Large time monotonicity of solutions of reaction-diffusion equations in $\mathbb{R}^N$

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Abstract

In this paper, we consider nonnegative solutions of spatially heterogeneous reaction-diffusion equations in the whole space. Under some assumptions on the initial conditions, including in particular the case of compactly supported initial conditions, we show that, above any arbitrary positive value, the solution is increasing in time at large times. Furthermore, in the one-dimensional case, we prove that, if the equation is homogeneous outside a bounded interval and the reaction is linear around the zero state, then the solution is time-increasing in the whole line at large times. The question of the monotonicity in time is motivated by a medical imagery issue.

Dans cet article nous étudions les solutions positives d'équations de réaction diffusion dans l'espace entier. Sous certaines conditions sur la donnée initiale, nous démontrons que, au dessus d'une certaine valeur arbitrairement petite, la solution est croissante en temps pour des temps assez grands.

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1. Introduction and main results

In this paper, we consider the Cauchy problem for the following reaction-diffusion equation set in the whole space $\mathbb{R}^N$

$$\begin{cases} u_t = \operatorname{div}(A(x)\nabla u) + f(x,u), & t>0,\ x\in\mathbb{R}^N, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Here u_t stands for $u_t(t,x) = \frac{\partial u}{\partial t}(t,x)$ and the divergence and the gradient act on the spatial variables x. We are interested in the monotonicity in time for large times, when

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the initial condition is localized and equation (1.1) is of the monostable. More precisely, the assumptions are listed below.

Framework and main assumptions

The initial condition u_0 is in $L^\infty(\mathbb{R}^N)$ with $0 \le u_0(x) \le 1$ a.e. in $\mathbb{R}^N$ and u_0 is non-trivial, in the sense that $\|u_0\|_{L^\infty(\mathbb{R}^N)} > 0$. We also assume that either there exists $\beta > 0$ such that

$$u_0(x) = O(e^{-\beta|x|^2}) \text{ as } |x| \to +\infty \tag{1.2}$$

(a particular important case is when u_0 is compactly supported), or there exist $0 < \gamma \le \delta$ and $\lambda > 0$ such that

$$\gamma\, e^{-\lambda|x|} \le u_0(x) \le \delta\, e^{-\lambda|x|} \text{ for all } |x| \text{ large enough,} \tag{1.3}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

The diffusion term A is assumed to be a symmetric matrix field $A = (A_{ij})_{1\le i,j\le N}$ of class $C^{1,\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$ and uniformly definite positive: there exists a constant $\nu \ge 1$ such that

$$\nu^{-1} I \le A(x) \le \nu I \text{ for all } x \in \mathbb{R}^N, \tag{1.4}$$

in the sense of symmetric matrices, where $I \in \mathbb{S}_N(\mathbb{R})$ is the identity matrix. One also assumes that A is locally asymptotically homogeneous at infinity, in the sense that

$$\forall\, 1 \le i,j \le N, \quad |\nabla A_{ij}(x)| \to 0 \text{ as } |x| \to +\infty. \tag{1.5}$$

A particular example of a $C^{1,\alpha}(\mathbb{R}^N)$ matrix field satisfying (1.5) is when $A_{ij}(x)$ converges to a constant as $|x| \to +\infty$ for every $1 \le i,j \le N$. An important subcase is that of a matrix field A which is independent of x. Notice that, since A is of class $C^{1,\alpha}(\mathbb{R}^N)$, the condition (1.5) is equivalent to the fact that the local oscillations of the functions A_{ij} converge to 0 at infinity, that is, for every $R > 0$ and $1 \le i,j \le N$,

$$\underset{B(x,R)}{\operatorname{osc}} A_{ij} := \max_{B(x,R)} A_{ij} - \min_{B(x,R)} A_{ij} \;\to\; 0 \text{ as } |x| \to +\infty,$$

where $B(x,R)$ denotes the open Euclidean ball of center x and radius R. However, notice that the matrix fields $A(x)$ satisfying this property should not converge as $|x| \to +\infty$ in general, even in dimension $N = 1$.

