Extinction vs. persistence in strong oscillating flows

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Abstract

In this paper, we give some conditions for finite-time extinction or persistence of the solutions of diffusion-advection equations in strong and oscillating flows, under Dirichlet boundary conditions. The enhancement of the diffusion rate depends on the interplay between strong advection and time-homogenization, and in particular on the ratio between the strength of the flow and its frequency parameter. Quantitative estimates of this ratio, which depend on the geometry of the domain, are provided in the case of a uniform flow. In the general time-space dependent case, the finite-time behavior of the solutions is related to on the existence of first integrals of the flow.

1 Introduction

This paper is concerned with the analysis of the influence of large advection on persistence or extinction property at finite times for the solutions of diffusion-advection equations

$$\begin{cases} \dfrac{\partial u_\gamma}{\partial t} = \Delta u_\gamma + A(\gamma) V(\gamma t, x) \cdot \nabla u_\gamma, \quad t > 0, \ x \in \Omega, \\ u_\gamma(t,x) = 0, \quad t > 0, \ x \in \partial\Omega, \\ u_\gamma(0,x) = u_{0,\gamma}(x), \quad x \in \Omega \end{cases} \tag{1.1}$$

with a large frequency parameter $\gamma > 0$. The domain $\Omega \subset \mathbb{R}^N$ is assumed to be bounded and of class C^2. The given flow V is in $L^\infty((0,+\infty) \times \Omega, \mathbb{R}^N)$, and its strength $A(\gamma)$ is typically large. Throughout the paper, we assume that $A(\gamma) \geq 0$ for all $\gamma > 0$ and that V is incompressible, that is divergence-free, for almost all times, in the sense that for a.e. $s \in \mathbb{R}_+ = [0, +\infty)$

$$\forall \ \psi \in W_0^{1,1}(\Omega), \quad \int_\Omega V(s,x) \cdot \nabla \psi(x) \, dx = 0.$$

For any initial condition $u_{0,\gamma} \in L^2(\Omega)$, the solution $t \mapsto u_\gamma(t,\cdot)$ of (1.1) is continuous in $L^2(\Omega)$ for $t \geq 0$, and, from standard parabolic estimates, it belongs to $L^\infty((\varepsilon,+\infty), W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega))$ and $\frac{du_\gamma}{dt} \in L^\infty((\varepsilon,+\infty), L^p(\Omega))$ for all $1 \leq p < +\infty$ and $\varepsilon > 0$.

We are interested in comparing the roles of diffusion and advection when the latter is large. For the sake of simplicity of the presentation, the second-order term in (1.1) is the Laplacian, but the results of the present paper could easily be adapted to the case when the second-order term is of the divergence form $\mathrm{div}(a(x)\nabla)$, where a is a uniformly elliptic symmetric matrix field in $\overline{\Omega}$. For each $\gamma > 0$ and each initial condition $u_{0,\gamma}$, the solution u_γ of (1.1) always decays with respect to time in L^2 norm and it converges to 0 in L^2 norm as $t \to +\infty$. Here, the issue is to know whether, given an initial condition, the solutions persist at any finite time (the L^2 norm stays away from 0) or converge to 0 when the amplitude A and the frequency scale γ are large. Many results were concerned with large time estimates and homogenization limits of the solutions after rescaling, but it is of great importance to estimate how fast the diffusion rate is enhanced by strong flows at finite times, especially in some applications to nonlinear reaction-diffusion-advection problems. Roughly speaking, since the amplitude is expected to be large, the behaviors of the solutions u_γ shall depend on the mixing properties of the flow, but they shall also depend on the interplay between the strength $A(\gamma)$ of the flow and the temporal frequency γ.

When $V = V(x)$ depends on x only, the following result was proved in [3]: if V has a first integral $w \in H_0^1(\Omega)\backslash\{0\}$, that is

$$V \cdot \nabla w = 0 \ \text{ a.e. in } \Omega,$$

then there are initial conditions $u_0 \in L^2(\Omega)$ such that the solutions u^A of (1.1) with advection $A\,V(x)$ satisfy

$$\liminf_{A \to +\infty} \|u^A(t,\cdot)\|_{L^2(\Omega)} > 0$$

for each $t > 0$; on the other hand, if V has no first integral $w \in H_0^1(\Omega)\backslash\{0\}$, then

$$\lim_{A \to +\infty} \|u^A(t,\cdot)\|_{L^2(\Omega)} = 0$$

for all $t > 0$ and for all $u_0 \in L^2(\Omega)$.[1] For Hamiltonian systems in even dimensions, when $V = \overline{\nabla} w$ is the orthogonal gradient of a first integral w which satisfies additional non-degeneracy assumptions, the solutions u^A behave as $A \to +\infty$ like those of an effective diffusion equation on the Reeb graph of w (see [19]). For equations which are set on a smooth compact Riemannian manifold, assuming that the vector field V is Lipschitz-continuous, it was recently proved in [6] that the solutions u^A converge to their average at any finite time as $A \to +\infty$ if and only if the operator $V \cdot \nabla$ has no eigenfunction in $H^1(M)$, other than the constant functions. Conditions for the existence or non-existence of H_0^1 first integrals in the Dirichlet case or eigenfunctions of $V \cdot \nabla$ in the no-boundary case are discussed in [3] and [6]. A more abstract formulation can also be found in [6], and the

[1] Actually, only initial conditions $u_0 \in H_0^1(\Omega)$ were considered in [3], but the result immediately extends to initial data $u_0 \in L^2(\Omega)$.

infinite two-dimensional case is analyzed in [21]. Very recently, relaxation enhancement results have been established for equations of the type $u_t^A = \Delta u^A + A\, V(At,x)\cdot\nabla u^A$ with time-periodic flows on smooth compact manifolds, see [14]. Some applications and propagation speed estimates for nonlinear reaction-diffusion problems in strong flows have been derived in [1, 2, 3, 4, 5, 6, 8, 10, 11, 13, 15, 16, 17, 20, 22].

In this paper, we consider Dirichlet boundary conditions and the general situation when the flow depends on time too. Some of the aforementionned results can be easily extended to the case when the flow is of the type $AV(t,x)$ (see the comments at the end of Section 3). However, the flow may involve various time scales and strengths. Typically, in equation (1.1), the strength $A(\gamma)$ and the time-frequency γ are large but may not be the same. New phenomena shall appear here due to the interplay between enhancement by strong advection and time-homogenization. As we shall see, the qualitative properties of the solutions at any finite time shall depend on quantitative estimates on the ratio $A(\gamma)/\gamma$, even when the flow is spatially uniform. We first analyze this particular situation in Section 2 and next we deal with the general time-space dependent case in Section 3. We will see the role of the first integrals (if any) of the flow. More precise quantitative estimates will be given in the time-periodic case. However, our analysis is valid for general non-periodic or even non-almost-periodic flows.

2 Uniform flows

This section is concerned with the case when the flow V does not depend on x, namely

$$V(\gamma t,x)=V(\gamma t).$$

We shall see that the extinction or persistence in finite time as $\gamma\to+\infty$ –and the time-decay if there is extinction– firstly depends on the time-average of the flow, and on the ratio between the amplitude $A(\gamma)$ and the frequency γ when the average is zero. We first deal with the case when the average of V is not zero.

