Transition fronts and stretching phenomena for a general class of reaction-dispersion equations

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Dedicated to the memory of Professor Paul Fife

Abstract

We consider a general form of reaction-dispersion equations with non-local or non-linear dispersal operators and local reaction terms. Under some general conditions, we prove the non-existence of transition fronts, as well as some stretching properties at large time for the solutions of the Cauchy problem. These conditions are satisfied in particular when the reaction is monostable and when the dispersal operator is either the fractional Laplacian, a convolution operator with a fat-tailed kernel or a nonlinear fast diffusion operator.

1 Introduction

This note is concerned with fast propagation phenomena and large time qualitative properties of dispersion-reaction equations of the type

$$u_t(t,x) = \mathcal{D}u(t,x) + f(u(t,x)), \quad t \in \mathbb{R}, \ x \in \mathbb{R}, \tag{1.1}$$

where the reaction f and the dispersal operator $\mathcal{D}$, possibly non-local and nonlinear, are given. More precisely, we make the following assumptions. The given reaction function $f : [0,1] \to \mathbb{R}$ is of class C^1 and such that $f(0) = f(1) = 0$. Throughout the paper, we assume that, for any $u_0 \in L^\infty(\mathbb{R}; [0,1])$, the Cauchy problem

$$\begin{cases} u_t(t,x) &= \mathcal{D}u(t,x) + f(u(t,x)), \quad t > 0, \ x \in \mathbb{R}, \\ u(0, \cdot) &= u_0 \end{cases} \tag{1.2}$$

admits a unique mild solution, such that $u(t, \cdot) \in L^\infty(\mathbb{R}; [0,1])$ for every $t > 0$ and u is uniformly continuous with respect to t in $[\delta, +\infty) \times \mathbb{R}$ for every $\delta > 0$. The equation (1.1) is assumed to be autonomous in the sense that, for any solution u of the Cauchy problem (1.2) and for any $T > 0$, the function $(t,x) \mapsto u(T+t, x)$ solves (1.2) with initial condition $u(T, \cdot)$. We also assume that a comparison principle holds for (1.2), that is, for any two solutions u and v with respective initial conditions u_0 and v_0,

$$\big(0 \le u_0 \le v_0 \le 1 \text{ a.e. in } \mathbb{R} \big) \implies \big(\forall t > 0, \ 0 \le u(t, \cdot) \le v(t, \cdot) \le 1 \text{ a.e. in } \mathbb{R} \big). \tag{1.3}$$

Lastly, the solutions u of (1.2) are assumed to have infinite spreading speeds (to the right) in the sense that

$$\forall u_0 \in L^\infty(\mathbb{R}; [0,1]) \setminus \{0\}, \ \forall c > 0, \quad \operatorname*{ess\,inf}_{(0,ct)} u(t, \cdot) \to 1 \text{ as } t \to +\infty. \tag{1.4}$$

The first typical example of equations (1.1) for which these conditions are fulfilled is the monostable reaction-diffusion equation with fractional diffusion

$$u_t = -(-\Delta_x)^\alpha u + f(u), \tag{1.5}$$

where the reaction function f is monostable, that is

$$f(0) = f(1) = 0, \ f(u) > 0 \text{ for all } u \in (0,1), \ f'(0) > 0, \tag{1.6}$$

and the dispersal operator $\mathcal{D}$ is the fractional Laplacian $\mathcal{D}u = -(-\Delta_x)^\alpha u$ with $0 < \alpha < 1$, see [6] (see also [5] for similar equations in periodic media). In [6], the function f is also assumed to be concave in $[0,1]$, but the condition (1.4) actually holds under the condition (1.6) by the comparison principle and putting below f a concave function $g : [0, 1-\varepsilon] \to \mathbb{R}$ on the interval $[0, 1-\varepsilon]$, with $\varepsilon \in (0,1)$ as small as wanted. The set of functions f satisfying (1.6) includes the class of Fisher-KPP [21, 30] nonlinearities f for which, in addition to these assumptions, f satisfies $f(u) \le f'(0)u$ in $[0,1]$. It also contains the set of C^2 concave functions f with $f(0) = f(1) = 0$ and $f > 0$ on $(0,1)$, whose archetype is the logistic nonlinearity $f(u) = u(1-u)$.

A second important class consists of integro-differential equations with fat-tailed dispersal kernels

$$u_t = J * u - u + f(u), \tag{1.7}$$

where f is of the type (1.6) and the dispersal operator is a convolution operator

$$\mathcal{D}u(t,x) = J * u(t,x) - u(t,x) := \int_\mathbb{R} J(x-y)(u(t,y) - u(t,x)) \, dy,$$

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with a fat-tailed positive even kernel $J \in L^1(\mathbb{R})$ such that

$$
\begin{cases}
\|J\|_{L^1(\mathbb{R})} = 1, \ \int_{\mathbb{R}} |x| J(x) dx < +\infty, \\
J \text{ is of class } C^1 \text{ for large } |x| \text{ and } J'(x) = o(J(x)) \text{ as } |x| \to +\infty,
\end{cases}
$$

see [22]. Notice that such kernels J are called "fat tailed" since they decay slowly as $|x| \to +\infty$ in the sense that $J(x)e^{\eta|x|} \to +\infty$ as $|x| \to +\infty$ for every $\eta > 0$. Archetypes are $J(x) = c_{\alpha,\beta} e^{-\beta|x|^\alpha}$ with $0 < \alpha < 1$, $\beta > 0$ and some normalization constant $c_{\alpha,\beta} > 0$, or $J(x) = c_\alpha/(1 + |x|^\alpha)$ with $\alpha > 2$ and some $c_\alpha > 0$. Such operators and equations arise in many physical or ecological models, see e.g. [10, 17, 18, 19, 20, 31, 32, 33, 37].

