Classic Realizability (and how to get rid of it)

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“Classic” realizability?

Introduced by Kleene in 1945 as a semantics of intuitionistic logic.

“To interpret every proposition as a set of programs (the realizers) witnessing the corresponding proposition.”

S. C. Kleene. On the interpretation of intuitionistic number theory. The Journal of Symbolic Logic, 10(4):109–124, 1945.

G. Kreisel. Interpretation of analysis by means of constructive functionals of finite types. Constructivity in Mathematics, pp. 101–128, 1959.

W.W. Tait. A realizability interpretation of the theory of species. Logic Colloquium, Lectures Notes in Mathematics, Vol. 453, pp 240–251, 1975.

J.-Y. Girard. Interprétation fonctionnelle et élimination des coupures de l’arithmétique d’ordre supérieur, 1972.

J.-L. Krivine. Lambda-Calculus, Types and Models, 1993.

Syntactic Results

Proofs of Strong Normalization

The λ-calculus

$$M, N \quad ::= \quad x \mid \lambda x.M \mid MN$$

$$(\lambda x.M)N \quad \to_\beta \quad M[N/x]$$

with simple types

$$A, B \ ::= \ \alpha \mid A \to B$$

and inference rules

$$\frac{}{\Gamma, x : A \vdash x : A}\ (\mathtt{var}) \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ (\to_{\mathtt{I}})$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\ (\to_{\mathtt{E}})$$

is strongly normalizing.

Simple induction does not work

Setting

- SN $= \{M \in \Lambda \mid M \text{ is strongly normalizing}\}$,

we want to prove

$$\Gamma \vdash M : A \quad \Longrightarrow \quad M \in \mathrm{SN}$$

The problem is in the application case:

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \ (\to_{\mathrm{E}})$$

Indeed $M, N \in \mathrm{SN} \not\Rightarrow MN \in \mathrm{SN}$.

Solution: Associate with each type A a set $[\![A]\!] \subseteq \mathrm{SN}$ of λ-terms of that type, and show that $M : A$ implies $M \in [\![A]\!]$. Important, define:

$$[\![A \to B]\!] = [\![A]\!] \Rightarrow [\![B]\!] = \{M \in \mathrm{SN} \mid \forall N \in [\![A]\!], MN \in [\![B]\!]\}$$

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Still, induction does not go through

The problem is now in the abstraction case:

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ (\to_{\mathrm{I}})$$

Knowing $M \in [\![B]\!]$ is not enough to conclude $\lambda x.M \in [\![A \to B]\!]$.

It is sufficient to ensure that for Q strongly normalizing:

$$M[Q/x] \in [\![B]\!] \quad \Longrightarrow \quad (\lambda x.M)Q \in [\![B]\!]$$

This leads to the notion of saturated set.

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Saturated sets

Saturated sets

- $\mathcal{N} \subseteq$ SN is saturated if $(\forall x \in \mathrm{Var}, Q, \vec{N} \in \mathrm{SN})$:
  - $x\vec{N} \in \mathcal{N}$,
  - $M[Q/x]\vec{N} \in \mathcal{N} \implies (\lambda x.M)Q\vec{N} \in \mathcal{N}$
- $\mathrm{SAT} = \{\mathcal{N} \subseteq \Lambda \mid \mathcal{N} \text{ is saturated}\}$

Examples:

- $\mathrm{SN} \in \mathrm{SAT}$
- $\{M \in \Lambda \mid M \to_\beta^* x\vec{N}\} \in \mathrm{SAT}$,

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Properties:

- $\text{SN} \in \mathrm{SAT}$,
- $\mathcal{N} \in \mathrm{SAT} \implies \mathrm{Var} \subseteq \mathcal{N}$
- $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{SAT} \implies \mathcal{N}_1 \Rightarrow \mathcal{N}_2 \in \mathrm{SAT}$,
- For all types A, we have $[\![A]\!] \in \mathrm{SAT}$.