Proposition 2.1 *Assume that $V(\gamma t,x)=V(\gamma t)$ and that there exists $\overline{V}\in\mathbb{R}^N\backslash\{(0,\ldots,0)\}$ such that*

$$\frac{1}{t}\int_0^t V(s)ds\to\overline{V}\quad \text{as } t\to+\infty. \tag{2.1}$$

Let $(u_{0,\gamma})_{\gamma>0}$ be a bounded family in $L^2(\Omega)$ and let $(u_\gamma)_{\gamma>0}$ denote the solutions of (1.1)*. If* $\lim_{\gamma\to+\infty}A(\gamma)=+\infty$*, then*

$$\forall\ t>0,\quad \|u_\gamma(t,\cdot)\|_{L^2(\Omega)}\to 0\quad \text{as } \gamma\to+\infty. \tag{2.2}$$

Moreover, if $u_{0,\gamma}\not\equiv 0$, then

$$\forall\ t>0,\quad \limsup_{\gamma\to+\infty}\frac{\ln\left(\|u_\gamma(t,\cdot)\|_{L^2(\Omega)}\right)}{A(\gamma)^2}\le-\frac{\left|\overline{V}\right|^2 t}{4}<0. \tag{2.3}$$

In the periodic case, Proposition 2.1 immediately leads to the following

Corollary 2.2 *Assume that $V(\gamma t, x) = V(\gamma t)$ and that there exists $T > 0$ such that*

$$V(s + T) = V(s) \quad \textit{for a.e. } s \in [0, +\infty) \tag{2.4}$$

and $\int_0^T V(s)ds \neq 0$. If $\lim_{\gamma \to +\infty} A(\gamma) = +\infty$, then properties (2.2) *and* (2.3) *hold for any bounded family $(u_{0,\gamma})_{\gamma>0}$ in $L^2(\Omega)$.*

Proof of Proposition 2.1. For each $\gamma > 0$, call

$$v_\gamma(t, x) = u_\gamma(t, x) \times \exp\left(\frac{A(\gamma)\overline{V} \cdot x}{2}\right).$$

Denote

$$C_1 = \min_{x \in \overline{\Omega}} \left(\overline{V} \cdot x\right) \quad \text{and} \quad C_2 = \max_{x \in \overline{\Omega}} \left(\overline{V} \cdot x\right).$$

It is immediate to see that, for each $t \geq 0$,

$$e^{-C_2 A(\gamma)} \times \int_\Omega (v_\gamma(t, x))^2 \, dx \leq \int_\Omega (u_\gamma(t, x))^2 \, dx \leq e^{-C_1 A(\gamma)} \times \int_\Omega (v_\gamma(t, x))^2 \, dx. \tag{2.5}$$

On the other hand, the functions v_γ satisfy

$$\frac{\partial v_\gamma}{\partial t} = \Delta v_\gamma + A(\gamma)\left(V(\gamma t) - \overline{V}\right) \cdot \nabla v_\gamma - \frac{A(\gamma)^2}{4} \times \left(2\overline{V} \cdot V(\gamma t) - \left|\overline{V}\right|^2\right) \times v_\gamma \tag{2.6}$$

and $v_\gamma(t, \cdot) \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ for all $1 \leq p < +\infty$, for a.e. $t > 0$. Denote

$$\phi_\gamma(t) = \int_\Omega (v_\gamma(t, x))^2 dx$$

for each $t \geq 0$ and $\gamma > 0$. The functions ϕ_γ are continuous in $[0, +\infty)$ and differentiable a.e. in $(0, +\infty)$. Multiply (2.6) by v_γ and integrate by parts over Ω. It follows that

$$\frac{(\phi_\gamma)'(t)}{2} = -\int_\Omega |\nabla v_\gamma(t, x)|^2 dx - \frac{A(\gamma)^2}{4} \times \left(2\overline{V} \cdot V(\gamma t) - \left|\overline{V}\right|^2\right) \times \phi_\gamma(t)$$

for a.e. $t > 0$. Hence,

$$\phi_\gamma(t) \leq \phi_\gamma(0) \times \exp\left\{-\frac{A(\gamma)^2}{2} \times \left(2\int_0^t \overline{V} \cdot V(\gamma s)\, ds - \left|\overline{V}\right|^2 t\right)\right\}$$

for all $t \geq 0$. Because of (2.5), there holds that, for all $t \geq 0$,

$$\|u_\gamma(t, \cdot)\|^2_{L^2(\Omega)} \leq C_3 \times \exp\left\{(C_2 - C_1)A(\gamma) - \frac{A(\gamma)^2}{2} \times \left(2\int_0^t \overline{V} \cdot V(\gamma s)\, ds - \left|\overline{V}\right|^2 t\right)\right\},$$

where $C_3 = \sup_{\gamma>0} \|u_{0,\gamma}\|^2_{L^2(\Omega)} < +\infty$. Assumption (2.1) implies that

$$\forall\, t > 0, \quad 2\int_0^t \overline{V} \cdot V(\gamma s)\, ds \to 2\left|\overline{V}\right|^2 t \ \text{ as } \gamma \to +\infty.$$

One concludes that $\|u_\gamma(t,\cdot)\|_{L^2(\Omega)} \to 0$ and that (2.3) holds as $\gamma \to +\infty$ for each $t > 0$, since $\lim_{\gamma\to+\infty} A(\gamma) = +\infty$ and $\overline{V} \neq 0$. $\square$

Proposition 2.1 and Corollary 2.2 show the extinction of the solutions u_γ when γ and $A(\gamma)$ are large and when the average of the flow is not zero. When the average of V is zero, the behavior of the solutions u_γ turns out to be more involved, even in the time-periodic case. It depends on the ratio between the strength $A(\gamma)$ of the flow and its temporal frequency γ. The following result covers this situation. But it also holds in a more general setting, for non-periodic flows, just assuming the boundedness of the antiderivative of V which is defined in $\mathbb{R}_+$ by

$$\forall\, t \geq 0, \quad W(t) = \int_0^t V(s)\, ds. \tag{2.7}$$

Theorem 2.3 *Assume that $V(\gamma t, x) = V(\gamma t)$ and that the continuous function W defined in $\mathbb{R}_+$ by* (2.7) *is bounded.*

1) *There is a constant $q_* \in (0, +\infty)$ such that, for any $q \in [0, q_*)$, there is $u_0 \in L^2(\Omega)$ such that*

$$\forall\, t > 0, \qquad \inf_{\gamma>0,\ 0\leq A(\gamma)\leq q\gamma} \|u_\gamma(t,\cdot)\|_{L^2(\Omega)} > 0, \tag{2.8}$$

where $(u_\gamma)_{\gamma>0}$ denote the solutions of (1.1) *with initial condition $u_{0,\gamma} = u_0$. In particular, if $\limsup_{\gamma\to+\infty} A(\gamma)/\gamma < q_*$, then*

$$\forall\, t > 0, \quad \liminf_{\gamma\to+\infty} \|u_\gamma(t,\cdot)\|_{L^2(\Omega)} > 0.$$

2) *If there exist two sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(\tau'_n)_{n\in\mathbb{N}}$ such that*

$$\begin{cases} \forall\, n \in \mathbb{N}, \quad 0 \leq \tau_n < \tau'_n \leq \tau_{n+1}, \\ \sup\limits_{n\in\mathbb{N}} (\tau'_n - \tau_n) + \sup\limits_{n\in\mathbb{N},\ n\geq 1} \left(\dfrac{\tau_n}{n}\right) < +\infty, \quad \inf\limits_{n\in\mathbb{N}} |W(\tau_n) - W(\tau'_n)| > 0, \end{cases} \tag{2.9}$$

there is $q^* \in [q_*, +\infty)$ such that, if $\liminf_{\gamma\to+\infty} A(\gamma)/\gamma > q^*$, then the extinction property (2.2) *holds for any $u_0 \in L^2(\Omega)$.*