Lastly, the conditions (1.3) and (1.4) are also fulfilled when the dispersal operator $\mathcal{D}$ corresponds to fast nonlinear diffusion. Using a detailed formal analysis of the equation

$$
u_t = (u^\gamma)_{xx} + f(u), \tag{1.8}
$$

with $0 < \gamma < 1$ and where f satisfies (1.6), it was shown in [29] that the assumption (1.4) was fulfilled. In [47], this result was extended to the case of nonlinear fractional diffusion equations:

$$
u_t = -(-\Delta_x)^\alpha(u^\gamma) + f(u), \tag{1.9}
$$

with $0 < \alpha < 1$ and $\max(1 - 2\alpha, 0) < \gamma$ and where f satisfies (1.6) and is a concave function. Due to the comparison principle (1.3), which is valid for both (1.8) and (1.9), it can be shown that the condition (1.4) holds without the concavity assumption, as for the standard fractional Laplacian.

Given the few general assumptions (1.3) and (1.4), the goal of the paper is twofold. Firstly, we will prove the non-existence of front-like entire solutions. Secondly, further stretching properties of the solutions of the Cauchy problem (1.2) at large time will be shown. As a corollary of the second main theorem, we will show that the solutions of (1.2) are in some sense flat at some large times in left and right neighborhoods of any level set. We again insist on the fact that these results will hold for the four main examples (1.5), (1.7), (1.8) and (1.9).

2 Non-existence of transition fronts

When propagating solutions are mentioned, one immediately has in mind standard traveling fronts $u(t,x) = \varphi(\pm x - ct)$, with velocity $c \in \mathbb{R}$ and front profile $\varphi : \mathbb{R} \to [0,1]$ such that $\varphi(-\infty) = 1$ and $\varphi(+\infty) = 0$. Without the assumption (1.4), standard traveling fronts $\varphi_c(\pm x - ct)$ are known to exist when we consider equations (1.1) with the standard (local) Laplacian $\mathcal{D}u = \Delta_x u = u_{xx}$ or a nonlocal convolution operator $\mathcal{D}u = J * u - u$ with a thin-tailed dispersal kernel J which is exponentially bounded in the sense that $\int_{\mathbb{R}} J(x) e^{\lambda x} dx < +\infty$ for some $\lambda > 0$. More precisely, when f is of the type (1.6), these fronts exist for every speed $c \geq c^*$, with a positive minimal speed c^*. Furthermore, uniqueness of the profile φ_c and stability results for a given $c \geq c^*$ have been shown, with possibly heterogeneous nonlinearities $f(x, u)$, see [1, 7, 12, 13, 15, 16, 30, 42, 43, 45, 46, 48]. However,

even for the reaction-diffusion equation $u_t = u_{xx} + f(u)$ with local diffusion and f of the type (1.6), fast propagation phenomena with infinite spreading speed as in (1.4) is known to occur when u_0 decays to 0 as $x \to \pm\infty$ more slowly than any exponentially decaying function, see [26]. Lastly, some existence, uniqueness and stability results of standard traveling fronts have also been shown for some nonlocal equations of the type (1.1) with fractional Laplacians or fat-tailed dispersal convolution kernels J, but for other types of nonlinearities $f(u)$ or $f(x, u)$ than (1.6), see e.g. [2, 8, 9, 12, 14, 15, 16, 23, 24, 36].

Here, the assumptions (1.3) and (1.4) forbid standard traveling fronts to exist, since all non-trivial solutions of the Cauchy problem accelerate with infinite speed as $t \to +\infty$ in the sense of (1.4). This non-existence result has already been known in the case of the fractional Laplacian $\mathcal{D}u = -(-\Delta_x)^\alpha u$ [6, 23] as well as in the convolution case $\mathcal{D}u = J * u - u$ with non-exponentially-bounded kernels J [22, 49]. Non-existence of standard traveling fronts has also been derived for the fast nonlinear diffusion equations (1.8) and (1.9) [29, 47]. However, other propagating solutions connecting 0 and 1, more general than the standard traveling fronts, can be investigated. These solutions, called transition fronts, have been defined in [3, 4] in more general situations. In the one-dimensional situation considered here, as a wave-like solution defined in [44], the following definition holds.

Definition 2.1 (Transition front) *A transition front connecting 0 (say, on the right) and 1 (on the left) for (1.1) is a time-global solution $u : \mathbb{R} \times \mathbb{R} \to [0, 1]$ (that is, the unique mild solution of the Cauchy problem (1.2) starting at any time $t_0 \in \mathbb{R}$ with "initial" condition $u(t_0, \cdot)$) such that there exists a family $(\xi_t)_{t \in \mathbb{R}}$ of real numbers with*

$$\begin{cases} \operatorname*{ess\,inf}_{(-\infty, x)} u(t, \xi_t + \cdot) \to 1 & as\ x \to -\infty, \\ \operatorname*{ess\,sup}_{(x, +\infty)} u(t, \xi_t + \cdot) \to 0 & as\ x \to +\infty, \end{cases} \quad uniformly\ in\ t \in \mathbb{R}. \qquad (2.1)$$

The real numbers ξ_t therefore reflect the positions of a transition front u as a function of time. However, the ξ_t's are not uniquely defined since any family $(\widetilde{\xi}_t)_{t \in \mathbb{R}}$ satisfies (2.1) as soon as $(\xi_t)_{t \in \mathbb{R}}$ does and $(\widetilde{\xi}_t - \xi_t)_{t \in \mathbb{R}}$ remains bounded. Roughly speaking, the condition (2.1) means that the diameter of the transition zone between the sets where $u \simeq 1$ and $u \simeq 0$ is uniformly bounded in time in the sense that, for every $0 < a \le b < 1$, there is a nonnegative real number C such that, for all $t \in \mathbb{R}$, up to a negligible set,

$$\big\{ x \in \mathbb{R};\ a \le u(t, x) \le b \big\} \subset \big[X(t) - C, X(t) + C \big].$$