The proof

Define:

$$[\![\alpha]\!] = \mathrm{SN} \qquad [\![A \to B]\!] = [\![A]\!] \Rightarrow [\![B]\!]$$

Theorem (Adequacy for Λ^{ST})

If

$$x_1 : A_1, \dots, x_n : A_n \vdash M : B$$

then $\forall N_1 \in [\![A_1]\!], \dots, N_n \in [\![A_n]\!]$

$$M[N_1/x_1, \dots, N_n/x_n] \in [\![B]\!]$$

In particular, $M = M[x_1/x_1, \dots, x_n/x_n] \in [\![B]\!] \subseteq \mathrm{SN}$

Corollary

The simply typed λ-calculus enjoys SN.

System F - Second Order λ-Calculus

Types2: $A, B ::= \cdots \mid \forall\alpha.A$

Inference Rules:

$$\frac{\Gamma \vdash M : A \quad \alpha \notin \Gamma}{\Gamma \vdash M : \forall\alpha.A}\ (\forall_I) \qquad \frac{\Gamma \vdash M : \forall\alpha.A}{\Gamma \vdash M : A[B/\alpha]}\ (\forall_E)$$

What can we do more?

For instance, now we can type $\Delta = \lambda x.xx$:

$$\dfrac{\dfrac{\dfrac{x : \forall\alpha.\alpha \vdash x : \beta \to \alpha \quad x : \forall\alpha.\alpha \vdash x : \beta}{x : \forall\alpha.\alpha \vdash xx : \alpha}}{x : \forall\alpha.\alpha \vdash xx : \forall\alpha.\alpha}}{\vdash \lambda x.xx : (\forall\alpha.\alpha) \to (\forall\alpha.\alpha)}$$

Pretty scary, because $\Delta\Delta = \Omega$ (the paradigmatic unsolvable).

System F — Reducibility Candidates

Given a collection $\{\mathcal{N}_i\}_{i\in\mathcal{I}}$ of saturated sets, we have:

$$\bigcap_{i\in\mathcal{I}} \mathcal{N}_i \in \mathrm{SAT}$$

Definition

A SAT-valuation is any function ρ :Type Variables $\to \mathrm{SAT}$.

- $[\![\alpha]\!]_\rho = \rho(\alpha)$,
- $[\![A \to B]\!]_\rho = [\![A]\!]_\rho \Rightarrow [\![B]\!]_\rho$,
- $[\![\forall\alpha.A]\!]_\rho = \bigcap_{\mathcal{N}\in\mathrm{SAT}} [\![A]\!]_{\rho[\mathcal{N}/\alpha]}$

For every $A \in$ *Types*2 and SAT-valuation ρ, we have $[\![A]\!]_\rho \in \mathrm{SAT}$.

Theorem (Adequacy)

If $x_1 : A_1, \dots, x_n : A_n \vdash M : B$ *then* $\forall N_1 \in [\![A_1]\!]_\rho, \dots, N_n \in [\![A_n]\!]_\rho$ *then*

$$M[N_1/x_1, \dots, N_n/x_n] \in [\![B]\!]_\rho$$

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Corollary

System F is strongly normalizing.

Corollary

Consistency of 2nd-order Peano arithmetic.

Advantage — A versatile approach

Saturated sets can be used to show, e.g.:

- Simply Typed λ-calculus: SN,
- System F (2nd order): SN,
- System F3 (3rd order): SN,

  $\vdots$
- System Fn (order n): SN,
- System Fω (limit): SN,
- Intersection types (without ω): SN,
- Intersection types (with ω): Head Normalization,
- etc.

Disadvantage — An impredicative approach

nice, nice, BUT...

What does actually decrease?

Some open problems

"Are there combinatorial proofs of such results?"

Gödel's koan (TLCA list, Problem 26)

Is there a 'natural' assignment $\# : \Lambda^{ST} \to \mathrm{Ordinals}$, satisfying

$$M \to_\beta N \quad \Longrightarrow \quad \#M > \#N \,?$$

Lévy's koan

Is there a (less natural?) assignment for System F?