3) *If Ω is convex, if V is periodic in the sense of* (2.4) *and if $V \not\equiv 0$, then one can choose $q_* = q^*$.*

Remark 2.4 *It is reasonable to conjecture that, under condition* (2.9)*, the extinction property* (2.2) *is still valid if $\liminf_{\gamma\to+\infty} A(\gamma)/\gamma = q^*$. However, this case does not follow immediately from the proof of part* 2) *of Theorem* 2.3 *and it is left here as an open problem.*

Before doing the proof of Theorem 2.3, we state a corollary which is a reformulation in the periodic case and the proof of which will be given at the end of this section. We can also derive the explicit value of $q_* = q^*$ when Ω is an euclidean ball as well as some general bounds on q_* and q^* in arbitrary domains.

Corollary 2.5 *Assume that $V(\gamma t, x) = V(\gamma t)$, that V is periodic in the sense of* (2.4)*, that $\overline{V} = 0$ and that $V \not\equiv 0$. Then there are two constants $0 < q_* \le q^* < +\infty$ which only depend on Ω and V such that:*

1) *For any $q \in [0, q_*)$, there is $u_0 \in L^2(\Omega)$ such that the solutions $(u_\gamma)_{\gamma>0}$ of* (1.1) *with initial condition $u_{0,\gamma} = u_0$ satisfy the persistence property* (2.8)*, that is*

$$\forall\ t > 0, \qquad \inf_{\gamma>0,\ 0\le A(\gamma)\le q\gamma} \|u_\gamma(t,\cdot)\|_{L^2(\Omega)} > 0.$$

2) *If $\liminf_{\gamma\to+\infty} A(\gamma)/\gamma > q^*$, then the functions $(u_\gamma)_{\gamma>0}$ satisfy the extinction property* (2.2) *for any $u_0 \in L^2(\Omega)$, that is $\lim_{\gamma\to+\infty} \|u_\gamma(t,\cdot)\|_{L^2(\Omega)} = 0$ for all $t > 0$.*

3) *If Ω is an euclidean ball of radius R, then*

$$q_* = q^* = \frac{2R}{\mathrm{osc}(W)},$$

where $\mathrm{osc}(W) = \sup_{t,t'\ge0} |W(t) - W(t')|$ is the oscillation of the function W defined in (2.7)*. In particular, for an interval $\Omega = I$ in dimension $N = 1$, then $q_* = q^* = L/\mathrm{osc}(W)$, where L denotes the length of the interval I.*

4) *In general domains Ω, the following estimates hold*

$$\frac{2R_{\min}(\Omega)}{\mathrm{osc}(W)} \le q_* \le q^* \le \frac{\mathrm{diam}(\Omega)}{\mathrm{osc}(W)}, \tag{2.10}$$

where $R_{\min}(\Omega)$ denotes the inner radius of Ω and $\mathrm{diam}(\Omega)$ its diameter.

Proof of Theorem 2.3. For every $q \ge 0$ and $\tau \ge 0$, call

$$\Omega_{q,\tau} = \Omega + qW(\tau) = \{x + qW(\tau),\ x \in \Omega\}.$$

Remember that the function W is assumed to be bounded in $\mathbb{R}_+$. Therefore, there exist $0 < q_* < +\infty$ such that, for any $q \in [0, q_*)$, there is a non-empty open subset $\Omega' \subset \Omega$ such that

$$\forall\ q' \in [0, q],\ \forall\ \tau \ge 0, \quad \Omega' \subset \Omega_{q',\tau}.$$

On the other hand, for any two sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(\tau'_n)_{n\in\mathbb{N}}$ such that

$$\inf_{n\in\mathbb{N}} |W(\tau_n) - W(\tau'_n)| > 0,$$

there is $q^* \in [q_*, +\infty)$ such that

$$\forall\ q > q^*, \qquad \inf_{q'\ge q,\ n\in\mathbb{N}} \mathrm{dist}\left(\Omega_{q',\tau_n}, \Omega_{q',\tau'_n}\right) > 0, \tag{2.11}$$

where $\mathrm{dist}(A, B)$ denotes the euclidean distance between two sets A and B. Furthermore, it follows from Helly's theorem that, if Ω is convex, if V satisfies (2.4) and if $V \not\equiv 0$, then one can choose $q_* = q^*$ (independently from the sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(\tau'_n)_{n\in\mathbb{N}}$).

For every $\gamma > 0$ and $\tau \ge 0$, call

$$v_\gamma(\tau,\cdot) = u_\gamma(\gamma^{-1}\tau, \cdot - \gamma^{-1}A(\gamma)W(\tau)) \ \text{ in } \Omega_{\gamma^{-1}A(\gamma),\tau}.$$

The functions v_γ solve the heat equation with diffusion coefficient γ^{-1}:

$$\frac{\partial v_\gamma}{\partial \tau} = \gamma^{-1}\Delta v_\gamma \quad \text{in } \Omega_{\gamma^{-1}A(\gamma),\tau} \tag{2.12}$$

for a.e. $\tau > 0$. Furthermore, for any $\gamma > 0$, $v_\gamma(\tau,\cdot) \in W_0^{1,p}(\Omega_{\gamma^{-1}A(\gamma),\tau}) \cap W^{2,p}(\Omega_{\gamma^{-1}A(\gamma),\tau})$ for all $1 \le p < +\infty$ and for a.e. $\tau > 0$, while $v_\gamma(0,\cdot) = u_0$ in $\Omega_{\gamma^{-1}A(\gamma),0} = \Omega$.

1) Assume first that $q \in [0, q_*)$ and call

$$\Gamma_q = \{\gamma > 0,\ 0 \le A(\gamma) \le \gamma q\}$$

(remember that $A(\gamma)$ is assumed to be nonnegative throughout the paper). It follows then from the characterization of q_* that there is a non-empty open subset $\Omega' \subset \Omega$ such that

$$\forall\ \gamma \in \Gamma_q,\ \forall\ \tau \ge 0, \quad \Omega' \subset \Omega_{\gamma^{-1}A(\gamma),\tau}.$$

Choose now any $u_0 \in L^2(\Omega)$ such that $u_0 \ge 0$ almost everywhere in Ω and the restriction u_0' of u_0 in Ω' is not zero. Fix any $\gamma \in \Gamma_q$. Call v_γ' the solution of

$$\begin{cases} \dfrac{\partial v_\gamma'}{\partial \tau} = \gamma^{-1}\Delta v_\gamma', & \tau > 0,\ x \in \Omega', \\ v_\gamma'(\tau,x) = 0, & \tau > 0,\ x \in \partial\Omega', \\ v_\gamma'(0,x) = u_0'(x), & x \in \Omega'. \end{cases}$$

It follows from the maximum principle that, for all $\tau \ge 0$,

$$v_\gamma(\tau,\cdot) \ge 0 \ \text{ a.e. in } \Omega_{\gamma^{-1}A(\gamma),\tau}, \quad \text{and } v_\gamma(\tau,\cdot) \ge v_\gamma'(\tau,\cdot) \ge 0 \ \text{ a.e. in } \Omega'.$$