Transition fronts connecting 0 on the left and 1 on the right can be defined similarly by permuting the limits as $x \to \pm\infty$ in (2.1). We will therefore only consider here transition fronts connecting 0 on the right and 1 on the left. Obviously, any standard traveling front $u(t, x) = \varphi(x - ct)$ with $\varphi(-\infty) = 1$ and $\varphi(+\infty) = 0$ would be a transition front connecting 0 and 1 with $\xi_t = ct$, but, as already emphasized, the assumption (1.4) excludes the existence of standard traveling fronts for (1.1). Nevertheless, transition fronts can a priori be much more general than standard traveling fronts, since no assumption is made on the family of front positions $(\xi_t)_{t \in \mathbb{R}}$. In particular, transition fronts different from the

standard traveling fronts have been constructed recently for some homogeneous or hetero-geneous local reaction-diffusion equations [27, 28, 34, 35, 38, 39, 40, 50, 51], and fronts with global speed ξ_t/t having different limits as $t \to \pm\infty$ are also known to exist even for the homogeneous local Fisher-KPP reaction-diffusion equation $u_t = u_{xx} + f(u)$ [25, 27, 50]. For our equation (1.1), transition fronts with global speed ξ_t/t growing arbitrarily as $t \to \pm\infty$ are a priori not excluded.

Our first result shows actually that this is anyway impossible, since transition fronts cannot exist whatever the family of positions $(\xi_t)_{t\in\mathbb{R}}$ may be.

Theorem 2.2 *Under the assumptions* (1.3) *and* (1.4)*, equation* (1.1) *does not admit any transition front connecting* 0 *and* 1.

Theorem 2.2 generalizes the known non-existence results of standard traveling fronts for the equations of the type (1.5) with the fractional Laplacian, (1.7) with a fat-tailed dispersal kernel J,and (1.8) and (1.9) with fast nonlinear diffusion since the standard fronts are particular classes of transition fronts.

Roughly speaking, this non-existence result can be heuristically explained as follows: for any transition front connecting 0 and 1, the uniformity (with respect to time) of the limits (2.1) prevents the front from traveling too fast (see also [27] for a similar phenomenon for local reaction-diffusion equations), and this last property is finally in contradiction with the spreading properties (1.4) combined with the comparison principle (1.3).

Proof of Theorem 2.2. We argue by contradiction. So assume that, under the assumptions (1.3) and (1.4), equation (1.1) admits a transition front u connecting 0 and 1. Let $(\xi_t)_{t\in\mathbb{R}}$ be a family of real numbers such that (2.1) holds.

For every $t \in \mathbb{R}$, denote

$$\xi_t^- = \sup\left\{x \in \mathbb{R}; \ \operatorname*{ess\,inf}_{(-\infty,x)} u(t,\cdot) \geq \frac{2}{3}\right\}$$

and

$$\xi_t^+ = \inf\left\{x \in \mathbb{R}; \ \operatorname*{ess\,sup}_{(x,+\infty)} u(t,\cdot) \leq \frac{1}{3}\right\}.$$

It follows from (2.1) that $\xi_t^\pm$ are real numbers. The definitions of $\xi_t^\pm$ imply that

$$\operatorname*{ess\,inf}_{(-\infty,\xi_t^-)} u(t,\cdot) \geq \frac{2}{3}, \quad \operatorname*{ess\,inf}_{(\xi_t^-,\xi_t^-+1)} u(t,\cdot) < \frac{2}{3} \tag{2.2}$$

and

$$\operatorname*{ess\,sup}_{(\xi_t^+,+\infty)} u(t,\cdot) \leq \frac{1}{3}, \quad \operatorname*{ess\,sup}_{(\xi_t^+-1,\xi_t^+)} u(t,\cdot) > \frac{1}{3} \tag{2.3}$$

together with $\xi_t^- \leq \xi_t^+$. Property (2.1) also yields

$$\sup_{t\in\mathbb{R}} \left(\xi_t - \xi_t^-\right) < +\infty.$$

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Indeed, otherwise, there would exist a sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\xi_{t_n} - \xi_{t_n}^- \to +\infty \text{ as } n \to +\infty,$$

whence ess $\inf_{(-\infty, \xi_{t_n}^- + 1)} u(t_n, \cdot) \to 1$ as $n \to +\infty$, contradicting the property

$$\operatorname*{ess\,inf}_{(\xi_{t_n}^-, \xi_{t_n}^- + 1)} u(t_n, \cdot) < \frac{2}{3}.$$

Similarly, there holds

$$\inf_{t\in\mathbb{R}} \left(\xi_t - \xi_t^+\right) > -\infty.$$

Eventually, using again $\xi_t^- \leq \xi_t^+$, one infers that the families $(\xi_t - \xi_t^{\pm})_{t\in\mathbb{R}}$ and $(\xi_t^+ - \xi_t^-)_{t\in\mathbb{R}}$ are all bounded.

From the general regularity assumptions made in the paper, the function u is actually uniformly continuous with respect to t in $[T, +\infty) \times \mathbb{R}$, for every $T \in \mathbb{R}$. By considering $T = 0$, it follows in particular that there exists $\delta > 0$ such that, for all $t \geq 0$, $s \geq 0$ and $x \in \mathbb{R}$,

$$\left(|t - s| \leq \delta\right) \implies \left(|u(t, x) - u(s, x)| \leq \frac{1}{12}\right).$$

As a consequence, for every $t \geq 0$, denoting $A_t^{\pm}$ the non-negligible measurable sets defined as

$$A_t^- = \left\{x \in (\xi_t^-, \xi_t^- + 1);\ u(t, x) < \frac{2}{3}\right\} \text{ and } A_t^+ = \left\{x \in (\xi_t^+ - 1, \xi_t^+);\ u(t, x) > \frac{1}{3}\right\},$$

it follows that

$$u(t + \delta, x) < \frac{3}{4} \text{ for all } x \in A_t^- \quad \text{and} \quad u(t + \delta, x) > \frac{1}{4} \text{ for all } x \in A_t^+. \tag{2.4}$$