About Fω

Is every λ-term typable in Fω, already typeable in $F3$?

Semantic Results

Denotational Semantics

The first model of λ-calculus — Scott's $\mathcal{D}_\infty$ (filter model).

Intersection types:

$$A, B \ ::= \ 0 \mid \omega \mid A \to B \mid A \wedge B$$

Inference Rules:

$$\frac{}{\Gamma \vdash M : \omega}\ (U) \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \wedge B}\ (\wedge_I) \qquad \frac{\Gamma \vdash M : A \quad A \leq B}{\Gamma \vdash M : B}\ (\wedge_I)$$

where $\leq$ is a subtyping relation satisfying moreover:

$$\omega \to 0 \leq 0 \qquad 0 \leq \omega \to 0$$

The interpretation of a λ-term M

$$[\![M]\!] = \{A \mid \exists \Gamma \,.\, \Gamma \vdash M : A\}$$

is a filter w.r.t. $\leq$.

Syntactic Approximants

- The set $\mathcal{A}$ of finite approximants is generated by:

$$P, Q \ ::= \ \lambda x_1 \dots x_n.y P_1 \cdots P_k \mid \bot$$

  where $\bot \sqsubseteq M$ for all λ-terms M.

- Given a λ-term M, define the set of its approximants as:

$$\mathcal{A}(M) = \{P \mid M \to_\beta^* N \wedge P \sqsubseteq N\}$$

- The Böhm tree of M is given by taking:

$$BT(M) = \bigvee_{P \in \mathcal{A}(M)} P$$

Approximation Theorem

Theorem (Approximation Theorem)

$$\Gamma \vdash M : A \iff \exists P \in \mathcal{A}(M) \,.\, \Gamma \vdash P : A$$

Also here the problem is in the application:

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$P \in \mathcal{A}(M)$ and $Q \in \mathcal{A}(N) \not\Rightarrow PQ \in \mathcal{A}(PQ)$.

Corollary

1. *If $[\![M]\!] \neq \{\omega\}$ then M solvable.*
2. *If $\mathcal{D}_\infty \models M = N$ then $BT(M) = BT(N)$.*
3. *If $\mathcal{D}_\infty \models M = N$ then M, N are observationally indistinguishable.*

Approximation Theorem via Realizability

Realizability interpretation

$$
\begin{array}{lcl}
\mathcal{A}_{\Gamma;A} & = & \{P \in \mathcal{A} \mid \Gamma \vdash P : A\} \\
[\![0]\!]_\Gamma & = & \{M \mid \forall \vec{N} \,.\, M\vec{N} \in \mathcal{A}_{\Gamma;0}\} \\
[\![A \to B]\!]_\Gamma & = & \{M \mid \forall \Delta, \vec{N} \in [\![A]\!]_\Delta \,.\, M\vec{N} \in \mathcal{A}_{\Gamma\wedge\Delta;B}\} \\
[\![A \wedge B]\!]_\Gamma & = & [\![A]\!]_\Gamma \cap [\![B]\!]_\Gamma
\end{array}
$$

Lemma

$[\![A]\!]_\Gamma \subseteq \mathcal{A}_{\Gamma;A}$.

Proposition

$x_1 : A_1, \ldots, x_n : A_n \vdash M : B \;\Rightarrow\; \forall N_i \in [\![A_i]\!]_{\Gamma_i}, M[\vec{N}/\vec{x}] \in [\![B]\!]_{\Gamma_1\wedge\cdots\wedge\Gamma_n}$

Corollary

The Approximation Theorem.

How to get rid of Realizability?

Nota Bene. The purpose is to get rid of impredicativity.

An Answer: Using Linear Logic

Scott's model *à la sauce de* Ehrhard.