Therefore,

$$\forall\ \tau \ge 0, \quad \|u_\gamma(\gamma^{-1}\tau,\cdot)\|_{L^2(\Omega)} = \|v_\gamma(\tau,\cdot)\|_{L^2(\Omega_{\gamma^{-1}A(\gamma),\tau})} \ge \|v_\gamma'(\tau,\cdot)\|_{L^2(\Omega')}.$$

As a conclusion, $\|u_\gamma(t,\cdot)\|_{L^2(\Omega)} \ge \|w'(t,\cdot)\|_{L^2(\Omega')} > 0$ for all $\gamma \in \Gamma_q$ and $t \ge 0$, where w' does not depend on γ and solves the heat equation

$$\begin{cases} \dfrac{\partial w'}{\partial t} = \Delta w', & t > 0,\ x \in \Omega', \\ w'(t,x) = 0, & t > 0,\ x \in \partial\Omega', \\ w'(0,x) = u_0'(x), & x \in \Omega' \end{cases}$$

in Ω'. This yields

$$\inf_{\gamma \in \Gamma_q} \|u_\gamma(t,\cdot)\|_{L^2(\Omega)} > 0 \ \text{ for all } t > 0,$$

which completes the proof of part 1) of Theorem 2.3. Actually, the lower bound can be made more explicit for some special choices of u_0. Indeed, assuming that Ω' is of class C^2 (this is possible without loss of generality, even if it means decreasing Ω'), if u_0' denotes a nonzero principal eigenfunction of $-\Delta$ in Ω' with Dirichlet boundary conditions and if $\lambda' > 0$ denotes the principal eigenvalue, then

$$w'(t,\cdot) = e^{-\lambda' t} u_0'$$

for all $t \geq 0$, whence $\|u_\gamma(t,\cdot)\|_{L^2(\Omega)} \geq e^{-\lambda' t}\|u_0'\|_{L^2(\Omega')}$ for all $t \geq 0$ and $\gamma \in \Gamma_q$.

2) Let us now assume that there exist two sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(\tau_n')_{n\in\mathbb{N}}$ satisfying (2.9). Let $q^* \in [q_*, +\infty)$ such that (2.11) holds, and assume that

$$\liminf_{\gamma\to+\infty} \frac{A(\gamma)}{\gamma} > q^*.$$

There exist then $\gamma_0 > 0$ and $\eta > 0$ such that

$$\forall\, \gamma \geq \gamma_0,\ \forall\, n \in \mathbb{N}, \quad \mathrm{dist}\left(\Omega_{\gamma^{-1}A(\gamma),\tau_n}, \Omega_{\gamma^{-1}A(\gamma),\tau_n'}\right) \geq \eta.$$

Fix any $\varepsilon > 0$. Let $U_0 \in C(\overline{\Omega})$ such that $\|u_0 - U_0\|_{L^2(\Omega)} \leq \varepsilon$, and call U_γ the solution of (1.1) with initial condition U_0 at $t = 0$. Since V is divergence-free (because it is independent of x here), the $L^2(\Omega)$ norm of any solution of (1.1) is nonincreasing with respect to time. In particular,

$$\forall\, \gamma > 0,\ \forall\, t \geq 0, \quad \|u_\gamma(t,\cdot) - U_\gamma(t,\cdot)\|_{L^2(\Omega)} \leq \varepsilon. \tag{2.13}$$

On the other hand, there exists $M > 0$ such that $-M \leq U_0 \leq M$ in Ω, and the maximum principle implies that

$$\forall\, \gamma > 0,\ \forall\, t \geq 0, \quad -M \leq U_\gamma(t,\cdot) \leq M \ \text{ a.e. in } \Omega.$$

For all $\gamma > 0$ and $\tau \geq 0$, call

$$V_\gamma(\tau,\cdot) = U_\gamma(\gamma^{-1}\tau, \cdot - \gamma^{-1}A(\gamma)W(\tau)) \ \text{ in } \Omega_{\gamma^{-1}A(\gamma),\tau}.$$

The functions V_γ solve the heat equation (2.12) in the domains $\Omega_{\gamma^{-1}A(\gamma),\tau}$ with Dirichlet boundary conditions and there holds

$$\forall\, \gamma > 0,\ \forall\, \tau \geq 0, \quad -M \leq V_\gamma(\tau,\cdot) \leq M \ \text{ a.e. in } \Omega_{\gamma^{-1}A(\gamma),\tau}. \tag{2.14}$$

Fix now any $\gamma \geq \gamma_0$. Call $\overline{V}_\gamma$ the solution of

$$\begin{cases} \dfrac{\partial \overline{V}_\gamma}{\partial \tau} = \gamma^{-1}\Delta \overline{V}_\gamma, & \tau > \tau_0,\ y \in \Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'}, \\ \overline{V}_\gamma(\tau,y) = M, & \tau > \tau_0,\ y \in \partial\Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'}, \\ \overline{V}_\gamma(\tau_0,y) = 0, & y \in \Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'} \end{cases}$$

in the domain

$$\Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'} = \Omega_{\gamma^{-1}A(\gamma),\tau_0'} + B(0,\eta/2),$$

where $B(0,\eta/2)$ denotes the open euclidean ball of centre 0 and radius $\eta/2$. Notice that $\overline{V}_\gamma(\tau,y)$ is increasing in $\tau\ge\tau_0$ for each $y\in\Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'}$, whence

$$\overline{V}_\gamma(\tau,y)>0 \ \text{ for all } \tau>\tau_0 \text{ and } y\in\Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'}.$$

Since $\overline{\Omega_{\gamma^{-1}A(\gamma),\tau_0}}\cap\overline{\Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'}}=\emptyset$ due to the choice of η, it follows from (2.14) and the maximum principle that

$$\forall\,\tau\in[\tau_0,\tau_0'],\quad V_\gamma(\tau,\cdot)\le\overline{V}_\gamma(\tau,\cdot)\ \text{ a.e. in } \Omega_{\gamma^{-1}A(\gamma),\tau}\cap\Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'}.$$

As a consequence,

$$V_\gamma(\tau_0',\cdot)\le\overline{V}_\gamma(\tau_0',\cdot)\ \text{ a.e. in } \Omega_{\gamma^{-1}A(\gamma),\tau_0'}. \tag{2.15}$$

