Shear flows of an ideal fluid and elliptic equations in unbounded domains^{*}

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Abstract

We prove that, in a two-dimensional strip, a steady flow of an ideal incompressible fluid with no stationary point and tangential boundary conditions is a shear flow. The same conclusion holds for a bounded steady flow in a half-plane. The proofs are based on the study of the geometric properties of the streamlines of the flow and on one-dimensional symmetry results for solutions of some semilinear elliptic equations. Some related rigidity results of independent interest are also shown in n-dimensional slabs in any dimension n.

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1 Introduction and main results

Let $v(x) = (v^1(x), \ldots, v^n(x))$, $x \in \overline{\Omega} \subset \mathbb{R}^n$, be a $C^2(\overline{\Omega})$ velocity field of a steady flow of an ideal fluid, where Ω is an open connected subset of $\mathbb{R}^n$. The vector field v is a solution of the system of Euler equations:

$$\begin{cases} v \cdot \nabla v + \nabla p = 0 & \text{in } \Omega, \\ \text{div } v = 0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

The flow v is called a shear flow if there is an orthogonal coordinate system $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that v is independent of x_1 and $v^2 = \cdots = v^n = 0$. It is easy to see that the flow v is a shear flow if and only if the pressure p is a constant.

In this paper we give a characterization of shear flows in dimension $n = 2$, in the strip $\Omega_2 \subset \mathbb{R}^2$ defined by

$$\Omega_2 = \mathbb{R} \times (0, 1) = \left\{ x = (x_1, x_2) \in \mathbb{R}^2, \ 0 < x_2 < 1 \right\} \tag{1.2}$$

and in the half-plane

$$\mathbb{R}_+^2 = \mathbb{R} \times (0, +\infty) = \left\{ x = (x_1, x_2) \in \mathbb{R}^2, \ x_2 > 0 \right\}.$$

Throughout the paper, we denote $|\cdot|$ the Euclidean norm in $\mathbb{R}^m$.

Our first main result is concerned with the strip Ω_2.

Theorem 1.1 *Suppose that the flow v solving (1.1) is defined in the closed strip $\overline{\Omega_2}$ with $v^2 = 0$ on $\partial\Omega_2$ and $\inf_{\Omega_2} |v| > 0$. Then v is a shear flow, that is,*

$$v(x) = (v^1(x_2), 0) \quad in \ \overline{\Omega_2}. \tag{1.3}$$

The condition $v^2 = 0$ on $\partial\Omega_2$ simply means that v is assumed to be tangential on the boundary of Ω_2. Notice also that, in the assumptions of Theorem 1.1, the flow v is not assumed to be a priori bounded in $\overline{\Omega_2}$. However, since v is (at least) continuous in $\overline{\Omega_2}$ and the interval $[0, 1]$ is bounded, the conclusion of Theorem 1.1 implies that v is necessarily bounded.

The assumption $\inf_{\Omega_2} |v| > 0$ means that the flow v has no stationary point in $\overline{\Omega_2}$ nor limiting stationary point at infinity as $x_1 \to \pm\infty$. In other words, Theorem 1.1 means that any $C^2(\overline{\Omega_2})$ non-shear flow which is tangential on $\partial\Omega_2$ must have a stationary point in $\overline{\Omega_2}$ or at infinity. These stationary points may well be in $\overline{\Omega_2}$ or only at infinity. For instance, on the one hand, for any $\alpha \neq 0$, the non-shear cellular flow of the type

$$v(x) = \nabla^\perp \big(\sin(\alpha x_1) \sin(\pi x_2) \big) = \big(-\pi \sin(\alpha x_1) \cos(\pi x_2), \alpha \cos(\alpha x_1) \sin(\pi x_2) \big), \tag{1.4}$$

which solves (1.1) with $p(x) = (\pi^2/4) \cos(2\alpha x_1) + (\alpha^2/4) \cos(2\pi x_2)$ and is tangential on $\partial\Omega_2$, has stationary points in $\overline{\Omega_2}$. On the other hand, the flow

$$v(x) = \nabla^\perp \big(\sin(\pi x_2) e^{x_1} \big) = \big(-\pi \cos(\pi x_2) e^{x_1}, \sin(\pi x_2) e^{x_1} \big),$$

which solves (1.1) with $p(x) = -(\pi^2/2) e^{2x_1}$ and is tangential on $\partial\Omega_2$, has no stationary point in $\overline{\Omega_2}$ ($|v| > 0$ in $\overline{\Omega_2}$), but $\inf_{\Omega_2} |v| = 0$.

Lastly, we point out that, the sufficient condition $\inf_{\Omega_2} |v| > 0$ is obviously not equivalent to being a shear flow, since any shear flow $v(x) = (v^1(x_2), 0)$ for which v^1 does not have a constant strict sign does not satisfy the condition $\inf_{\Omega_2} |v| > 0$ (however, under the conditions of Theorem 1.1, the first component v^1 in (1.3) has a constant strict sign in $\overline{\Omega_2}$).

Remark 1.2 Theorem 1.1 does not hold in the dimension 3. Concretely, define in a cylinder

$$\Omega = \left\{ x \in \mathbb{R}^3, \ x_2^2 + x_3^2 < 1 \right\}$$

a vector field $v(x) = (1, -x_3, x_2)$ and a scalar function $p(x) = (x_2^2 + x_3^2)/2$. Then the pair (v, p) is a solution of the Euler equations (1.1). The velocity v is tangent to the boundary $\partial\Omega$ and satisfies the inequalities $1 \leq |v| \leq \sqrt{2}$ in Ω, but it is not a shear flow.

The second main result deals with the case of the half-plane $\mathbb{R}_+^2$.

Theorem 1.3 *Suppose that the flow v solving (1.1) is defined in the closed half-plane $\overline{\mathbb{R}_+^2}$ with $v^2 = 0$ on $\partial\mathbb{R}_+^2$ and*

$$0 < \inf_{\mathbb{R}_+^2} |v| \leq \sup_{\mathbb{R}_+^2} |v| < +\infty. \tag{1.5}$$

Then v is a shear flow, which means here that $v(x) = (v^1(x_2), 0)$ in $\overline{\mathbb{R}_+^2} = \mathbb{R} \times [0, +\infty)$. In particular, the first component v^1 has a constant strict sign in $\overline{\mathbb{R}_+^2}$.

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Some comments are in order on the condition (1.5). None of the strict inequalities can be dropped for the conclusion to hold in general. More precisely, first, any cellular flow v of the type (1.4), which solves (1.1) and is tangential on $\partial\mathbb{R}_+^2$, satisfies $\sup_{\mathbb{R}_+^2} |v| < +\infty$ and $\inf_{\mathbb{R}_+^2} |v| = 0$, and it is not a shear flow. Second, the flow

$$v(x) = \nabla^\perp\big(x_2 \cosh(x_1)\big) = (-\cosh(x_1), x_2 \sinh(x_1)),$$

which solves (1.1) with $p(x) = -\cosh(2x_1)/4 + x_2^2/2$ and is tangential on $\partial\mathbb{R}_+^2$, satisfies $\inf_{\mathbb{R}_+^2} |v| > 0$ and $\sup_{\mathbb{R}_+^2} |v| = +\infty$, and it is not a shear flow.

An interesting question would be to see whether the conclusion of Theorem 1.3 still holds if the condition 1.5 is replaced by the following weaker one

$$\forall A > 0, \quad 0 < \inf_{\mathbb{R}\times(0,A)} |v| \le \sup_{\mathbb{R}\times(0,A)} |v| < +\infty.$$

We leave it as an open problem and refer to Remark 3.4 below and the end of the proof of Theorem 1.3 in Section 3 for further comments.

The proofs of Theorems 1.1 and 1.3 rely on the study of the geometric properties of the streamlines of the flow v and of the orthogonal trajectories of the gradient flow defined by the potential u of the flow v (see definition (2.1) below). The main point is to show that all streamlines of v go from $-\infty$ to $+\infty$ in the direction x_1 (and are bounded in the direction x_2 in the case of the half-plane $\mathbb{R}_+^2$). To do so, we use a continuation argument. Therefore, the streamlines of v are shown to foliate the domain and, since the vorticity $\frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}$ is constant along the streamlines of the flow v, the potential function u will be proved to satisfy a semilinear elliptic equation of the type $\Delta u = f(u)$.

To conclude the proof of Theorem 1.3 (the case of the half-plane), we use some monotonicity and one-dimensional symmetry results for solutions with bounded gradient of such semilinear elliptic equations [1, 6]. Regarding the case of the strip Ω_2 (with bounded cross section), we will reduce Theorem 1.1 to a new Liouville type result for these elliptic equations. More precisely, we will show the following theorem.