Tensor Types:

$$
\begin{array}{rcl}
A, B & ::= & \star \mid \mu \multimap A \\
\mu, \nu & ::= & 1 \mid \mu \otimes A
\end{array}
$$

The tensor product is associative, commutative, has 1 as neutral element $A \otimes 1 = A$, it is **not** idempotent $A \otimes A \neq A$.
Context $\Gamma = x_1 : \mu_1, \ldots, x_n : \mu_n$.

$$
\frac{}{x : A \vdash x : A}\ (ax) \qquad \frac{\Gamma, x : \mu \vdash M : \mu \multimap B}{\Gamma \vdash M : \mu \multimap B}\ (\multimap_I) \qquad \frac{\Gamma \vdash M : A \quad A \simeq B}{\Gamma \vdash M : B}\ (eq)
$$

$$
\frac{\Gamma \vdash M : (A_1 \otimes \cdots \otimes A_n) \multimap B \quad \Delta_i \vdash N : A_i}{\Gamma \otimes (\otimes_i \Delta_i) \vdash MN : B}\ (\multimap_E)
$$

where $\simeq$ is generated by $1 \to \star \simeq \star$. Types are otherwise unordered!

Interpretation of a λ-term: $[\![M]\!] = \{(\Gamma, A) \mid \Gamma \vdash M : A\}$

Resource Sensitiveness

A program $M : (A_1 \otimes A_2 \otimes A_3) \to B$ is able to produce a result of type B by consuming 3 resources of type A_1, A_2, A_3 during its evaluation.

Example. For all A, B, we have a proof $\pi :=$

$$\dfrac{\dfrac{x : A \multimap B \quad x : A}{x : (A \multimap B) \otimes A \vdash xx : B}}{\vdash \lambda x.xx : ((A \multimap B) \otimes A) \multimap B}$$

For $M : A \otimes (A \multimap B)$ we have $(\lambda x.xx)M : B$ and $(\lambda x.xx)M \to_\beta MM$.

$$\dfrac{\dfrac{\pi}{\vdash \lambda x.xx : ((A^A \multimap A^A) \otimes A^A) \multimap A^A} \quad \vdash \mathrm{I} : A^A \multimap A^A \quad \vdash \mathrm{I} : A^A}{\vdash (\lambda x.xx)\mathrm{I} : A \multimap A}$$

The contractum has a simpler proof:

$$\dfrac{\vdash \mathrm{I} : A^A \multimap A^A \quad \vdash \mathrm{I} : A^A}{\vdash \mathrm{I}\,\mathrm{I} : A \multimap A}$$

Qualitative Properties

Assume

$$M \to_\beta N$$

The tensor type systems satisfy the standard properties:

1. Subject Reduction. If $\Gamma \vdash M : A$ then $\Gamma \vdash N : A$.
2. Subject Expansion. If $\Gamma \vdash N : A$ then $\Gamma \vdash M : A$.

Quantitative Properties

Consider

$$(\lambda x.M)N \to_\beta M[N/x]$$

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but also more refined properties:

If π is a proof of

$$\Gamma \vdash (\lambda x.M)N : A$$

the there exists a proof π' of

$$\Gamma \vdash M[N/x] : A$$

such that $|\pi| < |\pi'|$, where $|\pi| := \#$ rules $(\multimap_E)$ in π.

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Quantitative Results

This property is preserved under “head” context (oh, joy!):

$$\lambda\vec{x}.\underline{(\lambda y.P)Q}\vec{N} \to_h \lambda\vec{x}.P[y/Q]\vec{N}$$

1. $\forall$ proof π of $\Gamma \vdash \lambda\vec{x}.(\lambda y.P)Q\vec{N} : A$
2. $\exists$ proof π' of $\Gamma \vdash \lambda\vec{x}.P[Q/y]\vec{N} : A$

Corollary - The model is sensible

$$[\![M]\!] \neq \emptyset \iff M \text{ has a head normal form}$$

Proof. ($\Leftarrow$) Easy.
($\Rightarrow$) Assume $\Gamma \vdash M : A$, then:

$$\begin{array}{ccccc} M \to_h & M_1 \to_h & M_2 \to_h & M_3 \to_h & \cdots \\ |\pi| < & |\pi_1| < & |\pi_2| < & |\pi_3| < & \cdots \end{array}$$

Impossible to have an infinite chain, the head reduction must terminate. $\square$

Unfortunately...