Call $w(t,x)$ the solution of

$$\begin{cases}\dfrac{\partial w}{\partial t}=\Delta w, & t>0,\ x\in\Omega^{\eta/2},\\ w(t,x)=1, & t>0,\ x\in\partial\Omega^{\eta/2},\\ w(0,x)=0, & x\in\Omega^{\eta/2}\end{cases}$$

in the domain

$$\Omega^{\eta/2}=\Omega+B(0,\eta/2).$$

The function w, which does not depend on γ, is increasing in t for each $x\in\Omega^{\eta/2}$. Furthermore, $w\le 1$, and $\max_{x\in K}w(t,x)<1$ for each $t>0$ and any compact set $K\subset\Omega^{\eta/2}$, from the strong maximum principle. In particular,

$$0<\delta:=\max_{x\in\overline{\Omega}}\ w(\gamma_0^{-1}\overline{\tau},x)<1,$$

where $\overline{\tau}=\sup_{n\in\mathbb{N}}(\tau_n'-\tau_n)\in(0,+\infty)$ from (2.9). Observe now that

$$\overline{V}_\gamma(\tau,y)=M\,w(\gamma^{-1}(\tau-\tau_0),y-\gamma^{-1}A(\gamma)W(\tau_0'))$$

for all $\tau>\tau_0$ and $y\in\overline{\Omega^{\eta/2}_{\gamma^{-1}A(\gamma),\tau_0'}}$. Thus,

$$\begin{aligned}\forall\ y\in\overline{\Omega_{\gamma^{-1}A(\gamma),\tau_0'}},\qquad \overline{V}_\gamma(\tau_0',y)&= M\,w(\gamma^{-1}(\tau_0'-\tau_0),y-\gamma^{-1}A(\gamma)W(\tau_0'))\\ &\le M\,w(\gamma_0^{-1}\overline{\tau},y-\gamma^{-1}A(\gamma)W(\tau_0'))\\ &\le M\delta\end{aligned}$$

since $\gamma\ge\gamma_0>0$ and owing to the definitions of δ and $\Omega_{\gamma^{-1}A(\gamma),\tau_0'}$. It follows from (2.15) that

$$V_\gamma(\tau_0',\cdot)\le M\delta\ \text{ a.e. in } \Omega_{\gamma^{-1}A(\gamma),\tau_0'}.$$

The lower bound $V_\gamma(\tau_0',\cdot)\ge -M\delta$ a.e. in $\Omega_{\gamma^{-1}A(\gamma),\tau_0'}$ holds similarly. Eventually,

$$\|V_\gamma(\tau_0',\cdot)\|_{L^\infty(\Omega_{\gamma^{-1}A(\gamma),\tau_0'})}\le M\delta,$$

that is $\|U_\gamma(\gamma^{-1}\tau_0',\cdot)\|_{L^\infty(\Omega)} \le M\delta$. Since $\tau_0' \le \tau_1$, the maximum principle yields $\|U_\gamma(\gamma^{-1}\tau_1,\cdot)\|_{L^\infty(\Omega)} \le M\delta$.

An immediate induction leads to

$$\forall\, n\in\mathbb{N},\quad \|U_\gamma(\gamma^{-1}\tau_n,\cdot)\|_{L^\infty(\Omega)} \le M\delta^n,$$

whence

$$\forall\, n\in\mathbb{N},\quad \|U_\gamma(\gamma^{-1}\tau_n,\cdot)\|_{L^2(\Omega)} \le M\delta^n|\Omega|^{1/2},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

Fix any $t>0$. Call $C=\sup_{n\ge 1}\tau_n/n\in(0,+\infty)$ from (2.9). For γ ($\ge\gamma_0$) large enough, there exists $n\in\mathbb{N}\backslash\{0\}$ such that

$$t\ge\frac{Cn}{\gamma}\ge\frac{\tau_n}{\gamma}\ \ \text{and}\ \ n\ge\frac{\gamma t}{C}-1.$$

Since the map $s\mapsto\|U_\gamma(s,\cdot)\|_{L^2(\Omega)}$ is nonincreasing, it follows that, for any fixed $t>0$,

$$\|U_\gamma(t,\cdot)\|_{L^2(\Omega)}\le\|U_\gamma(\gamma^{-1}\tau_n,\cdot)\|_{L^2(\Omega)}\le M\,\delta^{\gamma t/C-1}|\Omega|^{1/2}$$

for γ large enough. Since $\delta\in(0,1)$ is independent of γ, one concludes that

$$\|U_\gamma(t,\cdot)\|_{L^2(\Omega)}\to 0\ \ \text{as}\ \gamma\to+\infty$$

and that, using (2.13), $\|u_\gamma(t,\cdot)\|_{L^2(\Omega)}\le 2\varepsilon$ for γ large enough. That completes the proof of Theorem 2.3. □

Notice that the assumptions of Theorem 2.3 yield

$$\overline{V}:=\lim_{t\to+\infty}t^{-1}\int_0^t V(s)ds=0.$$

The arguments used in the proof of part 2) imply that if (2.9) is fulfilled and if $q:=\liminf_{\gamma\to+\infty}A(\gamma)/\gamma>q^*$, then

$$\forall\, t>0,\quad \limsup_{\gamma\to+\infty}\ \frac{\ln\big(\|u_\gamma(t,\cdot)\|_{L^2(\Omega)}\big)}{\gamma}\le\frac{t\,\ln(\delta)}{C}<0,$$

for any $u_0\in L^\infty(\Omega)\backslash\{0\}$, where $C=\sup_{n\ge1}\tau_n/n\in(0,+\infty)$, and $\delta\in(0,1)$ depends only on the domain Ω, on the flow V and on q. In particular, if $A(\gamma)=q\gamma$ with $q>q^*$, then, for each $t>0$, the quantity $\ln(\|u_\gamma(t,\cdot)\|_{L^2(\Omega)})$ is bounded from above by a negative constant times $A(\gamma)$ as $\gamma\to+\infty$. This decay is much slower than the one obtained in the general case when $\overline{V}$ exists and is not zero: the quantity $\ln(\|u_\gamma(t,\cdot)\|_{L^2(\Omega)})$ is then bounded from above by a negative constant times $A(\gamma)^2$ as $\gamma\to+\infty$ (see Proposition 2.1).

Proof of Corollary 2.5. Parts 1) and 2) immediately follow from Theorem 2.3, and part 3) follows from part 4). Let us then prove (2.10). First, if $q>\mathrm{diam}(\Omega)/\mathrm{osc}(W)$, then there exist $0\le\tau_1<\tau_2$ such that $|qW(\tau_1)-qW(\tau_2)|>\mathrm{diam}(\Omega)$, whence

$$\inf_{q'\ge q}\ \mathrm{dist}(\Omega+q'W(\tau_1),\Omega+q'W(\tau_2))>0.$$

It follows from the construction of q^* in the proof of Theorem 2.3 that this real number can be chosen so that $q^* \le q$. Since this is true for any $q > \mathrm{diam}(\Omega)/\mathrm{osc}(W)$, the upper bound in (2.10) follows. Let now B be an open euclidean ball of radius $R_{\min}(\Omega)$ which is included in Ω. Choose $q_0 = 2R_{\min}/\mathrm{osc}(W)$. Thus $|q_0 W(\tau) - q_0 W(\tau')| \le 2R_{\min}$, whence

$$(\overline{B} + q_0 W(\tau)) \cap (\overline{B} + q_0 W(\tau')) \neq \emptyset$$

for all $\tau, \tau' \ge 0$. If follows from Helly's theorem that, for any $0 \le q < q_0$, there exists a non-empty open set $\Omega' \subset B \subset \Omega$ such that

$$\forall\ q' \in [0, q],\ \forall\ \tau \ge 0, \quad \Omega' \subset B + q'W(\tau) \subset \Omega + q'W(\tau) = \Omega_{q',\tau}.$$

Therefore, it follows from the proof of Theorem 2.3 that one can choose q_* so that $q_* \ge 2R_{\min}/\mathrm{osc}(W)$. $\square$

Remark 2.6 *Notice from the above proof that the lower bound $q_* \ge 2R_{\min}/\mathrm{osc}(W)$ holds even when V is not assumed to be periodic.*

3 Non-uniform flows

In this section, the flow V is not assumed to be uniform in space anymore. We shall see that the conditions for persistence or extinction at any positive time as $\gamma \to +\infty$ are related to the existence or non-existence of first integrals of the flow. We first give a sufficient condition for the solutions of (1.1) not to converge to 0 in finite time as $\gamma \to +\infty$.