On the other hand, from (2.1), there is $M \geq 0$ such that, for every $t \in \mathbb{R}$,

$$\begin{cases} u(t, x) \geq \dfrac{3}{4} & \text{for a.e. } x \in (-\infty, \xi_t - M), \\[2mm] u(t, x) \leq \dfrac{1}{4} & \text{for a.e. } x \in (\xi_t + M, +\infty). \end{cases}$$

Since, for every $t \geq 0$, the sets $A_t^{\pm}$ have a positive measure, one infers from (2.4) that

$$\xi_t^- + 1 > \xi_{t+\delta} - M$$

and

$$\xi_t^+ - 1 < \xi_{t+\delta} + M.$$

Remembering that the quantities $(\xi_t - \xi_t^{\pm})_{t\in\mathbb{R}}$ are bounded, one gets that

$$\sup_{t\geq 0} |\xi_{t+\delta} - \xi_t| < +\infty. \tag{2.5}$$

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Finally, notice that $1 \geq u(0, \cdot) \geq (2/3)\mathbb{1}_{(-\infty, \xi_0^-)}$ a.e. in $\mathbb{R}$ by (2.2), where $\mathbb{1}_E$ denotes the characteristic function of a measurable set E. The comparison principle (1.3) implies that, for every $t > 0$,

$$1 \geq u(t, \cdot) \geq v(t, \cdot) \text{ a.e. in } \mathbb{R},$$

where v denotes the solution of the Cauchy problem (1.2) with initial condition $(2/3)\mathbb{1}_{(-\infty, \xi_0^-)}$. It follows then from the spreading property (1.4) that ess $\inf_{(0, ct)} v(t, \cdot) \to 1$ as $t \to +\infty$ for every $c > 0$, whence

$$\operatorname*{ess\,inf}_{(0, ct)} u(t, \cdot) \to 1 \text{ as } t \to +\infty \text{ for every } c > 0.$$

For any given $c > 0$, property (2.3) then yields $\xi_t^+ \geq ct$ for all $t > 0$ large enough, whence $\liminf_{t \to +\infty} \xi_t^+/t \geq c$. Therefore,

$$\liminf_{t \to +\infty} \frac{\xi_t}{t} \geq c \text{ for all } c > 0,$$

since $(\xi_t - \xi_t^+)_{t \in \mathbb{R}}$ is bounded. As a conclusion, $\xi_t/t \to +\infty$ as $t \to +\infty$. But (2.5) yields the existence of a real number M' such that $|\xi_{t+\delta} - \xi_t| \leq M'$ for all $t \geq 0$. Hence

$$\frac{\xi_{n\delta}}{n\,\delta} \leq \frac{\xi_0 + n\,M'}{n\,\delta}$$

for all $n \in \mathbb{N}$ and $\liminf_{t \to +\infty} \xi_t/t \leq M'/\delta < +\infty$. One has then reached a contradiction and the proof of Theorem 2.2 is thereby complete. $\qquad\square$

3   Stretching at large time for the Cauchy problem (1.2)

The second main result is concerned with flattening and stretching properties at large time for the solutions of the Cauchy problem (1.2). We assume in this section that the equation (1.1) is homogeneous, in the sense that, for any $u_0 \in L^\infty(\mathbb{R}; [0, 1])$ and for any $h \in \mathbb{R}$, there holds

$$u(t, \cdot + h) = v(t, \cdot) \text{ a.e. in } \mathbb{R} \text{ for every } t > 0, \tag{3.1}$$

where u and v denote the solutions of the Cauchy problem (1.2) with initial conditions u_0 and $v_0 := u_0(\cdot + h)$. The property (3.1) is satisfied by the previous examples of dispersal operators $\mathcal{D}u$: fractional Laplacian $\mathcal{D}u = -(-\Delta_x)^\alpha u$, the convolution operator $\mathcal{D}u = J * u - u$, the nonlinear fast diffusion $\mathcal{D}u = (u^\gamma)_{xx}$ and the nonlinear fractional diffusion $\mathcal{D}u = -(-\Delta_x)^\alpha(u^\gamma)$. If follows from assumptions (1.3) and (3.1) that if $u_0 \in L^\infty(\mathbb{R}; [0, 1])$ is nonincreasing, then

$$0 \leq u_0(\cdot + h) \leq u_0 \leq 1 \text{ a.e. in } \mathbb{R} \text{ for every } h > 0,$$

whence $0 \leq u(t, \cdot + h) \leq u(t, \cdot) \leq 1$ a.e. in $\mathbb{R}$ for every $t > 0$. In other words, $u(t, \cdot)$ is nonincreasing for every $t > 0$. Under these conditions, for every $t \geq 0$, let us denote

$$m(t) = \operatorname*{ess\,inf}_{\mathbb{R}} u(t, \cdot) = \operatorname*{ess\,inf}_{(x, +\infty)} u(t, \cdot) \text{ for every } x \in \mathbb{R}$$

and
$$M(t) = \operatorname*{ess\,sup}_{\mathbb{R}} u(t, \cdot) = \operatorname*{ess\,sup}_{(-\infty, x)} u(t, \cdot) \quad \text{for every } x \in \mathbb{R},$$

and observe that $0 \le m(t) \le M(t) \le 1$. Lastly, for every $\lambda \in (0, 1)$, set

$$x_\lambda(t) = \inf \left\{ x \in \mathbb{R}; \ \operatorname*{ess\,sup}_{(x, +\infty)} u(t, \cdot) < \lambda \right\}. \tag{3.2}$$