This does not work in general:

$$\lambda x.x\Omega \to_\beta \lambda x.x\Omega$$

and this term is typeable:

$$\dfrac{\dfrac{x : \omega \multimap A \vdash x : \omega \multimap A}{x : \omega \multimap A \vdash x\Omega : A}}{\vdash \lambda x.x\Omega : (\omega \multimap A) \multimap A}$$

Deep in the Desperation Pit...

The counterexample generalizes:

$$\lambda x.x(c_{100}\mathsf{I}) \to_\beta \lambda x.x(\mathsf{I}\cdots\mathsf{I}) \text{ 100 times}$$

these terms are typeable in the same way:

$$\cfrac{\cfrac{x:\omega \multimap A \vdash x:\omega \multimap A}{x:\omega \multimap A \vdash x(c_{100}\mathsf{I}):A}}{\vdash \lambda x.x(c_{100}\mathsf{I}):(\omega \multimap A) \multimap A}$$

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and we cannot do better.

Nothing has decreased

I only want to construct an approximant of that type!

Given a type derivation for M:

$$\Gamma \vdash \lambda x.x\underline{\Omega} : (\omega \multimap A) \multimap A$$

we can substitue the contracted redex with $\bot$, and obtain a term M'

$$\cfrac{\cfrac{x : \omega \multimap A \vdash x : \omega \multimap A}{x : \omega \multimap A \vdash x\Omega : A}}{\vdash \lambda x.x\bot : (\omega \multimap A) \multimap A}$$

typeable with π', morally the "same" derivation π.

We get:

- $|\pi| = |\pi'|$,
- $\#$ β-redexes in M $<$ $\#$ β-redexes in M'

Approximation Theorem — Quantitative Proof

Approximation Theorem

$$\Gamma \vdash M : A \iff \exists P \in \mathcal{A}(M) \,.\, \Gamma \vdash P : \bot$$

Proof.

($\Leftarrow$) Easy.
($\Rightarrow$) Start from a derivation π of

$$\Gamma \vdash M : A$$

Proceed by induction on the pair $(|\pi|, \#\beta\text{-redexes})$. At every step $M = C[(\lambda x.N)Q] \to_\beta C[N[Q/x]]$, we get:

- Either $\exists$ a derivation of $C[N[Q/x]]$ having smaller "weight",
- or $\exists$ a derivation of M having the same "weight", that also works for $M' = C[\bot]$ (having less redexes than M).

This cannot go on forever! At the end, we get the approximant P. □

Where does the magic come from?

Well,

A magician never reveals his tricks (Magician's code)

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From the hat of
Differential Linear Logic!

Idea: Replace *qualitative* with *quantitative* methods

Qualitative methods

Quantitative methods

Continuous semantics

- $[\![M]\!] =$ cont. function $A \to B$
- $[\![M]\!] = \bigvee_{A \in App(M)} [\![A]\!]$ Appr. Thm.

Böhm's approximants

$\lambda \vec{x}.yA_1 \cdots A_n$ (finite tree)

Böhm trees

$$\mathrm{BT}(M) = \bigcup_{P \in \mathcal{A}(M)} P$$

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$(D^n(s) \cdot (t_1, \ldots, t_n))0$ (linear term)