Theorem 3.1 *Assume that there is a function $w \in H_0^1(\Omega)\backslash\{0\}$ such that*

$$V(s, \cdot) \cdot \nabla w = 0 \quad \textit{a.e. in } \Omega \tag{3.1}$$

for a.e. $s \in \mathbb{R}$. Then, there is $u_0 \in L^2(\Omega)$ such that the solutions $(u_\gamma)_{\gamma>0}$ of (1.1) with initial condition $u_{0,\gamma} = u_0$ satisfy

$$\forall\ t \ge 0, \quad \inf_{\gamma>0}\ \|u_\gamma(t, \cdot)\|_{L^2(\Omega)} > 0. \tag{3.2}$$

Proof. Choose $u_0 = 1$. For each $\gamma > 0$, the solution u_γ of (1.1) with initial condition u_0 satisfies

$$0 < u_\gamma(t, \cdot) < 1 \ \text{ a.e. in } \Omega \tag{3.3}$$

for all $t > 0$, from the strong maximum principle. Let us prove that the functions u_γ satisfy the persistence property (3.2) at any time $t > 0$. Assume not. That is, assume there is $t > 0$ and a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of positive numbers such that $\|u_{\gamma_n}(t, \cdot)\|_{L^2(\Omega)} \to 0$ as $n \to +\infty$. Up to extraction of a subsequence, one can then assume that

$$u_{\gamma_n}(t, x) \to 0 \ \text{ as } n \to +\infty, \text{ for a.e. } x \in \Omega. \tag{3.4}$$

Let $n \in \mathbb{N}$ be any integer and let $\eta > 0$ and $\varepsilon > 0$ be any positive numbers such that $0 < \eta < t$. Multiply the equation (1.1) by $w^2/(u_{\gamma_n} + \varepsilon)$ and integrate over $(\eta, t) \times \Omega$. Notice that the function $(0,+\infty) \ni s \mapsto 1/(u_{\gamma_n}(s,\cdot)+\varepsilon)$ (resp. $(0,+\infty) \ni s \mapsto w^2/(u_{\gamma_n}(s,\cdot)+\varepsilon)$, resp. $(0,+\infty) \ni s \mapsto w^2/(u_{\gamma_n}(s,\cdot)+\varepsilon) \times \frac{du_\gamma}{dt}(s,\cdot)$) is in $L^\infty((t_0,+\infty), W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega))$ for all $1 \le p < +\infty$ (resp. in $L^\infty((t_0,+\infty), W_0^{1,1}(\Omega))$, resp. in $L^\infty((t_0,+\infty), L^1(\Omega))$) for all $t_0 > 0$. It is found that

$$\begin{aligned}\int_\Omega w^2 \ln(u_{\gamma_n}(t,\cdot)+\varepsilon) - \int_\Omega w^2 \ln(u_{\gamma_n}(\eta,\cdot)+\varepsilon) &= -\iint_{(\eta,t)\times\Omega} \nabla u_{\gamma_n} \cdot \nabla\left(\frac{w^2}{u_{\gamma_n}+\varepsilon}\right) \\ &\quad + A(\gamma_n) \iint_{(\eta,t)\times\Omega} \frac{w^2\, V(\gamma_n s, x)\cdot \nabla u_{\gamma_n}}{u_{\gamma_n}+\varepsilon}.\end{aligned}$$

The second term of the right-hand side vanishes because of (3.1) and V is divergence-free for almost all times. Expanding the first term of the right-hand side and using Young's inequality leads to

$$\begin{aligned}-\iint_{(\eta,t)\times\Omega} \nabla u_{\gamma_n} \cdot \nabla\left(\frac{w^2}{u_{\gamma_n}+\varepsilon}\right) &= -2\iint_{(\eta,t)\times\Omega} \frac{w\nabla w \cdot \nabla u_{\gamma_n}}{u_{\gamma_n}+\varepsilon} + \iint_{(\eta,t)\times\Omega} \frac{w^2 |\nabla u_{\gamma_n}|^2}{(u_{\gamma_n}+\varepsilon)^2} \\ &\ge -\iint_{(\eta,t)\times\Omega} |\nabla w|^2 = -(t-\eta)\int_\Omega |\nabla w|^2.\end{aligned}$$

Therefore,

$$\int_\Omega w^2 \ln(u_{\gamma_n}(t,\cdot)+\varepsilon) - \int_\Omega w^2 \ln(u_{\gamma_n}(\eta,\cdot)+\varepsilon) \ge -(t-\eta)\int_\Omega |\nabla w|^2 \tag{3.5}$$

for all $n \in \mathbb{N}$, $\varepsilon > 0$ and $\eta \in (0,t)$.

Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Since $u_{\gamma_n}(\eta,\cdot) \to u_0 = 1$ in $L^2(\Omega)$ as $\eta \to 0$, there exists a sequence $(\eta_k)_{k\in\mathbb{N}}$ of numbers in $(0,t)$ such that $\eta_k \to 0$ and $u_{\gamma_n}(\eta_k, x) \to 1$ for a.e. $x \in \Omega$, as $k \to +\infty$. By (3.3) and Lebesgue's dominated convergence theorem, it follows as $\eta = \eta_k \to 0$ in (3.5) that

$$\int_\Omega w^2 \ln(u_{\gamma_n}(t,\cdot)+\varepsilon) - \ln(1+\varepsilon)\int_\Omega w^2 \ge -t\int_\Omega |\nabla w|^2.$$

Next, pass to the limit as $n \to +\infty$ and use (3.3) and (3.4). It follows that, for each $\varepsilon > 0$,

$$[\ln\varepsilon - \ln(1+\varepsilon)]\int_\Omega w^2 \ge -t\int_\Omega |\nabla w|^2.$$

The limit as $\varepsilon \to 0$ leads to a contradiction. As a conclusion, the persistence property (3.2) is satisfied and the proof of Theorem 3.1 is complete. Actually, it follows from the proof that the conclusion holds good for any $u_0 \in L^2(\Omega)$ such that $u_0 \ge 0$ (or $u_0 \le 0$) a.e. in Ω and $\ln(|u_0|)w^2 \in L^1(\Omega)$. □

Remark 3.2 *In the time-periodic case, the assumptions of Theorem* 3.1 *yield an additional property, which is concerned with the principal eigenvalues of the parabolic operators given* *in* (1.1)*. Namely, besides* (3.1)*, assume now that the flow V is time-periodic, that is there exists $T > 0$ such that*

$$V(s+T,\cdot) = V(s,\cdot) \quad \text{for a.e. } s \in [0,+\infty). \tag{3.6}$$