Notice that $x_\lambda(t) = +\infty$ if $\lambda \le m(t)$, while $x_\lambda(t) = -\infty$ if $\lambda > M(t)$. Moreover,

$$-\infty \le x_b(t) \le x_a(t) \le +\infty \text{ if } 0 < a \le b < 1.$$

Theorem 3.1 *Assume that* (1.1) *satisfies* (1.3), (1.4) *and* (3.1) *and let u be the solution of the Cauchy problem* (1.2) *with a nonincreasing initial condition $u_0 \in L^\infty(\mathbb{R}; [0,1]) \backslash \{0\}$ such that $m(t) \to 0$ and $M(t) \to 1$ as $t \to +\infty$. Then, for every $0 < a < b < 1$,*

$$\limsup_{t \to +\infty} \big(x_a(t) - x_b(t) \big) = +\infty. \tag{3.3}$$

Theorem 3.1 means that the function $u(t, \cdot)$ becomes regularly as stretched as wanted as $t \to +\infty$, that is, for any given $0 < \varepsilon < 1/2$, the "reaction" zone where $u(t, \cdot)$ ranges between ε and $1 - \varepsilon$ cannot stay bounded at large time. However, the following reasonable conjecture remains open:

Conjecture 3.2 *Under the assumptions of Theorem 3.1, $x_a(t) - x_b(t) \to +\infty$ as $t \to +\infty$ for any $0 < a < b < 1$.*

The assumptions $m(t) \to 0$ and $M(t) \to 1$ are satisfied for instance if $\mathcal{D}u = -(-\Delta_x)^\alpha u$ with $0 < \alpha < 1$, $\mathcal{D}u = J * u - u$, $\mathcal{D}u = (u^\gamma)_{xx}$ with $0 < \gamma < 1$, and $\mathcal{D}u = -(-\Delta_x)^\alpha(u^\gamma)$ with $0 < \alpha < 1$ and $\max(1 - 2\alpha, 0) < \gamma$, and if $m(0) = 0$ and $M(0) = 1$ (remember that the $C^1([0,1])$ function f is such that $f(0) = f(1) = 0$). These conditions are also satisfied for the same type of dispersal $\mathcal{D}u$ in the case $m(0) = 0$, $0 < M(0) \le 1$ with $f > 0$ on $(0, 1)$.

Theorem 3.1 is in spirit similar to a result obtained in [26] for the solutions of the Cauchy problem of the equation

$$u_t = u_{xx} + f(u). \tag{3.4}$$

Namely, if f satisfies (1.6) and $f(u)/u$ is nonincreasing over $(0, 1]$ and if the initial condition $u_0 : \mathbb{R} \to (0, 1]$ satisfies $\liminf_{x \to -\infty} u_0(x) > 0$, $u_0(+\infty) = 0$, together with $\lim_{x \to +\infty} \big(u_0(x) e^{\eta x} \big) = +\infty$ for every $\eta > 0$ and $u_0'/u_0 \in L^p(\mathbb{R}) \cap C^{2,\theta}(\mathbb{R})$ for some $p \in (1, +\infty)$ and $\theta \in (0, 1)$, then the solution u of (3.4) with initial condition u_0 is such that $\|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to +\infty$. This last conclusion is clearly stronger than (3.3). But, for our equation (1.1), we make weaker regularity assumptions on u_0. In particular, u_0 may well be discontinuous and, since there is in general no spatial regularizing effect for (1.1), $u(t, \cdot)$ may not be differentiable or even continuous (for instance, there is no regularizing effect for (1.7) – however, the other examples mentioned in Section 1 have a regularizing effect, see also Remark 3.5 below). But the property (3.3) still gives some information about the profile of $u(t, \cdot)$ at large time. In particular, even if $u(t, \cdot)$ may be discontinuous at some points, the discontinuity jumps cannot stay large as time goes to $+\infty$.

Remark 3.3 The example of the previous paragraph shows that there are some equations of the type (1.1) (namely, (3.4)) for which the assumption (1.4) is not satisfied but for which there are *some* solutions u satisfying the hypotheses and conclusions of Theorem 3.1. However, under the assumption (1.4) (together with (1.3) and (3.1)), Theorem 3.1 says that *all* solutions u of (1.1) satisfying the hypotheses of Theorem 3.1 at $t = 0$ satisfy the conclusion (3.3). We point out that this result does not hold in general without the condition (1.4). Indeed, for instance, equation (3.4) with (1.6) admits standard traveling fronts $\varphi_c(x - ct)$ connecting 0 and 1 for every $c \geq c^*$, for some $c^* > 0$. So if $u_0 = \varphi_c$, then the solution $u(t, x) = \varphi_c(x - ct)$ of (3.4) with initial condition u_0 satisfies the assumptions of Theorem 3.1 but not the conclusion (3.3).

Moreover, we can describe the profile of the solution u around the position of each level sets. More precisely, for any level $\lambda \in (0, 1)$, $u(t, \cdot + x_\lambda(t))$ becomes regularly as flat as wanted at large times and becomes as close as wanted to λ locally on the left and on the right of the position $x_\lambda(t)$.

Corollary 3.4 *Under the same assumptions as in Theorem 3.1, for every $0 < \lambda < 1$ and $R > 0$, there holds*

$$\begin{cases} \liminf_{t \to +\infty} \left(\underset{x \in [-R,0]}{\text{ess sup}} |u(t, x + x_\lambda(t)) - \lambda| \right) = 0, \\ \liminf_{t \to +\infty} \left(\underset{x \in [0,R]}{\text{ess sup}} |u(t, x + x_\lambda(t)) - \lambda| \right) = 0. \end{cases} \tag{3.5}$$

Proof of Theorem 3.1. Let u be a solution of the Cauchy problem (1.2) with a nonincreasing initial condition $u_0 \in L^\infty(\mathbb{R}; [0, 1]) \setminus \{0\}$. As already emphasized, $u(t, \cdot)$ is nonincreasing in $\mathbb{R}$ for every $t > 0$. In particular, if $m(t) < \lambda < M(t)$, then $x_\lambda(t)$ defined in (3.2) is a real number and

$$\underset{(x_\lambda(t),+\infty)}{\text{ess sup}} u(t, \cdot) \leq \lambda \quad \text{and} \quad \underset{(-\infty,x_\lambda(t))}{\text{ess inf}} u(t, \cdot) \geq \lambda. \tag{3.6}$$