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Taylor Expansion

$$\mathcal{T}(M) = \sum_{t \in \mathcal{A}_{res}(M)} \frac{1}{\mathrm{m}(t)} t$$

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Ehrhard & Regnier's Commutation Theorem

$$\mathcal{T}(\mathrm{BT}(M)) = \mathrm{NF}_\beta \mathcal{T}(M)$$

Back to Syntax

The Resource Calculus

Resource approximants:

$$\begin{array}{rcl} t & ::= & x \mid \lambda x.t \mid t\,b \\ b & ::= & [t_1, \ldots, t_n] \qquad \text{where } n \geq 0 \end{array}$$

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Terms

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\end{array}
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The Resource Calculus

Resource approximants:

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Bags

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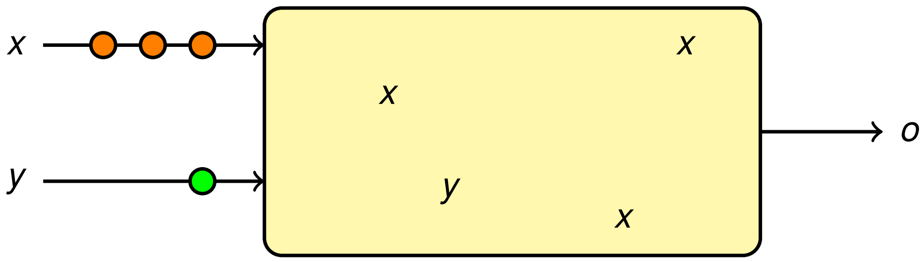
$$(\lambda xy.t)[s_{11}, s_{12}, s_{13}][s_{21}]$$



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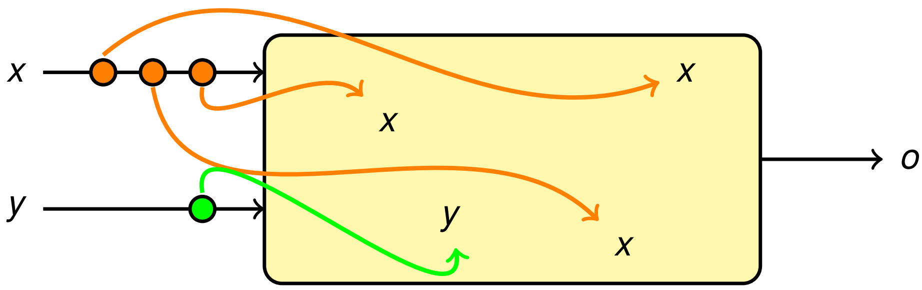

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Reduction:
If the number of occurrences of x in t equals k

$$(\lambda x.t)[s_1, \ldots, s_k] \quad \to_\beta \quad \sum_{p \in \mathfrak{S}_k} t\left\{s_{p(1)}/x_1, \ldots, s_{p(k)}/x_k\right\}$$

Otherwise:

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Linear & Confluent & Strongly Normalizable

Taylor Expansion

$$\mathcal{T} : \lambda\text{-terms} \quad \to \quad \text{(infinite) series of resource approximants}$$

$$MN \quad \mapsto \quad \sum_{k=0}^{\infty} \frac{1}{k!} M[\underbrace{N, \ldots, N}_{k \text{ times}}]$$

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Definition (Taylor expansion)

$$\mathcal{T}(x) = x \qquad \mathcal{T}(\lambda x.M) = \sum_{t \in \mathcal{T}(M)} \lambda x.t$$

$$\mathcal{T}(MN) = \sum_{k \in \mathbb{N}, t \in \mathcal{T}(M), s_1,\ldots,s_k \in \mathcal{T}(N)} t\left[s_1,\ldots,s_k\right]$$

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Notice that $t \in \mathcal{T}(M)$ probably goes to $\emptyset$.

Dynamics of Taylor Expansion

Thanks to SN,

$$\mathsf{NF}_\beta \mathcal{T}(M) = \bigcup \{ nf(t) \mid t \in \mathcal{T}(M) \}$$

always exists (it can be empty).