Theorem 1.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function and let u be a $C^2(\overline{\Omega_2})$ bounded solution of the equation*

$$\Delta u + f(u) = 0 \tag{1.6}$$

in the strip Ω_2 defined in (1.2). Suppose that u is equal to some constants on the boundary lines of Ω_2: $u = 0$ on $\{x_2 = 0\}$ and $u = c$ on $\{x_2 = 1\}$, with $c > 0$, and that

$$0 < u < c \tag{1.7}$$

in Ω_2. Then u is a function of x_2 only, that is, $u(x_1, x_2) = \widetilde{u}(x_2)$ in $\overline{\Omega_2}$, and $\widetilde{u}'(x_2) > 0$ for all $0 < x_2 < 1$.

Remark 1.5 The conclusion is sharp in the sense that $\widetilde{u}'(0)$ and/or $\widetilde{u}'(1)$ may well be equal to 0 in general. For instance, the function $u(x_1, x_2) = \widetilde{u}(x_2) := 1 - \cos(\pi x_2)$ solves (1.6) in Ω_2 with $f(s) = \pi^2 s$, $u = 0$ on $\{x_2 = 0\}$, $u = 2$ on $\{x_2 = 1\}$, $0 < u < 2$ in Ω_2, $\widetilde{u}'(x_2) > 0$ in $(0,1)$, but $\widetilde{u}'(0) = \widetilde{u}'(1) = 0$.

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As a matter of fact, given the assumptions of Theorem 1.4, it follows from [1, Theorem 1.1']
applied to u (resp. $c - u(x_1, 1 - x_2)$) that

$$u_{x_2} := \frac{\partial u}{\partial x_2} > 0 \ \text{ in } \left\{(x_1, x_2) \in \mathbb{R}^2, \ 0 < x_2 < \frac{1}{2}\right\}$$

(resp. $u_{x_2} > 0$ in $\{(x_1, x_2) \in \mathbb{R}^2, \ 1/2 < x_2 < 1\}$). Therefore, $u_{x_2} \geq 0$ in $\overline{\Omega_2}$ (we point out that this
monotonicity is only known in dimension $n = 2$). The new result in Theorem 1.4 is the fact that
the monotonicity property $u_{x_2} \geq 0$ in $\overline{\Omega_2}$ implies that u is one-dimensional, that is, u is a function
of x_2 only. Actually, it turns out that this last implication holds in any dimension $n \geq 2$, as the
following theorem shows:

Theorem 1.6 *Let $n \geq 2$, let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function and let u be a
$C^2(\overline{\Omega_n})$ bounded solution of the equation (1.6) in the n-dimensional slab Ω_n defined by*

$$\Omega_n = \mathbb{R}^{n-1} \times (0, 1) = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ 0 < x_n < 1\right\}. \tag{1.8}$$

*Suppose that u is equal to some constants on the boundary hyperplanes of Ω_n: $u = 0$ on $\{x_n = 0\}$
and $u = c$ on $\{x_n = 1\}$, with $c > 0$. Suppose also that u is non-decreasing with respect to the
variable x_n, namely $u_{x_n} \geq 0$ in $\overline{\Omega_n}$. Then u is a function of x_n only, that is, $u(x_1, \ldots, x_n) = \widetilde{u}(x_n)$
in $\overline{\Omega_n}$, and $\widetilde{u}'(x_n) > 0$ for all $0 < x_n < 1$.*

Remark 1.7 Notice that the monotonicity assumption $u_{x_n} \geq 0$ in $\overline{\Omega_n}$ and the strong maximum
principle imply that, for any $k \in \mathbb{N}$ with $k \geq 2$, either

$$u(x', x_n + 1/k) = u(x', x_n) \ \text{ for all } (x', x_n) \in \mathbb{R}^{n-1} \times [0, 1 - 1/k], \tag{1.9}$$

or

$$u(x', x_n + 1/k) > u(x', x_n) \ \text{ for all } (x', x_n) \in \mathbb{R}^{n-1} \times (0, 1 - 1/k), \tag{1.10}$$

where $x' = (x_1, \ldots, x_{n-1})$. Indeed, the $C^2(\mathbb{R}^{N-1} \times [0, 1 - 1/k])$ function w defined by $w(x', x_n) =
u(x', x_n + 1/k) - u(x', x_n)$ is a nonnegative solution of an elliptic equation of the type $\Delta w + a(x) w = 0$
in $\mathbb{R}^{n-1} \times (0, 1 - 1/k)$ for some function $a \in L^\infty(\mathbb{R}^{n-1} \times (0, 1 - 1/k))$. The case (1.9) would imply
that $0 = u(x', 0) = u(x', 1/k) = \cdots = u(x', 1) = c$, a contradiction with the assumption $c > 0$.
Therefore, (1.10) holds for all $k \in \mathbb{N}$ with $k \geq 2$. Hence u is actually (strictly) increasing with
respect to the variable x_n and

$$0 < u < c \ \text{ in } \Omega_n. \tag{1.11}$$

In other words, the monotonicity assumption $u_{x_n} \geq 0$ in $\overline{\Omega_n}$ and the boundary conditions $u = 0$
on $\{x_n = 0\}$ and $u = c$ on $\{x_n = 1\}$ with $c > 0$ make the assumption (1.7) in Ω_n redundant in the
statement of Theorem 1.6. On the other hand, in Theorem 1.4, this assumption (1.7), with strict
inequalities, is essential to get the monotonicity property $u_{x_2} \geq 0$ in Ω_2, as shown in [1, 5].

In Theorem 1.6, the monotonicity condition $u_{x_n} \geq 0$ in $\overline{\Omega_n}$ is a sufficient condition for the
one-dimensional symmetry to hold. Other sufficient conditions can be given, as the following result
shows.

Proposition 1.8 *Let $n \geq 2$, let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function and let $u
be a $C^2(\overline{\Omega_n})$ bounded solution of the equation (1.6) in the n-dimensional slab Ω_n defined in (1.8).*

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Suppose that u is equal to some constants on the boundary hyperplanes of Ω_n: $u = 0$ on $\{x_n = 0\}$ and $u = c$ on $\{x_n = 1\}$ with $c > 0$, and that condition (1.7) holds. If

$$\forall\, 0 < x_n < 1, \quad 0 < \inf_{x' \in \mathbb{R}^{n-1}} u(x', x_n) \leq \sup_{x' \in \mathbb{R}^{n-1}} u(x', x_n) < c \tag{1.12}$$

or if

$$f(c) \leq 0 \leq f(0) \tag{1.13}$$

then u is a function of x_n only, that is, $u(x_1, \ldots, x_n) = \widetilde{u}(x_n)$ in $\overline{\Omega_n}$, and $\widetilde{u}'(x_n) > 0$ for all $0 < x_n < 1$.

In Proposition 1.8, if condition (1.13) is assumed, then it follows from [1, Theorem 1.1] applied to u (resp. $c - u(x', 1 - x_n)$) that $u_{x_n} > 0$ in $\{x \in \mathbb{R}^n, \ 0 < x_n < 1/2\}$ (resp. $u_{x_n} > 0$ in $\{x \in \mathbb{R}^n, \ 1/2 < x_n < 1\}$). Hence $u_{x_n} \geq 0$ in $\overline{\Omega_n}$ and the conclusion of Proposition 1.8 in this case follows from Theorem 1.6. We point out that Theorem 1.6 holds in any dimension $n \geq 2$ and without any sign assumption on $f(0)$ and $f(c)$.

The remaining sections are organized as follows. In Section 2, we show that Theorem 1.1 can be reduced to Theorem 1.4, that is, Theorem 1.4 implies Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.3. Regarding Theorem 1.4, as already explained, it can be deduced from earlier results of the literature and from Theorem 1.6, the latter being proved in Section 4 with the use of the sliding method. The proof of Proposition 1.8 is also done in Section 4. From the previous paragraph, only the case of condition (1.12) will be considered in the proof of Proposition 1.8.

2   Proof of Theorem 1.1, assuming Theorem 1.4

In this section, $v : \overline{\Omega_2} \to \mathbb{R}^2$ is a $C^2(\overline{\Omega_2})$ flow solving (1.1) in the strip Ω_2 defined in (1.2). We assume that there is $\varepsilon_0 > 0$ such that

$$|v(x)| \geq \varepsilon_0 > 0 \quad \text{for all } x \in \overline{\Omega_2}$$

and that v is tangential on the boundary, that is, $v^2 = 0$ on $\partial\Omega_2$. Our goal is to show that v is a shear flow.