Let now a and b be two given real numbers such that $0 < a < b < 1$ and assume by contradiction that $\limsup_{t \to +\infty} \left(x_a(t) - x_b(t) \right) < +\infty$, that is, there exist $T > 0$ and $M > 0$ such that

$$0 \leq x_a(t) - x_b(t) \leq M \quad \text{for all } t \geq T. \tag{3.7}$$

Since $m(t) \to 0$ and $M(t) \to 1$ as $t \to +\infty$, one can assume without loss of generality that $m(t) < a < b < M(t)$ for all $t \geq T$, whence $x_a(t)$ and $x_b(t)$ are real numbers for $t \geq T$. Notice that

$$x_b(t) \leq x_{(a+b)/2}(t) \leq x_a(t) \quad \text{for all } t \geq T. \tag{3.8}$$

From the uniform continuity of u with respect to t in $[T, +\infty) \times \mathbb{R}$, there is $\delta > 0$ such that, for every $(t, s, x) \in [T, +\infty) \times [T, +\infty) \times \mathbb{R}$,

$$\left(|t - s| \leq \delta \right) \implies \left(|u(t, x) - u(s, x)| \leq \frac{b - a}{4} \right).$$

Now, for every $t \geq T$, one has $u(t, \cdot) \geq b$ a.e. in $(-\infty, x_b(t))$, whence

$$u(t + \delta, \cdot) \geq b - \frac{b - a}{4} = \frac{3b + a}{4} \quad \text{a.e. in } (-\infty, x_b(t)),$$

whereas $u(t, \cdot) \leq a$ a.e. in $(x_a(t), +\infty)$ and

$$u(t + \delta, \cdot) \leq a + \frac{b - a}{4} = \frac{3a + b}{4} \quad \text{a.e. in } (x_a(t), +\infty).$$

Since $(3a + b)/4 < (a + b)/2 < (3b + a)/4$, one infers from the definition of $x_{(a+b)/2}(t + \delta)$ that

$$x_b(t) \leq x_{(a+b)/2}(t + \delta) \leq x_a(t),$$

whence

$$\left| x_{(a+b)/2}(t + \delta) - x_{(a+b)/2}(t) \right| \leq M \quad \text{for all } t \geq T. \tag{3.9}$$

from (3.7) and (3.8).

On the other hand, since $u(T, \cdot)$ is nonincreasing and ranges in $[0, 1]$, it follows again from the definition of $x_{(a+b)/2}(T)$ that

$$1 \geq u(T, \cdot) \geq \frac{a + b}{2} \mathbb{1}_{(-\infty, x_{(a+b)/2}(T))} =: v_0 \quad \text{a.e. in } \mathbb{R}.$$

Therefore, the comparison principle (1.3) implies that, for every $t > 0$, $1 \geq u(t + T, \cdot) \geq v(t, \cdot)$ a.e. in $\mathbb{R}$, where v denotes the solution of the Cauchy problem (1.2) with initial condition $v_0 \in L^\infty(\mathbb{R}; [0, 1]) \setminus \{0\}$. The assumption (1.4) applied to v therefore yields ess $\inf_{(0,ct)} v(t, \cdot) \to 1$ as $t \to +\infty$ for every $c > 0$, whence

$$\operatorname*{ess\,inf}_{(0,ct)} u(t + T, \cdot) \to 1 \quad \text{as } t \to +\infty, \text{ for every } c > 0.$$

Finally, for any arbitrary positive real number c, there holds

$$\operatorname*{ess\,inf}_{(0,cn\delta)} u(T + n\delta, \cdot) > \frac{a + b}{2}$$

for $n \in \mathbb{N}$ large enough, whence $x_{(a+b)/2}(T + n\delta) \geq cn\delta$ for n large enough, since $u(T + n\delta, \cdot)$ is nonincreasing. As a consequence,

$$\liminf_{n \to +\infty} \frac{x_{(a+b)/2}(T + n\delta)}{n\delta} \geq c$$

and, since $c > 0$ is arbitrary, it follows that $x_{(a+b)/2}(T + n\delta)/(n\delta) \to +\infty$ as $n \to +\infty$. This contradicts (3.9) and the proof of Theorem 3.1 is thereby complete. $\qquad\square$

Remark 3.5 Consider here the case of equation (1.5) with the fractional Laplacian $\mathcal{D}u = -(-\Delta_x)^\alpha u$ with $0 < \alpha < 1$. For this equation, there is a regularizing effect in the spatial variable and, under the assumptions of Theorem 3.1, it follows that the solution u is of class C^1 in $[\delta, +\infty) \times \mathbb{R}$ for every $\delta > 0$, with bounded derivatives with respect to t and x.

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For any given $\lambda \in (0,1)$, there is a real number $T_\lambda > 0$ such that, for every $t \geq T_\lambda$, there is a unique $x_\lambda(t) \in \mathbb{R}$ such that $u(t, x_\lambda(t)) = \lambda$. We now claim that

$$\liminf_{t \to +\infty} |u_x(t, x_\lambda(t))| = 0.$$

Indeed, otherwise, there are $\varepsilon > 0$ and $T' \geq T_\lambda$ such that $u_x(t, x_\lambda(t)) \leq -\varepsilon$ for all $t \geq T'$. It follows then from the implicit function theorem that the function x_λ is of class $C^1([T', +\infty))$ and that

$$x_\lambda'(t) = -\frac{u_t(t, x_\lambda(t))}{u_x(t, x_\lambda(t))}$$

for all $t \geq T'$. Therefore, there is $M \geq 0$ such that $|x_\lambda'(t)| \leq M$ for all $t \geq T'$ and one can get a contradiction as in the end of the proof of Theorem 3.1.