The reaction term $f : \mathbb{R}^N \times [0,1] \to \mathbb{R}$ is a continuous function, of class $C^{0,\alpha}$ in x uniformly with respect to $u \in [0,1]$, and Lipschitz continuous in u, uniformly with respect to $x \in \mathbb{R}^N$. Throughout the paper, one assumes that

$$f(x,0) = f(x,1) = 0 \text{ for every } x \in \mathbb{R}^N \tag{1.6}$$

and that

$$u \mapsto \frac{f(x,1-u)}{u} \text{ is non-increasing in } (0,1] \tag{1.7}$$

for every $x \in \mathbb{R}^N$. One also assumes that there exist $\mu > 0$ and $s_0 \in (0,1)$ such that

$$f(x,s) \ge \mu\, s \text{ for all } (x,s) \in \mathbb{R}^N \times [0,s_0]. \tag{1.8}$$

These assumptions imply in particular that f is positive in $\mathbb{R}^N \times (0,1)$ and even that $\inf_{x\in\mathbb{R}^N} f(x,s) \geq \mu s > 0$ for every $s \in (0, s_0]$ and $\inf_{x\in\mathbb{R}^N} f(x,s) \geq \mu s_0(1-s)/(1-s_0) > 0$ for every $s \in [s_0, 1)$. Furthermore, f is assumed to be of class C^1 with respect to u in $\mathbb{R}^N \times ([0, s_0] \cup [s_1, 1])$ for some $s_1 \in (0,1)$ with $f_u = \frac{\partial f}{\partial u}$ bounded and uniformly continuous in $\mathbb{R}^N \times ([0, s_0] \cup [s_1, 1])$, and of class $C^{0,\alpha}$ with respect to x uniformly in $s \in [0, s_0] \cup [s_1, 1]$. Lastly, one assumes that $f_u(\cdot, 0)$ is locally asymptotically homogeneous at infinity, in the sense that, for every $R > 0$,

$$\operatorname*{osc}_{B(x,R)} f_u(\cdot, 0) \to 0 \ \text{ as } |x| \to +\infty. \tag{1.9}$$

Notice that (1.9) holds if $f_u(\cdot, 0) \in C^1(\mathbb{R}^N)$ and $|\nabla f_u(x, 0)| \to 0$ as $|x| \to +\infty$ or if $f_u(x, 0)$ converges to a constant as $|x| \to +\infty$ (in particular, if $f_u(\cdot, 0)$ is constant). An important class of examples of functions f satisfying the aforementioned hypotheses is when $f(x,u) = r(x)\, g(u)$, where g is of class C^1, concave in $[0,1]$, positive in $(0,1)$ with $g(0) = g(1) = 0$, and r is of class $C^{0,\alpha}(\mathbb{R}^N)$, locally asymptotically homogeneous at infinity and $0 < \inf_{\mathbb{R}^N} r \leq \sup_{\mathbb{R}^N} r < +\infty$. The archetype is the homogeneous logistic Fisher-KPP [9, 13] reaction $f(x, u) = u(1-u)$ with $r(x) = 1$ and $g(u) = u(1-u)$ as above. However, for general functions $f(x, u)$ satisfying the above assumptions, slow oscillations at infinity are not excluded, even in dimension $N = 1$ (see [11] for the study of one-dimensional equations of the type (1.1) with slow oscillations as $x \to \pm\infty$).

From the parabolic regularity theory, the solution u of (1.1) is well-defined for all $t > 0$ and it is classical in $(0, +\infty) \times \mathbb{R}^N$ with

$$0 < u(t,x) < 1 \ \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^N, \tag{1.10}$$

by the strong parabolic maximum principle. From the assumptions made on f, even without (1.9), it is shown in [5] that any stationary solution $p(x)$ of (1.1) such that $0 \leq p \leq 1$ in $\mathbb{R}^N$ is either identically equal to 0 in $\mathbb{R}^N$ or is bounded from below by a positive constant in $\mathbb{R}^N$. Since $\inf_{x\in\mathbb{R}^N} f(x,s) > 0$ for every $s \in (0,1)$, it then follows immediately in the latter case that p is identically equal to 1 in $\mathbb{R}^N$. Therefore, again from [5], the solution u of (1.1) satisfies $u(t,x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$.

Lastly, from [4], it is also known that there is $c > 0$ such that

$$\min_{|x| \leq ct} u(t,x) \to 1 \ \text{ as } t \to +\infty. \tag{1.11}$$

In other words, the state 1 invades the whole space as $t \to +\infty$ with at least a positive spreading speed $c > 0$. But, the asymptotic spreading speed of u may not be unique, in the sense that some oscillations of the spreading rates of the level sets of u between two different positive speeds are possible in general even for compactly supported initial conditions, see [11]. This means that, in general, there is no speed $c_0 > 0$ such that (1.11) holds for all $c \in [0, c_0)$ and $\max_{|x| \geq ct} u(t,x) \to 0$ as $t \to +\infty$ for all $c > c_0$. However, when the equation (1.1) is homogeneous and the initial condition is compactly supported, there exists such a positive spreading speed c_0, see e.g. [2].