Let us introduce a few important notations and definitions. First, let u be a potential function of the flow v, that is, $u : \overline{\Omega_2} \to \mathbb{R}$ is a $C^3(\overline{\Omega_2})$ function such that

$$u_{x_1} = v^2 \quad \text{and} \quad u_{x_2} = -v^1 \tag{2.1}$$

in $\overline{\Omega_2}$. Since Ω_2 is a simply connected domain, it follows that the potential function u is well and uniquely defined in $\overline{\Omega_2}$ up to a constant. We fix a unique function u such that $u(0,0) = 0$. Hence,

$$u(x_1, 0) = 0 \quad \text{for all } x_1 \in \mathbb{R} \tag{2.2}$$

since $v^2 = 0$ on $\partial\Omega_2$. For the same reason, there is a constant $c \in \mathbb{R}$ such that

$$u(x_1, 1) = c \quad \text{for all } x_1 \in \mathbb{R}. \tag{2.3}$$

We will show that u is bounded and ranges in Ω_2 between its two constant values 0 and c on $\partial\Omega_2$. To do so, we need to establish some further properties of the level curves of u. The level curves of the function u are understood as the connected components of the level sets of u. They are

actually non-parametrized streamlines of the flow v (trajectories of the flow v). For any $z \in \overline{\Omega_2}$, we denote by Γ_z the streamline of the flow v going through z. In other words, Γ_z is the level curve of u containing z, that is the connected component of the level set $\{x \in \overline{\Omega_2}, \ u(x) = u(z)\}$ containing z. Notice that, since $|v| > 0$ in $\overline{\Omega_2}$ and $v^2 = 0$ on $\partial \Omega_2$, a given streamline Γ_z of v cannot have an endpoint in $\overline{\Omega_2}$ and it always admits a C^1 parametrization $\gamma : \mathbb{R} \to \mathbb{R}^2$ ($\gamma(\mathbb{R}) = \Gamma_z$) such that $|\dot{\gamma}(t)| > 0$ for all $t \in \mathbb{R}$. Regarding the solution γ_z of $\dot{\gamma}_z(t) = v(\gamma_z(t))$ with $\gamma_z(0) = z$, it is defined on a maximal open interval $(a_-, a_+) \subset \mathbb{R}$ which may or may not be unbounded, depending on v.

Second, let us consider the gradient flow

$$\dot{y} = \nabla u = (v^2, -v^1)$$

in $\overline{\Omega_2}$, whose trajectories are orthogonal to the streamlines of the flow v. Since $|\nabla u| = |v| \geq \varepsilon_0 > 0$ in $\overline{\Omega_2}$, the following lemma holds immediately.

Lemma 2.1 *Let G be a nonempty open set included in Ω_2. Assume that u is bounded in G and let $g : \mathbb{R} \to \mathbb{R}^2$ be a parametrization of a trajectory Σ of the gradient flow $\dot{y} = \nabla u$ in G. Then Σ has a finite length such that*

$$\text{length}(\Sigma) \leq \frac{\text{osc}_G u}{\varepsilon_0} := \frac{\sup_G u - \inf_G u}{\varepsilon_0}$$

and it terminates on the boundary ∂G of G (that is, $\text{dist}(g(t), \partial G) = \inf_{x \in \partial G} |g(t) - x| \to 0$ as $t \to \pm\infty$).

Remark 2.2 Notice that Lemma 2.1 and the assumption $|\nabla u| = |v| \geq \varepsilon_0 > 0$ imply that, for any $z \in G$, the solution σ_z of $\dot{\sigma}_z(t) = \nabla u(\sigma_z(t))$ with $\sigma_z(0) = 0$ and $\sigma_z(t) \in G$ is defined on a maximal interval $(a_-, a_+) \subset \mathbb{R}$ which is necessarily bounded (however, the curve $\sigma_z((a_-, a_+))$ always admits parametrizations defined in $\mathbb{R}$).

The following three lemmas deal with some continuity (with respect to z) and asymptotic properties of the streamlines Γ_z of the flow v. In what follows, for any $r > 0$ and $z \in \overline{\Omega_2}$, we set

$$B_{z,r} = \{x \in \overline{\Omega_2}, \ |x - z| < r\}.$$

Lemma 2.3 *Let $z_1 \in \overline{\Omega_2}$ be given. For any $\varepsilon > 0$, there is $\delta > 0$ such that, for any $z_2 \in B_{z_1, \delta}$ and $z_3 \in B_{z_1, \delta}$, Γ_{z_3} lies in an ε-neighborhood of Γ_{z_2}.*

Proof. We fix $z_1 \in \overline{\Omega_2}$. For any $r > 0$, we define

$$G_{z_1, r} = \bigcup_{z \in B_{z_1, r}} \Gamma_z. \tag{2.4}$$

By definition of u and the streamlines Γ_z, there holds

$$\text{osc}_{G_{z_1, r}} u = \text{osc}_{B_{z_1, r}} u.$$

Hence, there exists a constant $C > 0$ only depending on z_1 and u such that, for all $0 < r < 1$,

$$\text{osc}_{G_{z_1, r}} u \leq C r.$$

Since $|\nabla u(z_1)| = |v(z_1)| > 0$, there is $r_1 \in (0,1]$ such that, for every $r \in (0,r_1)$, every level set of u has only one connected component in $B_{z_1,r}$ and can be written as the graph of a function depending on the variable $x \cdot v(z_1)$. Hence, for every $r \in (0,r_1)$, $z_2 \in B_{z_1,r}$ and $z_3 \in B_{z_1,r} \backslash \Gamma_{z_2}$, there holds $u(z_3) \neq u(z_2)$. Moreover, for any such r, z_2 and z_3, let us define

$$G'_{z_2,z_3,r} = \bigcup_{z \in B_{z_1,r}, \ \min(u(z_2),u(z_3)) < u(z) < \max(u(z_2),u(z_3))} \Gamma_z, \qquad (2.5)$$

which is a nonempty open set included in $G_{z_1,r} \cap \Omega_2$, with $\partial G'_{z_2,z_3,r} = \Gamma_{z_2} \cup \Gamma_{z_3}$. Notice that these properties hold even if $B_{z_1,r}$ intersects $\partial \Omega_2$: in that case, since ∇u is orthogonal to $\partial \Omega_2$ on $\partial \Omega_2$, the values 0 on $\{x_2 = 0\}$ or c on $\{x_2 = 1\}$ would be a global minimum or maximum of u in $B_{z_1,r}$, still for $r > 0$ small enough.

Now, for any $r \in (0,r_1)$, any $z_2 \in B_{z_1,r}$, any $z_3 \in B_{z_1,r} \backslash \Gamma_{z_2}$ and any $z_4 \in \Gamma_{z_3}$, let $\Sigma_{z_4} \subset G'_{z_2,z_3,r}$ be the trajectory of the gradient flow $\dot{y} = \nabla u$ with an end point at z_4 (this gradient curve Σ_{z_4} is well defined in $G'_{z_2,z_3,r}$ since by definition $\nabla u(z_4) \neq (0,0)$ is orthogonal to Γ_{z_3} at z_4). By Lemma 2.1, the second end point of Σ_{z_4} lies on $\partial G'_{z_2,z_3,r}$ and hence on Γ_{z_2} (it can not lie on Γ_{z_3} since u is strictly monotone along Σ_{z_4}). Moreover,

$$\text{length}(\Sigma_{z_4}) \leq \frac{\text{osc}_{G'_{z_2,z_3,r}} u}{\varepsilon_0} \leq \frac{\text{osc}_{G_{z_1,r}} u}{\varepsilon_0} \leq \frac{C\,r}{\varepsilon_0}.$$

Hence, for every $r \in (0,r_1)$, every $z_2 \in B_{z_1,r}$ and every $z_3 \in B_{z_1,r} \backslash \Gamma_{z_2}$, the distance between any point $z_4 \in \Gamma_{z_3}$ and Γ_{z_2} is less than Cr/ε_0 and the lemma follows. $\qquad \square$

Lemma 2.4 *For any given streamline Γ of the flow v and any C^1 parametrization $\gamma = (\gamma^1, \gamma^2) : \mathbb{R} \to \Gamma$ such that $|\dot{\gamma}(t)| > 0$ for all $t \in \mathbb{R}$, there holds $|\gamma^1(t)| \to +\infty$ as $|t| \to +\infty$.*

Proof. Assume that the conclusion does not hold. Then there are $y \in \overline{\Omega_2}$ and a sequence $(\tau_n)_{n \in \mathbb{N}}$ such that $|\tau_n| \to +\infty$ and $\gamma(\tau_n) \to y$ as $n \to +\infty$. Since $|v(y)| > 0$, it follows that, for each $k \in \mathbb{N}$ large enough, there is $t_k \in \mathbb{R}$ such that

$$v(y) \cdot (\gamma(t_k) - y) = 0, \quad |\gamma(t_k) - y| \leq k^{-1} \quad \text{and} \quad |t_k| > |t_{k-1}|. \qquad (2.6)$$

If there were some integers k and l such that $t_k < t_l$ and $\gamma(t_k) = \gamma(t_l)$, then the open set ω surrounded by $\gamma([t_k, t_l])$ would be nonempty (since $|\dot{\gamma}(t)| > 0$ for all $t \in \mathbb{R}$). By definition of γ, the function u is constant on the curve $\gamma([t_k, t_l]) = \partial \omega$. Thus, u has either an interior minimum or an interior maximum in ω, which is ruled out since u has no critical point. Therefore, the points $\gamma(t_k)$ are pairwise distinct.

