings*, Comment. Math. Helv. **76**, (2001), 300-313.

[Sa] M. Sakuma, *Uniqueness of symmetries of knots*, Math. Z. **192**, (1986), 225-242.

[Sco] P. Scott, *Homotopy implies isotopy for some Seifert fibre spaces*, Topology **24**, (1985), 341-351.

[Su] M. Suzuki, *Group theory II*, Grundlehren Math. Wiss. **248**, Springer-Verlag, New York, 1982.

[V] O. Ja. Viro *Nonprojecting isotopies and knots with homeomorphic coverings*, Zap. Nau. Semin. POMI **66**, 133147, 207208, Studies in topology, II (1976)

[Wa] F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II*, Invent. Math. **3**, (1967), 308333; ibid. **4** (1967), 87117.

[Z1] B. Zimmermann, *On the Hantzsche-Wendt Manifold*, Mh. Math. **110**, (1990) , 321-327.

[Z2] B. Zimmermann, *On hyperbolic knots with homeomorphic cyclic branched coverings*, Math. Ann. **311**, (1998), 665-673.

Aix-Marseille Univ, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France
michel.boileau@univ-amu.fr
Dipartimento di Matematica e Fisica "Niccolò Tartaglia", Università Cattolica del Sacro Cuore, Via Musei 41, 25121 Brescia, Italy
clara.franchi@unicatt.it
Dipartimento di Matematica e Geoscienze, Universitá degli studi di Trieste, Via Valerio 12/1, 34127 Trieste
mmecchia@units.it
Aix-Marseille Univ, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France
luisa.paoluzzi@univ-amu.fr
Dipartimento di Matematica e Geoscienze, Universitá degli studi di Trieste, Via Valerio 12/1, 34127 Trieste
zimmer@units.it