Main results

The main result of our paper is the following asymptotic time-monotonicity of the solutions of (1.1).

Theorem 1.1. *Let A be a symmetric matrix field $A = (A_{ij})_{1\le i,j\le N}$ of class $C^{1,\alpha}(R^N)$. Let $f : \mathbb{R}^N \times [0,1] \to R$ be a continuous function, of class $C^{0,\alpha}$ in x, uniformly with respect to $u \in [0,1]$, and Lipschitz continuous in u, uniformly with respect to $x \in R^N$. Under the above assumptions* (1.2) *or* (1.3)*, and* (1.4)*-*(1.9)*, the solution u of* (1.1) *satisfies*

$$\inf_{x\in\mathbb{R}^N} u_t(t,x) \to 0 \quad \text{as } t \to +\infty. \tag{1.12}$$

Furthermore, for every $0 < \varepsilon < 1$, there is a time $T_\varepsilon > 0$ such that

$$\forall\,(t,x) \in [T_\varepsilon, +\infty) \times \mathbb{R}^N, \quad u(t,x) \ge \varepsilon \implies u_t(t,x) > 0. \tag{1.13}$$

Property (1.13) means the monotonicity in time at large times in the time-dependent sets where u is bounded away from 0. On the other hand, in the sets where, say, $t \ge 1$ and u is close to 0, then u_t is close to 0 too.[2] Therefore, property (1.13) easily yields (1.12). Lastly, since

$$u(t,x) \to 0 \text{ as } |x| \to +\infty \text{ locally uniformly in } t \in [0,+\infty), \tag{1.14}$$

as will be easily seen in the proof of Theorem 1.1 (more precisely, see the proof of Lemma 2.1 below), property (1.13) implies that, for every $T \ge T_\varepsilon$, the set $\big\{(t,x) \in [T_\varepsilon, T] \times \mathbb{R}^N,\ u(t,x) \ge \varepsilon\big\}$ is compact, whence

$$\min_{(t,x)\in[T_\varepsilon,T]\times\mathbb{R}^N,\, u(t,x)\ge\varepsilon} u_t(t,x) > 0.$$

Let us now comment some earlier related references in the literature. In [16], the question of the time-monotonicity at large times had been addressed for the solutions of some reaction-diffusion equations in straight infinite cylinders with advection shear flows and with f being independent of the unbounded variable. Other time-monotonicity results have been obtained in [3] for time-global transition fronts of space-heterogeneous reaction-diffusion equations of the type (1.1) connecting two stable limiting points. In [19], the time-monotonicity of the solutions u of equations $u_t = \Delta u + f(x,u)$ with reactions f of the ignition type or involving a weak Allee effect has been established for large times in the set where $0 < \varepsilon \le u(t,x) \le 1 - \varepsilon < 1$, for any $\varepsilon > 0$ small enough. Lastly, we refer to [7] for some results on time-monotonicity for small t and large x for the solutions of the homogeneous equation $u_t = \Delta u + g(u)$ which are initially compactly supported.

For the heterogeneous Fisher-KPP type equation (1.1), we conjecture that, under the assumptions of Theorem 1.1, $u_t(t,\cdot) > 0$ in $\mathbb{R}^N$ for t large enough. This is still an open question. However, we can answer positively under some additional assumptions on (1.1) in dimension 1.

[2]Indeed, if $u(t_n,x_n) \to 0$ with $(t_n,x_n) \in [1,+\infty)\times\mathbb{R}^N$, then the functions $v_n(t,x) := u(t+t_n, x+x_n)$ converge locally in $C^{1,2}_{t,x}((-1,+\infty)\times\mathbb{R}^N)$, up to extraction of a subsequence, to a solution v of an equation of the type $v_t = \mathrm{div}(A_\infty(x)\nabla u) + f_\infty(x,u)$ for some diffusion and reaction coefficients A_∞ and f_∞ satisfying the same type of assumptions as A and f. Furthermore, $v(0,0) = 0$ and $0 \le v \le 1$ in $(-1,+\infty)\times\mathbb{R}^N$, whence $v = 0$ in $(-1,0]\times\mathbb{R}^N$ from the strong maximum principle and then $v = 0$ in $(-1,+\infty)\times\mathbb{R}^N$ from the uniqueness of the solutions of the associated Cauchy problem. Finally, $v_t(0,0) = 0$ and $u_t(t_n,x_n) = (v_n)_t(0,0) \to v_t(0,0) = 0$ as $n \to +\infty$.