Now, from (2.6), for each $k \in \mathbb{N}$ large enough, the nonzero vector $\gamma(t_k) - \gamma(t_{k-1})$ is parallel to the nonzero vector $\nabla u(y)$ (which is orthogonal to $v(y)$), whence

$$\frac{\gamma(t_k) - \gamma(t_{k-1})}{|\gamma(t_k) - \gamma(t_{k-1})|} = \pm \frac{\nabla u(y)}{|\nabla u(y)|}.$$

One infers that, for each k large enough, there is $\theta_k \in [0,1]$ such that

$$\begin{aligned}
\frac{|u(\gamma(t_k)) - u(\gamma(t_{k-1}))|}{|\gamma(t_k) - \gamma(t_{k-1})|} &= \frac{|\nabla u(\theta_k \gamma(t_k) + (1-\theta_k)\gamma(t_{k-1})) \cdot (\gamma(t_k) - \gamma(t_{k-1}))|}{|\gamma(t_k) - \gamma(t_{k-1})|} \\
&= \frac{|\nabla u(\theta_k \gamma(t_k) + (1-\theta_k)\gamma(t_{k-1})) \cdot \nabla u(y)|}{|\nabla u(y)|}.
\end{aligned}$$

But the left-hand side is equal to 0 since $u(\gamma(t_k)) = u(\gamma(t_{k-1}))$ by definition of γ, whereas the right-hand side converges to $|\nabla u(y)| > 0$ as $k \to +\infty$, a contradiction. The proof of Lemma 2.4 is thereby complete. $\qquad\square$

Now, we say that a streamline Γ of the flow v (that is, a level curve of u) is *regular* if it has a parametrization $\gamma : \mathbb{R} \to \Gamma$ such that

$$\gamma^1(t) \to \pm\infty \quad \text{as } t \to \pm\infty, \qquad (2.7)$$

with $\gamma(t) = (\gamma^1(t), \gamma^2(t))$.

Lemma 2.5 *All streamlines of the flow v are regular.*

Proof. Let a be an arbitrary real number. Since $v^2 = 0$ on the line $\Gamma = \{(x_1, 0), \ x_1 \in \mathbb{R}\}$ and v is continuous with $|v| \geq \varepsilon_0 > 0$ in $\overline{\Omega_2}$, the line Γ is equal to the streamline $\Gamma_{(a,0)}$ and it is regular. Let us consider

$$E = \big\{ b \in [0,1] : \Gamma_{(a,b)} \text{ is regular} \big\}.$$

The set E is not empty, since $0 \in E$ (notice also that $1 \in E$ for the same reason as for 0). We have to show that $E = [0,1]$. To do so, let us prove that E is both closed and relatively open in $[0,1]$.

First, let b be in E, that is, the streamline $\Gamma_{(a,b)}$ is regular. By Lemma 2.3, there is $\delta > 0$ such that $\Gamma_{(a,b)}$ lies in a 1-neighborhood of $\Gamma_{(a,b')}$ for every $b' \in [b - \delta, b + \delta] \cap [0,1]$. For any such b', by Lemma 2.4, the streamline $\Gamma_{(a,b')}$ has a C^1 parametrization $\gamma = (\gamma^1, \gamma^2) : \mathbb{R} \to \Gamma_{(a,b')}$ such that $|\dot{\gamma}(t)| > 0$ for every $t \in \mathbb{R}$ and $|\gamma^1(t)| \to +\infty$ as $|t| \to +\infty$. In particular, up to changing t into $-t$, one can assume without loss of generality that

$$\gamma^1(t) \to -\infty \quad \text{as } t \to -\infty.$$

Since the streamline $\Gamma_{(a,b)}$ is assumed to be regular and since it lies in a 1-neighborhood of $\Gamma_{(a,b')}$, it follows immediately that $\gamma^1(t) \to +\infty$ as $t \to +\infty$. Therefore, $\Gamma_{(a,b')}$ is regular for every $b' \in [b - \delta, b + \delta] \cap [0,1]$, and E is relatively open in $[0,1]$.

Second, let $b \in \overline{E}$. By Lemma 2.3, there is $\delta > 0$ such that $\Gamma_{(a,b')}$ lies in a 1-neighborhood of $\Gamma_{(a,b)}$ for every $b' \in [b - \delta, b + \delta] \cap [0,1]$. Since $b \in \overline{E}$, there is $b' \in E \cap [b - \delta, b + \delta]$. In particular, the streamline $\Gamma_{(a,b')}$ is regular. Thus, as in the previous paragraph, the streamline $\Gamma_{(a,b)}$ is regular too. In other words, $b \in E$ and E is closed.

As a conclusion, E is equal to the whole interval $[0,1]$. Since a was an arbitrary real number, one infers that all streamlines Γ_z (for any $z \in \overline{\Omega_2}$) of the flow v are regular. $\qquad\square$

We recall that u is constant along the lines $\{x_2 = 0\}$ and $\{x_2 = 1\}$, and satisfies (2.2) and (2.3), after normalization. Furthermore, since $v^2 = 0$ on $\partial\Omega_2$ and $|v| > 0$ in $\overline{\Omega_2}$, one can assume in the sequel without loss of generality that

$$v^1(0,0) < 0,$$

even if it means changing v into $-v$.

Lemma 2.6 *The function u is bounded in $\overline{\Omega_2}$, $c > 0$ and there holds*

$$0 < u < c \quad \text{in } \Omega_2.$$

Proof. Let $M \in [0, +\infty)$ be defined as

$$M = \max_S |u|,$$

where S is the segment $S = \{(0, x_2), \ x_2 \in [0,1]\}$. By Lemma 2.5 all level curves of u are regular and then intersect S. Therefore, $|u|$ is bounded by M in $\overline{\Omega_2}$.

Let now σ be the solution of

$$\dot{\sigma}(t) = \nabla u(\sigma(t)), \tag{2.8}$$

taking values in $\overline{\Omega_2}$, with

$$\sigma(0) = (0, 0). \tag{2.9}$$

Since u is bounded in Ω_2 and $|\nabla u| \geq \varepsilon_0 > 0$ in Ω_2 with $\nabla u(0,0) = (v^2(0,0), -v^1(0,0)) = (0, -v^1(0,0))$ and $-v^1(0,0) > 0$, it follows from Lemma 2.1 and Remark 2.2 that the function σ is defined on a maximal time interval $[0, \tau]$ with $\tau \in (0, +\infty)$ and that

$$\Sigma := \sigma([0, \tau]) \tag{2.10}$$

has finite length and ends on $\partial \Omega_2$. Since the function

$$\theta : t \mapsto u(\sigma(t)) \tag{2.11}$$

is (strictly) increasing in the interval $[0, \tau]$ ($\dot{\theta}(t) = |\nabla u(\sigma(t))|^2 \geq \varepsilon_0^2 > 0$ for all $t \in [0, \tau]$), the second end point $\sigma(\tau)$ of Σ lies on $\{x_2 = 1\}$, that is,

$$\sigma(\tau) = (\xi, 1)$$

for some $\xi \in \mathbb{R}$. Furthermore, since $\sigma(0) = u(0,0) = 0$ and $\sigma(\tau) = u(\xi, 1) = c$, one infers that $c > 0$ and that $0 < u(\sigma(t)) < c$ for all $t \in (0, \tau)$.