For each $\gamma > 0$, call φ_γ the principal eigenfunction of the parabolic equation (1.1) *with $\gamma^{-1}T$-periodicity in time, and denote λ_γ the principal eigenvalue. That is, $\varphi_\gamma \in L^\infty(\mathbb{R}, W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega))$, $\frac{d\varphi_\gamma}{dt} \in L^\infty(\mathbb{R}, L^p(\Omega))$ for all $1 \leq p < +\infty$ and φ_γ satisfies*

$$\begin{cases} \dfrac{\partial \varphi_\gamma}{\partial t} - \Delta\varphi_\gamma - A(\gamma)V(\gamma t, x)\cdot\nabla\varphi_\gamma = \lambda_\gamma\varphi_\gamma, \\ \varphi_\gamma\left(\cdot + \dfrac{T}{\gamma}, \cdot\right) = \varphi_\gamma \end{cases}$$

and $\varphi_\gamma(t,\cdot) > 0$ a.e. in Ω for all $t \in \mathbb{R}$. The maximum principle yields $\lambda_\gamma > 0$. It follows from the proof of Theorem 3.1 *that*

$$\lambda_\gamma \leq \frac{\displaystyle\int_\Omega |\nabla w|^2}{\displaystyle\int_\Omega w^2} =: R_w.$$

Hence, under assumptions (3.1) *and* (3.6)*, the principal eigenvalues λ_γ are bounded and*

$$\forall\ \gamma > 0, \quad 0 < \lambda_\gamma \leq \Lambda,$$

where Λ denotes the infimum of the Rayleigh quotient R_w among all first integrals $w \in H_0^1(\Omega)\backslash\{0\}$ satisfying (3.1) (*as a matter of fact, this infimum is reached from Rellich's theorem*)*. In the time-independent case, we refer to [3, 5, 7, 9, 12, 18] for further estimates on the principal eigenvalues in the limit of large drifts.*

It is worth pointing out that, in Theorem 3.1 and Remark 3.2, the real numbers γ and $A(\gamma)$ are arbitrary and may not be large. A fortiori, no relation between $A(\gamma)$ and γ is required, unlike in Theorem 2.3. The existence of first integrals $w \in H_0^1(\Omega)\backslash\{0\}$ satisfying (3.1) is a sufficient condition for the persistence property (3.2) to hold. However, this condition is not at all necessary in general, as follows from parts 1) of Theorem 2.3 or Corollary 2.5.

Next, we give a sufficient condition for the solutions of (1.1) to tend to extinction in any finite time.

Theorem 3.3 *Assume that there exists $T > 0$ such that, for any sequence $(\tau_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, there exist a subsequence $(\tau_{n_k})_{k\in\mathbb{N}}$, a Borel set $B \subset (0,T)$ with positive Lebesgue measure and $V_\infty \in L^\infty((0,T)\times\Omega,\mathbb{R}^N)$ such that $V(\tau+\tau_{n_k},x) \to V_\infty(\tau,x)$ in $L^2((0,T)\times\Omega,\mathbb{R}^N)$ as $k \to +\infty$ and, for all $\tau \in B$, $V_\infty(\tau,\cdot)$ has no first integral in $H_0^1(\Omega)\backslash\{0\}$. If*

$$\frac{A(\gamma)}{\gamma} \to +\infty \quad \text{as } \gamma \to +\infty,$$

then the extinction property (2.2) *holds, where $(u_\gamma)_{\gamma>0}$ denote the solutions of* (1.1) *with any initial condition $u_0 \in L^2(\Omega)$.*

Proof. Assume that the conclusion does not hold. There exist then $u_0 \in L^2(\Omega)$, $t_0 > 0$, $\varepsilon > 0$ and a sequence of positive real numbers $(\gamma_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \gamma_n = \lim_{n\to+\infty} A(\gamma_n)/\gamma_n = +\infty$ and

$$\|u_{\gamma_n}(t_0,\cdot)\|_{L^2(\Omega)} \geq \varepsilon \text{ for all } n \in \mathbb{N}. \tag{3.7}$$

By linearity, one can assume without loss of generality that $\|u_0\|_{L^2(\Omega)} = 1$ and $\varepsilon \in (0,1]$. Some of the arguments below are inspired from the proof of Theorem 3.1 of [3]. Call $u_{n,\pm}$ the solutions of problem (1.1) with $\gamma = \gamma_n$ and initial condition $u_0^\pm = \mathbf{1}_{\{\pm u_0 > 0\}} u_0$. There holds

$$u_{\gamma_n} = u_{n,+} + u_{n,-}.$$

On the other hand, the maximum principle implies that, for all $t > 0$, $\pm u_{n,\pm}(t,\cdot) \geq 0$ a.e. in Ω. Therefore, either $\|u_{n,+}(t_0,\cdot)\|_{L^2(\Omega)} \geq \varepsilon/2$ or $\|u_{n,-}(t_0,\cdot)\|_{L^2(\Omega)} \geq \varepsilon/2$. Up to extraction of some subsequence, and even if it means changing $\varepsilon/2$ into ε, one can assume without loss of generality that (3.7) holds with $u_0 \geq 0$ a.e. in Ω. Call now $u_0^M = \mathbf{1}_{\{u_0 < M\}} u_0$ and $M > 0$ large enough so that $\|u_0 - u_0^M\|_{L^2(\Omega)} \leq \varepsilon/2$. Let u_n^M and $\overline{u}_n$ be the solutions of (1.1) with $\gamma = \gamma_n$ and initial conditions u_0^M and $u_0 - u_0^M$. There holds

$$u_n = u_n^M + \overline{u}_n.$$

But $\|\overline{u}_n(t_0,\cdot)\|_{L^2(\Omega)} \leq \|\overline{u}_n(0,\cdot)\|_{L^2(\Omega)} = \|u_0 - u_0^M\|_{L^2(\Omega)} \leq \varepsilon/2$. Thus, $\|u_n^M(t_0,\cdot)\|_{L^2(\Omega)} \geq \varepsilon/2$. Therefore, even if it means changing $\varepsilon/2$ into ε, one can assume without loss of generality that (3.7) holds with $0 \leq u_0 \leq M$ a.e. in Ω, whence, for all $t > 0$ and for all $n \in \mathbb{N}$,

$$0 \leq u_{\gamma_n}(t,\cdot) \leq M \quad \text{a.e. in } \Omega.$$

For each $n \in \mathbb{N}$, call v_n the function defined by

$$\forall\, \tau \geq 0, \quad v_n(\tau,\cdot) = u_{\gamma_n}(\gamma_n^{-1}\tau,\cdot).$$

The functions v_n solve

$$\frac{\partial v_n}{\partial \tau} = \gamma_n^{-1}\Delta v_n + \gamma_n^{-1} A(\gamma_n) V(\tau,x)\cdot\nabla v_n.$$

Furthermore, for each $n \in \mathbb{N}$, $0 \leq v_n(\tau,\cdot) \leq M$ a.e. in Ω for all $\tau \geq 0$, $v_n \in L^\infty((\eta,+\infty), W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega))$, $\frac{dv_n}{d\tau} \in L^\infty((\eta,+\infty), L^p(\Omega))$ for all $1 \leq p < +\infty$ and $\eta > 0$, and

$$\|v_n(0,\cdot)\|_{L^2(\Omega)} = 1, \quad \|v_n(\gamma_n t_0,\cdot)\|_{L^2(\Omega)} \geq \varepsilon.$$

Furthermore, the functions $\tau \mapsto \|v_n(\tau,\cdot)\|_{L^2(\Omega)}$ are nonincreasing. Since $\lim_{n\to+\infty} \gamma_n = +\infty$, there exists $N \in \mathbb{N}$ large enough such that, for all $n \geq N$, there is $\tau_n \in [0, \gamma_n t_0 - T]$ such that

$$0 \leq \|v_n(\tau_n,\cdot)\|_{L^2(\Omega)} - \|v_n(\tau_n + T,\cdot)\|_{L^2(\Omega)} \leq \frac{2T}{\gamma_n t_0}. \tag{3.8}$$