Theorem 1.2. *In addition to* (1.2) *or* (1.3)*,* (1.4) *and* (1.6)*-*(1.8)*, assume that* $N = 1$*, that* $A'(x) = 0$ *for* $|x|$ *large enough and that there are* $\lambda^{\pm} > 0$*,* $\theta \in (0, 1)$ *and two functions* $f^{\pm} : [0, 1] \to \mathbb{R}$ *such that* $f(x, u) = f^{\pm}(u)$ *for* $\pm x$ *large enough and* $f^{\pm}(u) = \lambda^{\pm} u$ *for all* $u \in [0, \theta]$*. Then there is* $\tau > 0$ *such that the solution* u *of* (1.1) *satisfies*

$$u_t(t, x) > 0 \quad \text{for all } t \geq \tau \text{ and } x \in \mathbb{R}. \tag{1.15}$$

Let us now describe the main ideas of the proof of Theorems 1.1 and 1.2 and the outline of the paper. In Section 2, the solution u is proved to be T-monotone in time ($u(t + T, x) \geq u(t, x)$) at large time t and for all T large enough, by using the decay of u_0 at infinity and some Gaussian estimates for the fundamental solution associated with the linear equation obtained from (1.1). This T-monotonicity is then improved in Section 3 by compactness arguments in the region where u is away from 0 and from 1 and then in Section 4 by using in particular the assumption (1.7) and by an application of the maximum principle in some sets which are defined recursively. In Section 5, the monotonicity in time is proved in the region where u is close to 1 by using Harnack inequality applied to the function $1 - u$ and some passage to the limit. In Section 6, the τ-monotonicity in time, for any $\tau > 0$, is shown in the region where u is close to 0, by using some Gaussian estimates as well as some new quantitative inequalities for the fundamental solutions associated with families of linear equations similar to (1.1) (these new estimates are proved in Section 8). Section 7 is devoted to the proof of properties (1.12) and (1.13) of Theorem 1.1. Lastly, Section 9 is concerned with the proof of Theorem 1.2, where explicit estimates of the Green function associated to some one-dimensional initial and boundary value problem in half-lines are used.

Remark 1.3. Assume in this remark that, instead of the whole space $\mathbb{R}^N$, equation (1.1) is set on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with Neumann type boundary conditions $\mu(x) \cdot \nabla u(t, x) = 0$ on $\partial\Omega$, where μ is a continuous vector field such that $\mu(x) \cdot \nu(x) > 0$ for all $x \in \partial\Omega$ and ν denotes the outward normal vector field on $\partial\Omega$. Then it follows from the arguments used in the proof of Theorem 1.1 (see especially Section 5) that, under assumptions (1.4) and (1.6)-(1.8), any solution u with a nontrivial initial condition $0 \leq, \not\equiv u_0 \leq, \not\equiv 1$ is increasing in time in the whole set $\overline{\Omega}$ at large times.

Modeling and background

The question of the monotonicity of the solution for large times comes from a simple and natural medical imagery question. A natural way to model a tumor is to introduce a function $\phi(t, x)$ describing the density of tumor cells. In some types of cancers, tumor cells migrate and multiply. They migrate randomly and multiply according to logistic type laws. The simplest model of tumor is therefore the classical KPP equation, as described for instance in Murray [14]

$$\phi_t - \nu \Delta \phi = \lambda \phi (1 - \phi),$$

with positive coefficients ν and λ. Treatments like radiotherapy or chemotherapy induce the death of a part of tumor cells. A simple way to model a treatment at time t_0 is to say that ϕ is discontinuous at t_0 and

$$\phi(t_0^+, x) = \beta\, \phi(t_0^-, x)$$

for all x and for some $0 < \beta < 1$. The total tumoral mass is by definition

$$M(t) = \int_{\mathbb{R}^N} \phi(t,x)dx.$$

However the tumor size can be evaluated through medical imagery devices which detect tumor cells only if their density is large enough, above some threshold $\sigma > 0$. The measured size of the tumor is therefore

$$S(t) = \int_{\mathbb{R}^N} 1_{\phi(t,x)>\sigma} dx.$$

Note that $M(t)$ has a discontinuity during the treatment, but then immediately regrowth. However the clinicians do not measure this total tumoral mass, but only $S(t)$. A natural question then arises: do $S(t)$ and $M(t)$ have the same behavior ? Or is it possible that $S(t)$ decreases after the treatment before growing again ?