Again for (1.5) with the fractional Laplacian $\mathcal{D}u = -(-\Delta_x)^\alpha u$, and with a Fisher-KPP function f, Roquejoffre and Tarfulea [41] recently showed that the partial derivatives of u with respect to x are all bounded by u times an exponentially decaying function, that is, for every $k \in \mathbb{N}$, there are positive constants C_k and δ_k such that $\|\partial_x^k u(t, \cdot)\|_\infty \leq C_k u(t, x) e^{-\delta_k t}$ for large t. This result has been proved under the additional condition that u_0 is exponentially decaying as $|x| \to +\infty$, and it actually holds in any spatial dimension. It implies in particular that $\|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to +\infty$. Moreover, it is proved in [41] that the large time dynamics of the KPP problem (1.5) is in some sense the same as that of the ODE $\theta'(t) = f(\theta)$.

More generally speaking, under the general conditions (1.3), (1.4) and (3.1), we conjecture that fast propagation should lead to a flattening of the solution in the sense that, at least if $f > 0$ on $(0, 1)$, then the only x-monotone solutions $u : \mathbb{R} \times \mathbb{R} \to (0, 1)$ of (1.5) are actually x-independent, that is, they can be written as $u(t, x) = \theta(t)$, where $\theta'(t) = f(\theta(t))$ for all $t \in \mathbb{R}$ with $\theta(-\infty) = 0$ and $\theta(+\infty) = 1$.

An intermediate result that we prove here under the sole conditions (1.3), (1.4) and (3.1) is the flattening result stated in Corollary 3.4. We point out that this result holds for all equations (1.5), (1.7), (1.8) and (1.9) under the same conditions as in Section 1.

Proof of Corollary 3.4. Let u be a solution of the Cauchy problem (1.2) with a nonincreasing initial condition $u_0 \in L^\infty(\mathbb{R}; [0, 1]) \backslash \{0\}$. As already emphasized, $u(t, \cdot)$ is nonincreasing in $\mathbb{R}$ for every $t > 0$. In particular, if $m(t) < \lambda < M(t)$, then $x_\lambda(t)$ defined in (3.2) is a real number and (3.6) holds.

Let then λ be any real number such that $0 < \lambda < 1$, R be any positive real number and $\varepsilon > 0$ be any positive real number such that $0 < \lambda - \varepsilon < \lambda < \lambda + \varepsilon < 1$. From Theorem 3.1, there exist two increasing sequences $(t_n)_{n \in \mathbb{N}}$ and $(t_n')_{n \in \mathbb{N}}$ of positive real numbers such that

$$x_\lambda(t_n) - x_{\lambda+\varepsilon}(t_n) \to +\infty \quad \text{and} \quad x_{\lambda-\varepsilon}(t_n') - x_\lambda(t_n') \to +\infty \quad \text{as } n \to +\infty.$$

As a consequence, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\begin{cases} x_{\lambda+\varepsilon}(t_n) \leq x + x_\lambda(t_n) & \text{for all } x \in [-R, 0], \\ x + x_\lambda(t_n') \leq x_{\lambda-\varepsilon}(t_n') & \text{for all } x \in [0, R]. \end{cases}$$

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Thus, by (3.6) applied with $\lambda - \varepsilon$, λ and $\lambda + \varepsilon$, one infers that, for all $n \geq N$,

$$\begin{cases} \lambda \leq u(t_n, x + x_\lambda(t_n)) \leq \lambda + \varepsilon & \text{for a.e. } x \in [-R, 0], \\ \lambda - \varepsilon \leq u(t'_n, x + x_\lambda(t'_n)) \leq \lambda & \text{for a.e. } x \in [0, R], \end{cases}$$

whence

$$\operatorname*{ess\,sup}_{x \in [-R, 0]} |u(t_n, x + x_\lambda(t_n)) - \lambda| \leq \varepsilon \quad \text{and} \quad \operatorname*{ess\,sup}_{x \in [0, R]} |u(t'_n, x + x_\lambda(t'_n)) - \lambda| \leq \varepsilon$$

for all $n \geq N$. Since $\varepsilon > 0$ can be arbitrarily small and $\lim_{n \to +\infty} t_n = \lim_{n \to +\infty} t'_n = +\infty$, the proof of Corollary 3.4 is thereby complete. $\qquad\square$

Remark 3.6 It easily follows from the proof of Corollary 3.4 that, if Conjecture 3.2 is true, then for any level $\lambda \in (0, 1)$, the solution u of the Cauchy problem (1.2) converges to λ locally around each position of the level set as $t \to +\infty$, that is,

$$\operatorname*{ess\,sup}_{x \in [-R, R]} |u(t, x + x_\lambda(t)) - \lambda| \to 0 \quad \text{as } t \to +\infty$$

for every $R > 0$. This last convergence result would then hold automatically for the whole family $t \to +\infty$ and on both the left and right of $x_\lambda(t)$ simultaneously.

Lastly, in order to illustrate the theoretical results on the flattening and stretching properties of the solutions of the Cauchy problem (1.2), we have performed some numerical simulations that are depicted in figures 1 and 2. They have been done with the KPP nonlinearity $f(u) = u(1 - u)$, for the three examples $\mathcal{D}u = -(-\Delta_x)^\alpha u$ with $\alpha = 0.9$, $\mathcal{D}u = J * u - u$ with $J(x) = \exp(-\sqrt{|x|})/4$, and $\mathcal{D}u = (u^\gamma)_{xx}$ with $\gamma = 1/2$ (Fig.1, a, b and c, respectively). These properties, and the acceleration of the solutions in each of these three cases are to be compared with the well-known convergence to a standard travelling front which can be observed with the standard diffusion operator $\mathcal{D}u = u_{xx}$ (Fig. 1, d). To compute these solutions, we used (a and b): the Strang splitting, which consists in splitting the equation (1.2) into two simpler evolution problems [11]: $v_t = -(-\Delta_x)^\alpha v$ and $v_t = J * v - v$ which are treated with fast Fourier transform techniques and $w_t = w(1 - w)$ which solution can be computed explicitly; (c and d): the time-dependent finite element solver of Comsol Multiphysics©.