Thanks to the Commutation Theorem we have

$$\mathrm{BT}(M) = \mathrm{BT}(N) \iff \mathsf{NF}_\beta(\mathcal{T}(M)) = \mathsf{NF}_\beta(\mathcal{T}(N))$$

A proof of context closure via Taylor Expansion

So we can (re)prove properties of Böhm trees:

$$BT(M) = BT(N) \quad \Longrightarrow \quad \forall C[] \ . \ BT(C[M]) = BT(C[N])$$

by structural induction on $C[]$.

Proof. Assume $NF_\beta(\mathcal{T}(M)) = NF_\beta(\mathcal{T}(N))$. Consider $C[] = C_1[]C_2[]$. Take $t \in \mathrm{NF}_\beta\mathcal{T}(C_1[M]C_2[M])$. Then $\exists t' \in \mathcal{T}(C_1[M]C_2[M])$ such that

$$t' = c_1[b_1](c_2[b_2]) \twoheadrightarrow t + T$$

with $c_1[] \in \mathcal{T}(C_1[])$, $c_2[] \in \mathcal{T}(C_2[])$ and $b_i \in \mathcal{M}_f(\mathcal{T}(M))$. By confluence

$$t' \twoheadrightarrow c_1[nf(b_1)](c_2[nf(b_2)]) \twoheadrightarrow t + T \neq \emptyset$$

Therefore $\exists b'_1, b'_2 \in \mathcal{T}(N)$ s.t. $c_1[b'_1](c_2[b'_2])$ generates t. $\square$

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Therefore $\exists b_1', b_2' \in \mathcal{T}(N)$ s.t. $c_1[b_1'](c_2[b_2'])$ generates t. $\square$

Old Results with Simpler Proofs

Together with Davide Barbarossa, we had some fun:

Genericity Lemma

If $C[\Omega]$ has a β-normal form then $\forall N \in \Lambda \,.\, C[N] =_\beta C[\Omega]$.

Scott's continuity

For all $P \in \mathcal{A}(C[M])$, there exists $Q \in \mathcal{A}(M)$ such that $P \leq \mathrm{BT}(C[Q])$.

Berry's stability

Let $C[-_1, \dots, -_n]$. For all $i \in \mathcal{I}$, take $\emptyset \neq \mathcal{X}_i \subseteq \Lambda$ and $M_i \in \Lambda$. Assume, for all $i \in \mathcal{I}$, that $\mathcal{X}_i \uparrow$ and $\mathcal{A}(M_i) = \mathcal{X}_i$ then

$$\mathcal{A}(C[M_1, \dots, M_n]) = \inf\{C[N_1, \dots, N_n] \mid \forall i \in \mathcal{I} \,.\, N_i \in \mathcal{X}_i\}$$

The Perpendicular Lines Lemma

Let $n \geq 0$, $\mathcal{I} = \{1, \ldots, n\}$, $C[-_1, \ldots, -_n]$ be a n-context, $(M_{ij})_{(i,j)\in\mathcal{I}\times\mathcal{I}}$ and $(N_i)_{i\in\mathcal{I}}$ be sequences of λ-terms. Assume that

$$\forall Z \in \Lambda \left\{ \begin{array}{lcl} C[Z, \ \ M_{12}, \ldots\ldots, M_{1n}] & = & N_1 \\ C[M_{21}, Z, \ldots\ldots\ldots, M_{2n}] & = & N_2 \\ \qquad\qquad \ddots & \vdots & \vdots \\ C[M_{n1}, \ldots, M_{n(n-1)}, Z\,] & = & N_n \end{array} \right.$$

then $\forall Z_1, \ldots, Z_n \in \Lambda$, $C[Z_1, \ldots, Z_n] = N_1 = \cdots = N_n$.

Proof. Claim + confluence + strong normalization.

Claim. For all $c \in \mathcal{T}(C[\xi_1, \ldots, \xi_n])$, if $c \not\twoheadrightarrow_r 0$ then c cannot contain any hole.

Taylor Expansion, and how to get rid of other proof-techniques