In other words: can the observed size of a tumor decrease whereas its actual total mass $\int_{\mathbb{R}^N} \phi(t,x)dx$ increases ?

Let us detail the link between this question and the positivity of ϕ_t. For this let $\Omega(t) = \{x \in \mathbb{R}^N;\ \phi(t,x) > \sigma\}$, and let $x_0 \in \partial\Omega(t_0^+)$. We have

$$\begin{aligned}
\phi_t(t_0^+,x_0) &= \nu\Delta\phi(t_0^+,x_0) + \lambda\phi(t_0^+,x_0)(1-\phi(t_0^+,x_0)) \\
&= \nu\beta\Delta\phi(t_0^-,x_0) + \lambda\beta\phi(t_0^-,x_0)(1-\beta\phi(t_0^-,x_0)) \\
&= \beta\phi_t(t_0^-,x_0) - \lambda\beta\phi(t_0^-,x_0)(1-\phi(t_0^-,x_0)) + \lambda\beta\phi(t_0^-,x_0)(1-\beta\phi(t_0^-,x_0)) \\
&= \beta\phi_t(t_0^-,x_0) + \lambda\beta(1-\beta)\phi^2(t_0^-,x_0).
\end{aligned}$$

The second term is positive, hence if $\phi_t(t_0^-,x) > 0$ everywhere on $\partial\Omega(t_0^+)$, this implies that $\phi_t(t_0^+,\cdot)$ is positive on $\partial\Omega(t_0^+)$, hence that $S(t)$ is increasing just after t_0.

The medical imagery question therefore reduces to the study of the sign of ϕ_t. If $\phi_t \geq 0$ on the boundary of the medical image, then $S(t)$ can not decay after a treatment, and its qualitative behavior is similar to that of $M(t)$. Our results show that $\phi_t \geq 0$ if t is large enough, namely in old enough tumors.

2. *T*-monotonicity in time

Throughout this section and the next ones, one assumes that the conditions (1.4)-(1.9) are fulfilled and u denotes a solution of (1.1) with initial condition u_0 having Gaussian decay at infinity as in (1.2) or satisfying (1.3). The first step in the proof of Theorem 1.1 consists in showing that u is T-monotone in time.

Lemma 2.1. *There is $T > 0$ such that*

$$u(1+t,x) \geq u(1,x) \quad \text{for all } t \geq T \text{ and } x \in \mathbb{R}^N. \tag{2.1}$$

Proof. First of all, as already emphasized, the strong maximum principle implies that $u(1,x) < 1$ for all $x \in \mathbb{R}^N$. Remember also that $u(1,\cdot)$ is actually of class $C^2(\mathbb{R}^N)$. The strategy consists in bounding $u(1,x)$ from above as $|x| \to +\infty$ by a function having the same decay as u_0, and then in showing that $u(1+t,\cdot)$ is above $u(1,\cdot)$ in $\mathbb{R}^N$ for all $t > 0$ large enough. To do so, we will use some lower and upper bounds for the heat kernel associated with the linearized equation (2.3) below, as well as the spreading property (1.11). For the sake of clarity, the two cases – Gaussian decay for u_0 or (1.3)– will be treated separately.

Case 1: Gaussian decay. Assume here that u_0 has Gaussian decay at infinity, that is, there exists $\beta > 0$ such that $u_0(x) = O(e^{-\beta|x|^2})$ as $|x| \to +\infty$. Since $u_0 \in L^\infty(\mathbb{R}^N; [0,1])$, there is then $C > 0$ such that

$$0 \le u_0(x) \le C\, e^{-\beta|x|^2} \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Remember that the function f is globally Lipschitz continuous in its second variable, uniformly with respect to $x \in \mathbb{R}^N$. Since $f(\cdot, 0) = 0$ in $\mathbb{R}^N$, let then $L > 0$ be such that

$$f(x,s) \le Ls \quad \text{for all } (x,s) \in \mathbb{R}^N \times [0,1]. \tag{2.2}$$