Let now x be an arbitrary point in Ω_2. By Lemma 2.5, the streamline Γ_x is regular and therefore it intersects $\Sigma = \sigma([0, \tau])$. In other words, there is $t_x \in [0, \tau]$ such that $\sigma(t_x) \in \Gamma_x$, hence

$$u(\sigma(t_x)) = u(x)$$

by definition of u and Γ_x. Notice that t_x is unique since $t \mapsto u(\sigma(t))$ is (strictly) increasing on $[0, \tau]$. Furthermore, the streamline Γ_x lies entirely in the open set Ω_2 (it can not intersect $\partial\Omega_2 = \{x_2 = 0\} \cup \{x_2 = 1\}$ since both lines $\{x_2 = 0\}$ and $\{x_2 = 1\}$ are themselves streamlines of v). Thus, $\sigma(t_x) \in \Omega_2$. Hence $t_x \in (0, \tau)$ and $0 < u(\sigma(t_x)) < c$, that is,

$$0 < u(x) < c.$$

Since $x \in \Omega_2$ was arbitrary, the proof of Lemma 2.6 is thereby complete. $\quad\square$

End of the proof of Theorem 1.1. Let $\sigma : [0, \tau] \to \overline{\Omega_2}$ be the solution of (2.8) with $\sigma(0) = (0,0)$ and $\sigma(\tau) = (\xi, 1)$ for some $\xi \in \mathbb{R}$. Since u is of class $C^3(\overline{\Omega_2})$, the function σ is of class $C^3([0, \tau])$. The function $\theta : [0, \tau] \to [0, c]$ defined by (2.11) satisfies $\dot{\theta}(t) = |\nabla u(\sigma(t))|^2 > 0$ in $[0, \tau]$ and is therefore a C^3 diffeomorphism from $[0, \tau]$ to $[0, c]$. Let us now define

$$\begin{aligned} f : [0, c] &\to \mathbb{R} \\ s &\mapsto f(s) := -\Delta u\big(\sigma(\theta^{-1}(s))\big), \end{aligned}$$

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that is,
$$f(\theta(t)) = -\Delta u(\sigma(t))$$
for all $t \in [0, \tau]$. The function f is of class C^1 in $[0, c]$ (by extending it to $f(0)$ in $(-\infty, 0)$ and by $f(c)$ in $(c, +\infty)$, it then becomes a Lipschitz continuous function defined in $\mathbb{R}$).

Let us finally show that u is a classical solution of the elliptic equation $\Delta u + f(u) = 0$ in $\overline{\Omega_2}$. To do so, observe first that $\Delta u = v_{x_1}^2 - v_{x_2}^1$ in $\overline{\Omega_2}$, and that the $C^1(\overline{\Omega_2})$ function $v_{x_1}^2 - v_{x_2}^1$ satisfies
$$v \cdot \nabla(v_{x_1}^2 - v_{x_2}^1) = 0 \quad \text{in } \overline{\Omega_2},$$
by (1.1). Therefore, the function $v_{x_1}^2 - v_{x_2}^1$ is constant along any streamline of v, that is, along any level curve of u. Let now x denote any arbitrary point in $\overline{\Omega_2}$. As in the proof of Lemma 2.6, the regular streamline Γ_x intersects $\Sigma = \sigma([0, \tau])$ and there is a unique $t_x \in [0, \tau]$ such that $\sigma(t_x) \in \Gamma_x$ and
$$\theta(t_x) = u(\sigma(t_x)) = u(x).$$
Finally, since the function $v_{x_1}^2 - v_{x_2}^1$ is constant on the streamline Γ_x containing both x and $\sigma(t_x)$, one infers from the definitions of θ and f that
$$\Delta u(x) = v_{x_1}^2(x) - v_{x_2}^1(x) = v_{x_1}^2(\sigma(t_x)) - v_{x_2}^1(\sigma(t_x)) = \Delta u(\sigma(t_x)) = -f(\theta(t_x)) = -f(u(x)). \quad (2.12)$$

As a conclusion, the function u is a classical solution of (1.6) in $\overline{\Omega_2}$ with $u = 0$ on $\{x_2 = 0\}$, $u = c$ on $\{x_2 = 1\}$ and $0 < u < c$ in Ω_2, for some Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$. Theorem 1.4 implies that u depends only on the variable x_2, that is $u(x) = \widetilde{u}(x_2)$ in $\overline{\Omega_2}$, for some $C^3([0, 1])$ function $\widetilde{u}$ such that $\widetilde{u}'(x_2) > 0$ for all $x_2 \in (0, 1)$. In other words,
$$v(x) = (-\widetilde{u}'(x_2), 0) \quad \text{in } \overline{\Omega_2}.$$
Notice finally that $v^1(x) = -\widetilde{u}'(x_2)$ has a constant sign in $\overline{\Omega_2}$, including the boundary $\partial\Omega_2$, since $|v|$ is continuous and does not vanish in $\overline{\Omega_2}$. The proof of Theorem 1.1 is thereby complete. $\square$

Remark 2.7 It follows from the conclusion of Theorem 1.1 that, for any $z = (z_1, z_2) \in \overline{\Omega_2}$, the streamline Γ_z is nothing but the line $\{(x_1, x_2) \in \mathbb{R}^2, \ x_2 = z_2\}$. Furthermore, the gradient curve Σ defined in (2.10) is equal to the segment $S = \{(0, x_2), \ x_2 \in [0, 1]\}$.

3 Proof of Theorem 1.3

The proof of Theorem 1.3 follows the same scheme as that of Theorem 1.1, apart from some additional observations induced by the unboundedness of $\mathbb{R}_+^2$ in the direction x_2. For the sake of clarity, we preferred to put the two proofs in two different sections.

In this section, $v : \overline{\mathbb{R}_+^2} \to \mathbb{R}^2$ is a $C^2(\overline{\mathbb{R}_+^2})$ flow solving (1.1) in the half-plane $\mathbb{R}_+^2$. We assume that there are $0 < \varepsilon_0 \leq M < +\infty$ such that
$$0 < \varepsilon_0 \leq |v(x)| \leq M \quad \text{for all } x \in \overline{\mathbb{R}_+^2}$$
and that v is tangential on the boundary, that is, $v^2 = 0$ on $\partial\mathbb{R}_+^2 = \{(x_1, 0), \ x_1 \in \mathbb{R}\}$. Our goal is to show that v is a shear flow.

As in Section 2, let u be a potential function of the flow v, that is, $u : \overline{\mathbb{R}_+^2} \to \mathbb{R}$ is a $C^3(\overline{\mathbb{R}_+^2})$ function satisfying (2.1) in $\mathbb{R}_+^2$. The function u is uniquely fixed by the normalization $u(0, 0) = 0$, hence
$$u(x_1, 0) = 0 \quad \text{for all } x_1 \in \mathbb{R},$$

since $v^2 = 0$ on $\partial\mathbb{R}^2_+$. The level curves of u are the streamlines Γ_z of the flow v. Since $|v| > 0$ in $\overline{\mathbb{R}^2_+}$ and $v^2 = 0$ on $\partial\mathbb{R}^2_+$, any given streamline Γ_z of v can not have an endpoint in $\overline{\mathbb{R}^2_+}$ and it has a C^1 parametrization $\gamma : \mathbb{R} \to \mathbb{R}^2$ ($\gamma(\mathbb{R}) = \Gamma_z$) such that $|\dot\gamma(t)| > 0$ for all $t \in \mathbb{R}$. Since $|\nabla u| = |v| \geq \varepsilon_0 > 0$ in $\mathbb{R}^2_+$, Lemma 2.1 holds immediately, with $\mathbb{R}^2_+$ in place of Ω_2.

For any $r > 0$ and $z \in \overline{\mathbb{R}^2_+}$, we define the restricted ball $B_{z,r}$ as

$$B_{z,r} = \left\{ x \in \overline{\mathbb{R}^2_+}, \ |x - z| < r \right\}.$$

It is obvious to see that Lemma 2.3 holds, with $\mathbb{R}^2_+$ in place of Ω_2.

Now, for any $z \in \overline{\mathbb{R}^2_+}$, we say that the streamline Γ_z is *vertically bounded* if there is a real number $A_z > 0$ such that

$$\Gamma_z \subset \mathbb{R} \times [0, A_z]. \tag{3.1}$$

Notice that, for any $x_1 \in \mathbb{R}$, the streamline $\Gamma_{(x_1,0)} = \partial\mathbb{R}^2_+$ is vertically bounded. We shall prove that all streamlines Γ_z are vertically bounded and regular, in the sense of (2.7). To do so, we first observe that the analogue of Lemma 2.4 holds for vertically bounded streamlines.

Lemma 3.1 *For any given vertically bounded streamline Γ of the flow v and any C^1 parametrization $\gamma = (\gamma^1, \gamma^2) : \mathbb{R} \to \Gamma$ such that $|\dot\gamma(t)| > 0$ for all $t \in \mathbb{R}$, there holds $|\gamma^1(t)| \to +\infty$ as $|t| \to +\infty$.*

Proof. It is identical to that of Lemma 2.4 up to replacing $\overline{\Omega_2}$ by $\mathbb{R} \times [0, A]$, where $A > 0$ is such that $\Gamma \subset \mathbb{R} \times [0, A]$. □

Lemma 2.5 can now be extended as follows.

Lemma 3.2 *All streamlines Γ_z of the flow v are regular in the sense of (2.7) and vertically bounded in the sense of (3.1).*

Proof. Let a be an arbitrary real number. Since $v^2 = 0$ on the line $\Gamma = \{(x_1, 0), \ x_1 \in \mathbb{R}\} = \partial\mathbb{R}^2_+$ and v is continuous with $|v| \geq \varepsilon_0 > 0$ in $\overline{\mathbb{R}^2_+}$, $\Gamma = \Gamma_{(a,0)}$ and this streamline is regular and vertically bounded. Let us consider

$$E = \left\{ b \in [0, +\infty), \ \Gamma_{(a,b)} \text{ is regular and vertically bounded} \right\}. \tag{3.2}$$

The set E is not empty, since $0 \in E$. Let us now show that E is both closed and relatively open in $[0, +\infty)$.