For any $n \geq N$ and $\tau \geq -\tau_n$, call

$$w_n(\tau,\cdot) = v_n(\tau + \tau_n,\cdot).$$

The functions w_n satisfy $\varepsilon \le \|w_n(\tau,\cdot)\|_{L^2(\Omega)} \le 1$ for all $\tau \in [0,T]$ and $0 \le w_n \le M$ a.e. in Ω for all $\tau \ge -\tau_n$. Furthermore, they satisfy the equations

$$\frac{\partial w_n}{\partial \tau} = \gamma_n^{-1}\Delta w_n + \gamma_n^{-1}A(\gamma_n)V(\tau+\tau_n,x)\cdot\nabla w_n. \tag{3.9}$$

Multiply (3.9) by w_n and integrate over $(0,T)\times\Omega$. It is found that

$$\gamma_n^{-1}\int_{(0,T)\times\Omega}|\nabla w_n|^2 = \int_\Omega \frac{w_n(0,\cdot)^2}{2} - \int_\Omega \frac{w_n(T,\cdot)^2}{2} \le \|w_n(0,\cdot)\|_{L^2(\Omega)} - \|w_n(T,\cdot)\|_{L^2(\Omega)}.$$

Because of (3.8), it follows that

$$\int_{(0,T)\times\Omega}|\nabla w_n|^2 \le \frac{2T}{t_0}$$

for all $n \ge N$. Up to extraction of a subsequence, it follows from standard arguments that the functions w_n converge weakly in $L^2((0,T)\times\Omega)$ to a function w such that $\nabla w \in L^2((0,T)\times\Omega,\mathbb{R}^N)$ and such that the functions $\frac{\partial w_n}{\partial x_i}$ converge weakly in $L^2((0,T)\times\Omega)$ to $\frac{\partial w}{\partial x_i}$, for each $1 \le i \le N$. Furthermore, for a.e. $\tau \in (0,T)$, $w(\tau,\cdot) \in H_0^1(\Omega)$, $0 \le w(\tau,\cdot) \le M$ a.e. in Ω, and $\|\nabla w\|^2_{L^2((0,T)\times\Omega)} \le 2T/t_0$. Remember that $\gamma_n \to +\infty$, $\gamma_n^{-1}A(\gamma_n) \to +\infty$ as $n \to +\infty$ (whence $A(\gamma_n) \to +\infty$) and that, up to extraction of another subsequence, the functions $V(\cdot+\tau_n,\cdot)$ converge to a function V_∞ in $L^2((0,T)\times\Omega,\mathbb{R}^N)$. Divide (3.9) by $\gamma_n^{-1}A(\gamma_n)$, multiply by any test function $\varphi \in \mathcal{D}((0,T)\times\Omega)$, integrate by parts the first two quantities over $(0,T)\times\Omega$ and pass to the limit as $n \to +\infty$. From the above estimates, it follows that

$$\iint_{(0,T)\times\Omega}(V_\infty\cdot\nabla w)\,\varphi = 0$$

for any $\varphi \in \mathcal{D}((0,T)\times\Omega)$, whence $V_\infty\cdot\nabla w = 0$ a.e. in $(0,T)\times\Omega$, and then $V_\infty(\tau,\cdot)\cdot\nabla w(\tau,\cdot) = 0$ (in $L^2(\Omega)$) for a.e. $\tau \in (0,T)$ by Fubini's theorem.

On the other hand, for any Borel set $E \subset (0,T)$ with Lebesgue measure $|E| > 0$, there holds

$$\int_{E\times\Omega} w_n \ge \frac{1}{M}\int_{E\times\Omega} w_n^2 \ge \frac{|E|\,\varepsilon^2}{M} > 0.$$

By passing to the limit as $n \to +\infty$, one gets that

$$\int_{E\times\Omega} w \ge \frac{|E|\,\varepsilon^2}{M} > 0.$$

Thus, there exists a Borel set $E' \subset E$ such that $|E'| > 0$ and, for all $\tau \in E'$, $w(\tau,\cdot) \in H_0^1(\Omega)\backslash\{0\}$ and $V_\infty(\tau,\cdot)\cdot\nabla w(\tau,\cdot) = 0$ in $L^2(\Omega)$. Choosing $E = B$ (given in the assumptions of Theorem 3.3) leads to a contradiction. The proof is now complete. $\square$

Remark 3.4 *It follows from Theorem* 3.3 *that if V is time-periodic in the sense of* (3.6) *and if there is a non-negligible Borel set $B_0 \subset (0,T)$ such that, for all $\tau \in B_0$, $V(\tau,\cdot)$ has no first integral in $H_0^1(\Omega)\backslash\{0\}$, then $\lim_{\gamma\to+\infty}\|u_\gamma(t,\cdot)\|_{L^2(\Omega)} = 0$ for all $t > 0$ and for all $u_0 \in L^2(\Omega)$, as soon as $\lim_{\gamma\to+\infty}A(\gamma)/\gamma = +\infty$.*

Theorem 3.3 gives a sufficient condition for extinction which is more general than the statement of part 2) of Theorem 2.3. The proof of part 2) of Theorem 2.3 could have been done by using similar arguments as above. However, in Theorem 2.3, we used a more direct approach. That approach was also interessting because it provided quantitative estimates of the ratio $A(\gamma)/\gamma$ above which the solutions will tend to extinction at any finite time.

We complete this section with a few additional comments on the behavior of the solutions u^A of the equations

$$\begin{cases} \dfrac{\partial u^A}{\partial t} = \Delta u^A + A\, V(t,x) \cdot \nabla u^A, & t > 0,\ x \in \Omega, \\ u^A(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, & \\ u^A(0,x) = u_0^A(x), \quad x \in \Omega, & \end{cases} \tag{3.10}$$

as $A \to +\infty$. The bounded flow V is not assumed to be highly oscillating anymore. The same arguments as the ones used in the proof of Theorem 3.1 imply that if there is $w \in H_0^1(\Omega)\backslash\{0\}$ and $T > 0$ such that $V(t,\cdot)\cdot\nabla w = 0$ for a.e. $t \in (0,T)$, then there is $u_0 \in L^2(\Omega)$ such that the solutions u^A with initial condition $u_0^A = u_0$ satisfy $\inf_{A\geq 0} \|u^A(t,\cdot)\|_{L^2(\Omega)} > 0$ for all $t \in [0,T]$. On the other hand, it follows from the proof of Theorem 3.3 that if there is $T > 0$ and a non-negligible Borel set $B \subset (0,T)$ such that $V(t,\cdot)$ has no first integral in $H_0^1(\Omega)\backslash\{0\}$ for all $t \in B$, then $\lim_{A\to+\infty} \|u^A(t,\cdot)\|_{L^2(\Omega)} = 0$ for every $t \geq T$ and every initial condition $u_0^A = u_0 \in L^2(\Omega)$. In particular, when $V(t,x) = V(t)$ is uniform, then the same conclusion holds for any $t > 0$ such that $V \not\equiv 0$ in $(0,t)$. Notice lastly that in problem (3.10), the large amplitude parameter is A, while in Theorem 3.3 or Remark 3.4, we needed $A(\gamma)/\gamma$ to be large, because of the scaled time variable γt in the flow V.

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