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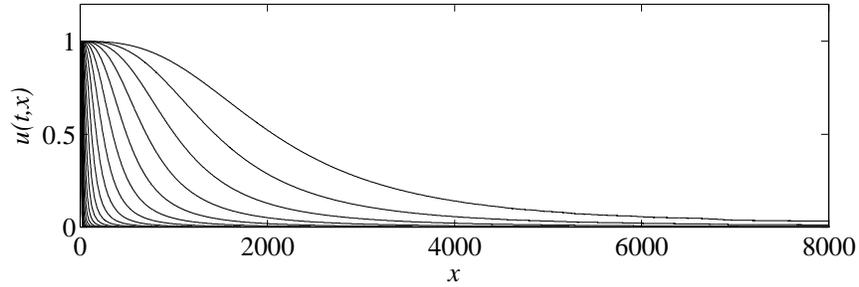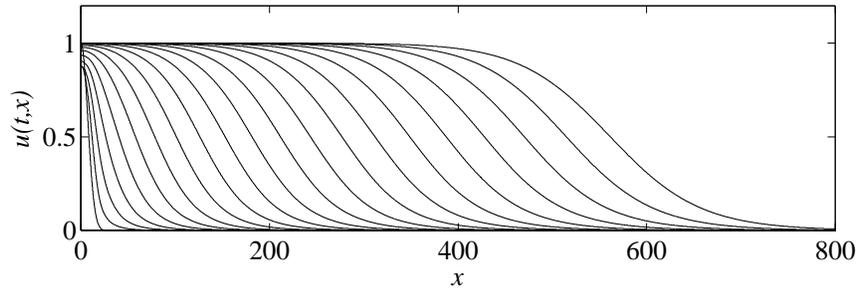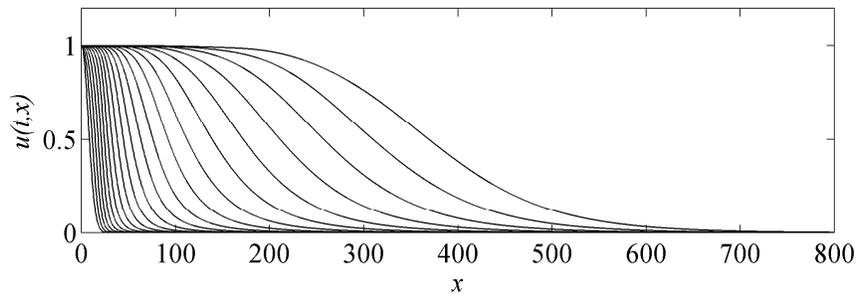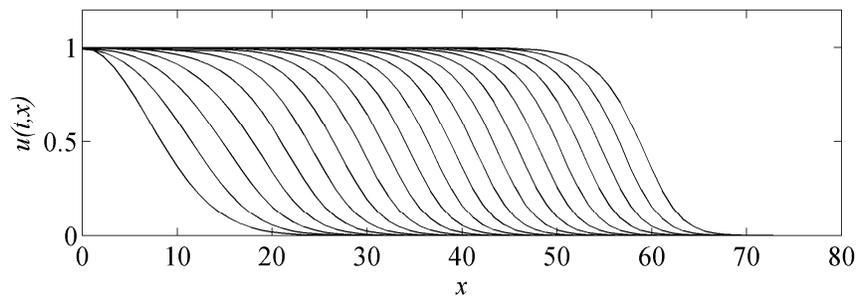
Figure 1: Propagation to the right of the solution of the Cauchy problem (1.2), at successive times $t = 0, 1, \ldots, 20$, with (a) $\mathcal{D}u = -(-\Delta_x)^\alpha u$ with $\alpha = 0.9$; (b) $\mathcal{D}u = J * u - u$ with $J(x) = \exp(-\sqrt{|x|})/4$; (c) $\mathcal{D}u = (u^\gamma)_{xx}$ with $\gamma = 1/2$; and (d) $\mathcal{D}u = u_{xx}$. In all cases, the initial condition was $u_0(x) = \exp(-x^2/100)$ and the function f was of the KPP type $f(u) = u\,(1 - u)$.

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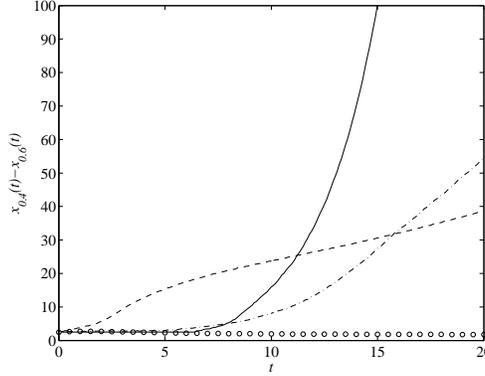
Figure 2: Distance $x_{0.4}(t) - x_{0.6}(t)$ between two level sets of the solution of the Cauchy problem (1.2), for $t \in (0, 20)$, with: (solid line) $\mathcal{D}u = -(-\Delta_x)^{\alpha}u$ with $\alpha = 0.9$; (dashed line) $\mathcal{D}u = J * u - u$ with $J(x) = \exp(-\sqrt{|x|})/4$; (dash-dot line) $\mathcal{D}u = (u^{\gamma})_{xx}$ with $\gamma = 1/2$; and (circles) $\mathcal{D}u = u_{xx}$. The assumptions on f and u_0 are the same as in Fig. 1.

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