First, let b be in E, that is, the streamline $\Gamma_{(a,b)}$ is regular and vertically bounded. By Lemma 2.3 (with $\mathbb{R}^2_+$ in place of Ω_2), there is $\delta > 0$ such that $\Gamma_{(a,b)}$ lies in a 1-neighborhood of $\Gamma_{(a,b')}$ and $\Gamma_{(a,b')}$ lies in a 1-neighborhood of $\Gamma_{(a,b)}$ for every $b' \in [b - \delta, b + \delta] \cap [0, +\infty)$. For any such b', the streamline $\Gamma_{(a,b')}$ is vertically bounded, since so is $\Gamma_{(a,b)}$. By Lemma 3.1, $\Gamma_{(a,b')}$ has a C^1 parametrization $\gamma = (\gamma^1, \gamma^2) : \mathbb{R} \to \Gamma_{(a,b')}$ such that $|\dot\gamma(t)| > 0$ for every $t \in \mathbb{R}$ and $|\gamma^1(t)| \to +\infty$ as $|t| \to +\infty$. Since $\Gamma_{(a,b)}$ is regular, one then concludes as in the proof of Lemma 2.5 that $\Gamma_{(a,b')}$ is regular. Therefore, $b' \in E$. Hence, $[b - \delta, b + \delta] \cap [0, +\infty) \subset E$ and E is relatively open in $[0, +\infty)$.

Second, let $b \in \overline{E}$. By Lemma 2.3 (with $\mathbb{R}^2_+$ in place of Ω_2), there is $\delta > 0$ such that $\Gamma_{(a,b)}$ lies in a 1-neighborhood of $\Gamma_{(a,b')}$ and $\Gamma_{(a,b')}$ lies in a 1-neighborhood of $\Gamma_{(a,b)}$ for every $b' \in [b - \delta, b + \delta] \cap [0, +\infty)$. Since $b \in \overline{E}$, there is $b' \in E \cap [b - \delta, b + \delta]$. In particular, the streamline $\Gamma_{(a,b')}$ is vertically bounded and regular. Thus, as in the previous paragraph, the streamline $\Gamma_{(a,b)}$ is vertically bounded and regular too. In other words, $b \in E$ and E is closed.

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As a conclusion, E is equal to the whole interval $[0, +\infty)$. Since a was an arbitrary real number, the conclusion of Lemma 3.2 follows. $\qquad\square$

We recall that u is constant along the lines $\Gamma = \{(x_1, 0), x_1 \in \mathbb{R}\} = \partial \mathbb{R}_+^2$, with $u = 0$ on $\partial \mathbb{R}_+^2$ after normalization. As in Section 2, since $v^2 = 0$ on $\partial \mathbb{R}_+^2$ and $|v| > 0$ in $\overline{\mathbb{R}_+^2}$, one can assume in the sequel without loss of generality that

$$v^1(0,0) < 0.$$

Lemma 3.3 *There holds*

$$u > 0 \text{ in } \mathbb{R}_+^2.$$

Proof. Let σ be the solution of (2.8) and (2.9) taking values in $\overline{\mathbb{R}_+^2}$, that is $\dot{\sigma}(t) = \nabla u(\sigma(t))$, and $\sigma(0) = (0,0)$. Since $\nabla u(0,0) = (v^2(0,0), -v^1(0,0)) = (0, -v^1(0,0))$ and $-v^1(0,0) > 0$, the function σ is defined on a maximal interval I of the type $[0, \tau]$ with $\tau \in (0, +\infty)$, or $[0, \tau)$ with $\tau \in (0, +\infty]$. The gradient curve $\Sigma = \sigma(I)$ cannot end at an (interior) point in $\mathbb{R}_+^2$, since $|\nabla u| = |v| > 0$ in $\mathbb{R}_+^2$. It cannot end on $\partial \mathbb{R}_+^2$ either, since the function

$$\theta : t \mapsto u(\sigma(t)) \tag{3.3}$$

is (strictly) increasing in I ($\dot{\theta}(t) = |\nabla u(\sigma(t))|^2 \geq \varepsilon_0^2 > 0$ for all $t \in I$) and u is constant on $\partial \mathbb{R}_+^2$. Therefore, Σ has infinite length and, since $|\dot{\sigma}(t)| = |\nabla u(\sigma(t))| \leq M$ for all $t \in I$, one infers that I is infinite, that is $I = [0, +\infty)$ and

$$\Sigma = \sigma([0, +\infty)). \tag{3.4}$$

Moreover, $u(\sigma(t)) > u(\sigma(0)) = u(0,0) = 0$ for all $t \in (0, +\infty)$.

Write $\sigma(t) = (\sigma^1(t), \sigma^2(t))$ for all $t \in [0, +\infty)$. We claim that

$$\limsup_{t \to +\infty} \sigma^2(t) = +\infty. \tag{3.5}$$

Indeed, otherwise, the function $t \mapsto \sigma^2(t)$ would be bounded in $[0, +\infty)$ and there would exist $A > 0$ such that $\Sigma \subset \mathbb{R} \times [0, A]$. But $u_{x_2} = -v^1$ is bounded in $\mathbb{R} \times [0, A]$ (since v is actually assumed to be bounded in $\mathbb{R}_+^2$) and $u = 0$ on $\{x_2 = 0\}$. Thus, u is bounded in $\mathbb{R} \times [0, A]$, hence $\theta(t) = u(\sigma(t))$ is bounded in $[0, +\infty)$. But $\dot{\theta}(t) = |\nabla u(\sigma(t))|^2 \geq \varepsilon_0^2 > 0$ for all $t \in [0, +\infty)$, leading to a contradiction. As a consequence, the claim (3.5) is proved.

Let now x be an arbitrary point in $\mathbb{R}_+^2$. By Lemma 3.2, the streamline Γ_x is regular and vertically bounded. Therefore, by (3.5) and $\sigma(0) = (0,0)$, Γ_x intersects $\Sigma = \sigma([0, +\infty))$. In other words, there is $t_x \in [0, +\infty)$ such that $\sigma(t_x) \in \Gamma_x$, hence

$$u(\sigma(t_x)) = u(x).$$

The real number t_x is unique since $t \mapsto u(\sigma(t))$ is (strictly) increasing. Furthermore, the streamline Γ_x lies entirely in the open set $\mathbb{R}_+^2$ (it can not intersect $\partial \mathbb{R}_+^2 = \{x_2 = 0\}$ since the line $\{x_2 = 0\}$ is itself a streamline of v). Thus, $\sigma(t_x) \in \mathbb{R}_+^2$. Hence $t_x \in (0, +\infty)$ and $u(\sigma(t_x)) > 0$, that is, $u(x) > 0$. The proof of Lemma 3.3 is thereby complete. $\qquad\square$

Remark 3.4 It is straightforward to see that, once Lemma 3.2 is proved, the conclusion of Lemma 3.3 holds if, instead of (1.5), one only assumes that

$$\forall A > 0, \quad 0 < \inf_{\mathbb{R} \times (0,A)} |v| \leq \sup_{\mathbb{R} \times (0,A)} |v| < +\infty, \tag{3.6}$$

even if the interval $[0, \tau)$ might be bounded in that case. Lemma 2.1 and 2.3 could be extended under this hypothesis, provided that $G \subset \mathbb{R}_+^2$ in Lemma 2.1 is vertically bounded and $\Gamma_{z_1} \subset \overline{\mathbb{R}_+^2}$ in Lemma 2.3 is vertically bounded. However, it is unclear to prove Lemma 3.2 with (3.6) instead of (1.5) (the proof of the closedness of the set E defined in (3.2) is not clear).

End of the proof of Theorem 1.3. Let $\sigma : [0, +\infty) \to \overline{\mathbb{R}_+^2}$ be the solution of (2.8) and (2.9). Since u is of class $C^3(\overline{\mathbb{R}_+^2})$, the function σ is of class $C^3([0, +\infty))$. The function $\theta : [0, +\infty) \to [0, +\infty)$ defined by (3.3) satisfies $\dot{\theta}(t) = |\nabla u(\sigma(t))|^2 \geq \varepsilon_0^2 > 0$ in $[0, +\infty)$ with $\theta(0) = u(\sigma(0)) = u(0, 0) = 0$ and is therefore a C^3 diffeomorphism from $[0, +\infty)$ to itself. Let us now define

$$
\begin{aligned}
f : [0, +\infty) &\to \mathbb{R} \\
s &\mapsto f(s) := -\Delta u\big(\sigma(\theta^{-1}(s))\big),
\end{aligned}
$$

that is, $f(\theta(t)) = -\Delta u(\sigma(t))$ for all $t \in [0, +\infty)$. The function f is of class C^1 in $[0, +\infty)$ (by extending it to $f(0)$ in $(-\infty, 0)$, it is then a locally Lipschitz continuous function defined in $\mathbb{R}$).

Let us finally show that u is a classical solution of the elliptic equation $\Delta u + f(u) = 0$ in $\overline{\mathbb{R}_+^2}$. As in the proof of Theorem 1.1, $\Delta u = v_{x_1}^2 - v_{x_2}^1$ and $v \cdot \nabla(v_{x_1}^2 - v_{x_2}^1) = 0$ in $\overline{\mathbb{R}_+^2}$. Therefore, the function $v_{x_1}^2 - v_{x_2}^1$ is constant along any streamline of v. Let now x denote an arbitrary point in $\overline{\mathbb{R}_+^2}$. From Lemma 3.2 and (3.5), the regular and vertically bounded streamline Γ_x intersects $\Sigma = \sigma\big([0, +\infty)\big)$ and there is a unique $t_x \in [0, +\infty)$ such that $\sigma(t_x) \in \Gamma_x$ and $\theta(t_x) = u(\sigma(t_x)) = u(x)$. Finally, since $v_{x_1}^2 - v_{x_2}^1$ is constant on the streamline Γ_x containing both x and $\sigma(t_x)$, one infers as in (2.12) that

$$
\Delta u(x) = -f(u(x)).
$$

As a conclusion, the function u is a classical solution of (1.6) in $\overline{\mathbb{R}_+^2}$ with $u = 0$ on $\partial \mathbb{R}_+^2 = \{x_2 = 0\}$ and $u > 0$ in $\mathbb{R}_+^2$, for some locally Lipschitz-continuous function $f : \mathbb{R} \to \mathbb{R}$. For any $A > 0$, the function u is positive and bounded in the strip $\mathbb{R} \times (0, A)$, it is then a classical solution of an equation of the type $\Delta u + f_A(u) = 0$ in $\mathbb{R} \times [0, A]$ for some globally Lipschitz continuous function $f_A : [0, +\infty) \to \mathbb{R}$ such that $f_A = f$ on the image of $\mathbb{R} \times [0, A]$ by u. It then follows from [1, Theorem 1.1'] that

$$
u_{x_2} > 0 \text{ in } \mathbb{R} \times \left(0, \frac{A}{2}\right),
$$

hence $u_{x_2} > 0$ in $\mathbb{R}_+^2$ since $A > 0$ can be arbitrary (see also [3, Theorem 2], [4, Remark 2] and [5, Theorem 1.1] for related results). Finally, since $|\nabla u| = |v|$ is bounded in $\mathbb{R}_+^2$ (the second strict inequality in (1.5) is used here), one concludes from [6, Theorem 1.3] (see also [5, Theorems 1.6]) that u depends only on the variable x_2, that is

$$
u(x) = \widetilde{u}(x_2) \text{ in } \overline{\mathbb{R}_+^2},
$$

for some $C^3([0, +\infty))$ function $\widetilde{u}$ such that $\widetilde{u}'(x_2) > 0$ for all $x_2 \in (0, +\infty)$. In other words,

$$
v(x) = (-\widetilde{u}'(x_2), 0) \text{ in } \overline{\mathbb{R}_+^2}.
$$

Furthermore, $v^1(x) = -\widetilde{u}'(x_2)$ has a constant sign in $\overline{\mathbb{R}_+^2}$, including the boundary $\partial \mathbb{R}_+^2$, since $|v|$ is continuous and does not vanish in $\overline{\mathbb{R}_+^2}$. The proof of Theorem 1.3 is thereby complete. $\square$

Remark 3.5 It follows from the conclusion of Theorem 1.3 that, for any $z = (z_1, z_2) \in \overline{\mathbb{R}_+^2}$, the streamline Γ_z is nothing but the line $\{(x_1, x_2) \in \mathbb{R}^2, \ x_2 = z_2\}$. Furthermore, the gradient curve Σ defined in (3.4) is equal to the half-line $\{(0, x_2), \ x_2 \in [0, +\infty)\}$.

4 Proof of Theorem 1.6 and Proposition 1.8

This section is devoted to the proof of the one-dimensional symmetry results concerned with the elliptic equation (1.6). As already emphasized in Section 1, Theorem 1.4 follows from Theorem 1.6 and from earlier results in the literature. It only remains to show Theorem 1.6 and Proposition 1.8.

Proof of Theorem 1.6. Let Ω_n be the n-dimensional slab defined in (1.8), with $n \geq 2$. Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function and let u be a $C^2(\overline{\Omega_n})$ bounded solution of the equation (1.6) in Ω_n with $u = 0$ on $\{x_n = 0\}$, $u = c$ on $\{x_n = 1\}$ and $c > 0$. We also assume that $u_{x_n} \geq 0$ in $\overline{\Omega_n}$.

In order to prove that u depends on the variable x_n only, we will show that it is increasing in any direction $\xi = (\xi_1, \cdots, \xi_n) = (\xi', \xi_n)$ with $\xi' = (\xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ and $\xi_n > 0$. We fix such a vector ξ and we will use a sliding method [2], by sliding u in the direction ξ and comparing the shifted function to the function u itself. Namely, for any $\tau \in (0, 1/\xi_n)$, we define

$$\Omega^\tau = \mathbb{R}^{n-1} \times (0, 1 - \tau\xi_n) \tag{4.1}$$

and

$$w^\tau(x) = u(x + \tau\xi) - u(x) \quad \text{for } x \in \overline{\Omega^\tau}. \tag{4.2}$$

Our aim is to show that $w^\tau > 0$ in $\overline{\Omega^\tau}$ for all $\tau \in (0, 1/\xi_n)$.

First, it follows from standard elliptic estimates up to the boundary that ∇u is bounded in $\overline{\Omega_n}$. Since $u(x', 0) = 0 < c = u(x', 1)$ for all $x' \in \mathbb{R}^{n-1}$, one gets the existence of $\tau^* \in (0, 1/\xi_n)$ such that

$$w^\tau > 0 \quad \text{in } \overline{\Omega^\tau} \quad \text{for all } \tau \in (\tau^*, 1/\xi_n).$$

Let us now define

$$\tau_* = \inf \left\{ \tau \in (0, 1/\xi_n), \ w^{\tau'} > 0 \ \text{in } \overline{\Omega^{\tau'}} \text{ for all } \tau' \in (\tau, 1/\xi_n) \right\}. \tag{4.3}$$

There holds $0 \leq \tau_* \leq \tau^* < 1/\xi_n$.

Let us assume by contradiction that $\tau_* > 0$. One has $w^\tau > 0$ in $\overline{\Omega^\tau}$ for all $\tau \in (\tau_*, 1/\xi_n)$ and $w^{\tau_*} \geq 0$ in $\overline{\Omega^{\tau_*}}$ by continuity. Moreover, there are a sequence $(\tau_k)_{k \in \mathbb{N}}$ in $(0, \tau_*]$ converging to τ_* and a sequence $(x^k)_{k \in \mathbb{N}}$ of points in $\mathbb{R}^n$ such that

$$x^k \in \overline{\Omega^{\tau_k}} \quad \text{and} \quad w^{\tau_k}(x^k) \leq 0$$

for all $k \in \mathbb{N}$. Write $x^k = (x'^k, x_n^k)$ with $x'^k \in \mathbb{R}^{n-1}$ and $x_n^k \in [0, 1 - \tau_k\xi_n]$, and define

$$\begin{cases} u_k(x) = u(x' + x'^k, x_n) & \text{for } x \in \overline{\Omega_n}, \\ w_k(x) = w^{\tau_*}(x' + x'^k, x_n) = u_k(x + \tau_*\xi) - u_k(x) & \text{for } x \in \overline{\Omega^{\tau_*}}. \end{cases} \tag{4.4}$$

From standard elliptic estimates, since f is locally Lipschitz continuous and u is a $C^2(\overline{\Omega_n})$ bounded solution of (1.6), the sequences of the functions u_k and w_k are actually bounded in $C_{loc}^{2,\alpha}(\overline{\Omega_n})$ and $C_{loc}^{2,\alpha}(\overline{\Omega^{\tau_*}})$ respectively, for every $\alpha \in [0, 1)$. Therefore, up to extraction of a subsequence, the functions u_k converge in $C_{loc}^2(\overline{\Omega_n})$ to a $C^2(\overline{\Omega_n})$ bounded solution U of (1.6). By passing to the limit, one gets that $U = 0$ on $\{x_n = 0\}$, $U = c$ on $\{x_n = 1\}$ and $U_{x_n} \geq 0$ in $\overline{\Omega_n}$ (hence, $0 \leq U \leq c$ in $\overline{\Omega_n}$). It follows then from Remark 1.7, as in (1.11), that

$$0 < U < c \quad \text{in } \Omega_n. \tag{4.5}$$

Now, by definition of w_k, the functions w_k then converge in $C^2_{loc}(\overline{\Omega^{\tau_*}})$ to the function W defined in $\overline{\Omega^{\tau_*}}$ by

$$W(x) = U(x + \tau_* \xi) - U(x) \text{ for } x \in \overline{\Omega^{\tau_*}}.$$

Furthermore, $W \geq 0$ in $\overline{\Omega^{\tau_*}}$ (since so is w^{τ_*}). Up to extraction of a subsequence, one can assume that $x_n^k \to \widetilde{x}_n$ as $k \to +\infty$ for some $\widetilde{x}_n \in [0, 1 - \tau_* \xi_n]$. Let us set $\widetilde{x} = (0, \cdots, 0, \widetilde{x}_n) \in \overline{\Omega^{\tau_*}}$. Lastly, $w^{\tau_k}(x^k) \leq 0$ means that

$$u(x'^k + \tau_k \xi', x_n^k + \tau_k \xi_n) \leq u(x'^k, x_n^k),$$

that is, $u_k(\tau_k \xi', x_n^k + \tau_k \xi_n) \leq u_k(0, x_n^k)$. By passing to the limit as $k \to +\infty$, one infers that $U(\tau_* \xi', \widetilde{x}_n + \tau_* \xi_n) \leq U(0, \widetilde{x}_n)$, that is $U(\widetilde{x} + \tau_* \xi) \leq U(\widetilde{x})$, i.e. $W(\widetilde{x}) \leq 0$. But $W \geq 0$ in $\overline{\Omega^{\tau_*}}$. Hence, $W(\widetilde{x}) = 0$, that is

$$U(\widetilde{x} + \tau_* \xi) = U(\widetilde{x}).$$

Several cases can occur, whether $\widetilde{x}$ be on $\partial \Omega^{\tau_*} = \{x_n = 0\} \cup \{x_n = 1 - \tau_* \xi_n\}$ or in Ω^{τ_*}. On the one hand, if $\widetilde{x}_n = 0$, then $U(\widetilde{x}) = 0$. Hence, $U(\widetilde{x} + \tau_* \xi) = 0$. But $\widetilde{x} + \tau_* \xi \in \Omega_n$ (it is an interior point) since $0 < \widetilde{x}_n + \tau_* \xi_n = \tau_* \xi_n < 1$, contradicting (4.5). On the other hand, if $\widetilde{x}_n = 1 - \tau_* \xi_n$, then $U(\widetilde{x} + \tau_* \xi) = c$. Hence, $U(\widetilde{x}) = c$. But $\widetilde{x} \in \Omega_n$ since $0 < \widetilde{x}_n = 1 - \tau_* \xi_n < 1$, contradicting again (4.5). Therefore, $\widetilde{x} \in \Omega^{\tau_*}$ (it is an interior point). But the function W is a nonnegative solution of an equation of the type $\Delta W + c(x) W = 0$ in Ω^{τ_*}, for some function $c \in L^\infty(\Omega^{\tau_*})$. The strong maximum principle implies that $W = 0$ in Ω^{τ_*}, and then on $\partial \Omega^{\tau_*}$ by continuity. This leads to a contradiction as in the beginning of this paragraph.

Finally, the assumption $\tau_* > 0$ is ruled out. In other words, $\tau_* = 0$ and

$$u(x + \tau \xi) > u(x) \text{ for all } x \in \overline{\Omega^\tau} \text{ and for all } 0 < \tau < 1/\xi_n.$$

This means that u is increasing in any direction $\xi = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that $\xi_n > 0$. By continuity, for any $\xi' \in \mathbb{R}^{n-1}$, u then is nondecreasing in the direction $\xi = (\xi', 0)$. So is it in the direction $\xi = (-\xi', 0)$. As a consequence, u does not depend on the direction $(\xi', 0)$ and, since $\xi' \in \mathbb{R}^{n-1}$ is arbitrary, u depends only on the variable x_n, that is $u(x) = \widetilde{u}(x_n)$ for some $C^2([0, 1])$ function $\widetilde{u}$. The function $\widetilde{u}$ is actually increasing in $[0, 1]$ from Remark 1.7. Lastly, for any $h \in (0, 1/2]$, the $C^2([0, h])$ function z defined by

$$z(x_n) = \widetilde{u}(2h - x_n) - \widetilde{u}(x_n)$$

is nonnegative and it satisfies an equation of the type $z''(x_n) + d(x_n) z(x_n) = 0$ in $[0, h]$ for some bounded function d. Furthermore, $z(h) = 0$, and $z > 0$ in $[0, h)$ since $\widetilde{u}$ is increasing in $[0, 1]$. Hopf lemma implies that $z'(h) < 0$, that is, $\widetilde{u}'(h) > 0$ for any $h \in (0, 1/2]$. Similarly, by working with the function $c - \widetilde{u}(1 - x_n)$, one infers that $\widetilde{u}'(h) > 0$ for all $h \in [1/2, 1)$. The proof of Theorem 1.6 is thereby complete. $\square$

Proof of Proposition 1.8. Let $n \geq 2$, let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function and let u be a $C^2(\overline{\Omega_n})$ bounded solution of (1.6) in Ω_n with $u = 0$ on $\{x_n = 0\}$, $u = c$ on $\{x_n = 1\}$ and $c > 0$. As emphasized in Section 1, we only need to consider the case where u satisfies (1.12). As in the proof of Theorem 1.6, we will show that u is increasing in any direction $\xi = (\xi', \xi_n) \in \mathbb{R}^n_+$ with $\xi' \in \mathbb{R}^{n-1}$ and $\xi_n > 0$.[1]

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[1] We could also prove with the same method that u is increasing in x_n and then use the conclusion of Theorem 1.6. But we preferred to prove directly the monotonicity in the direction ξ, since the notations will be the same as in the proof of Theorem 1.6.

We fix such a vector ξ. For any $\tau \in (0, 1/\xi_n)$, we define Ω^τ and w^τ as in (4.1) and (4.2). Since $u(x', 0) = 0 < c = u(x', 1)$ for all $x' \in \mathbb{R}^{n-1}$, one gets as in the proof of Theorem 1.6 the existence of $\tau^* \in (0, 1/\xi_n)$ such that $w^\tau > 0$ in $\overline{\Omega^\tau}$ for all $\tau \in (\tau^*, 1/\xi_n)$. We then define $\tau_* \in [0, \tau^*]$ as in (4.3) and assume by contradiction that $\tau_* > 0$. One has $w^\tau > 0$ in $\overline{\Omega^\tau}$ for all $\tau \in (\tau_*, 1/\xi_n)$ and $w^{\tau_*} \geq 0$ in $\overline{\Omega^{\tau_*}}$. Moreover, there are a sequence $(\tau_k)_{k \in \mathbb{N}}$ in $(0, \tau_*]$ converging to τ_* and a sequence $(x^k)_{k \in \mathbb{N}}$ of points in $\mathbb{R}^n$ such that $x^k \in \overline{\Omega^{\tau_k}}$ and $w^{\tau_k}(x^k) \leq 0$ for all $k \in \mathbb{N}$. Define u_k and w_k as in (4.4). Up to extraction of a subsequence, the functions u_k converge in $C^2_{loc}(\overline{\Omega_n})$ to a $C^2(\overline{\Omega_n})$ bounded solution U of (1.6) such that $U = 0$ on $\{x_n = 0\}$ and $U = c$ on $\{x_n = 1\}$. Furthermore, U satisfies

$$\forall\, 0 < x_n < 1, \quad 0 < \inf_{x' \in \mathbb{R}^{n-1}} U(x', x_n) \leq \sup_{x' \in \mathbb{R}^{n-1}} U(x', x_n) < c$$

since u satisfies this assumption (1.12) and this condition is invariant by translation in the directions x'. In particular, $0 < U < c$ in Ω_n, that is, U still satisfies (4.5). One then gets a contradiction exactly as in the proof of Theorem 1.6.

Therefore, $\tau_* = 0$ and $u(x + \tau\xi) > u(x)$ for all $\xi = (\xi', \xi_n)$ with $\xi_n > 0$ and for all $\tau \in (0, 1/\xi_n)$ and $x \in \overline{\Omega^\tau}$. As in the proof of Theorem 1.6, one concludes that $u(x) = \widetilde{u}(x_n)$ only depends on x_n and that $\widetilde{u}'(x_n) > 0$ for all $x_n \in (0, 1)$. $\qquad\square$

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