The maximum principle yields

$$0 \le u(1,x) \le e^L\, v(1,x) \quad \text{for all } x \in \mathbb{R}^N,$$

where v denotes the solution of the Cauchy problem

$$\begin{cases} v_t &= \operatorname{div}(A(x)\nabla v), \quad t > 0,\ x \in \mathbb{R}^N, \\ v(0,\cdot) &= u_0. \end{cases} \tag{2.3}$$

Therefore,

$$0 \le u(1,x) \le C\, e^L \int_{\mathbb{R}^N} p(1,x;y)\, e^{-\beta|y|^2}\, dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $p(t,x;y)$ denotes the heat kernel associated to the linear equation (2.3), that is, for every $y \in \mathbb{R}^N$, $p(\cdot,\cdot;y)$ solves (2.3) with the Dirac distribution δ_y at y as initial condition. It follows from the bounds of p in [15] (see also [1, 6, 8, 10] for related results) that there is a real number $K \ge 1$ such that

$$\frac{e^{-K|x-y|^2/t}}{K\, t^{N/2}} \le p(t,x;y) \le \frac{K\, e^{-|x-y|^2/(Kt)}}{t^{N/2}} \quad \text{for all } t > 0 \text{ and } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{2.4}$$

In particular,

$$0 \le u(1,x) \le K\, C\, e^L \int_{\mathbb{R}^N} e^{-|x-y|^2/K - \beta|y|^2} dy$$

Let $\eta \in (0,1)$ be such that $\eta < \beta\, K\,(1-\eta)$ and denote $\rho = \beta - \eta/(K(1-\eta)) > 0$. By writing

$$\begin{aligned} -\frac{|x-y|^2}{K} = -\frac{|x|^2}{K} + \frac{2(x\cdot y)}{K} - \frac{|y|^2}{K} \quad &\le \quad -\frac{|x|^2}{K} + \frac{(1-\eta)|x|^2}{K} + \frac{|y|^2}{K(1-\eta)} - \frac{|y|^2}{K} \\ &= \quad -\frac{\eta\,|x|^2}{K} + \frac{\eta\,|y|^2}{K\,(1-\eta)}, \end{aligned}$$

it follows that

$$0 \le u(1,x) \le K\, C\, e^L\, e^{-\eta|x|^2/K} \int_{\mathbb{R}^N} e^{-\rho|y|^2} dy \quad \text{for all } x \in \mathbb{R}^N.$$

To sum up, since the continuous function $u(1,\cdot)$ is less than 1 in $\mathbb{R}^N$ by (1.10), one infers that there exist some real numbers $\theta\in(0,1)$ and $\omega>0$ such that

$$u(1,x)\le \min\big(\theta,\omega\, e^{-\eta|x|^2/K}\big)\ \text{ for all } x\in\mathbb{R}^N. \tag{2.5}$$

Let us now show that $u(1+t,\cdot)$ is above $u(1,\cdot)$ in $\mathbb{R}^N$ for all $t>0$ large enough. Since f is nonnegative in $\mathbb{R}^N\times[0,1]$, one infers from the maximum principle that $u(1+t,x)\ge v(1+t,x)$ for all $t\ge0$ and $x\in\mathbb{R}^N$, where v solves (2.3). Since u_0 is nonnegative a.e. in $\mathbb{R}^N$ and non-trivial, there is $R>0$ such that

$$\sigma:=\int_{B(0,R)}u_0(y)\,dy>0$$

and

$$u(1+t,x)\ge\int_{\mathbb{R}^N}p(1+t,x;y)\,u_0(y)\,dy\ge\frac{1}{K\,(1+t)^{N/2}}\int_{B(0,R)}e^{-K|x-y|^2/(1+t)}\,u_0(y)\,dy$$

for all $t\ge0$ and $x\in\mathbb{R}^N$, from (2.4). By writing

$$-\frac{K\,|x-y|^2}{1+t}\ge-\frac{2\,K\,|x|^2}{1+t}-\frac{2\,K\,|y|^2}{1+t}\ge-\frac{2\,K\,|x|^2}{1+t}-2KR^2$$

for all $t\ge0$, $x\in\mathbb{R}^N$ and $y\in B(0,R)$, one gets that

$$u(1+t,x)\ge\frac{e^{-2KR^2}\,e^{-2K|x|^2/(1+t)}}{K\,(1+t)^{N/2}}\int_{B(0,R)}u_0(y)\,dy=\frac{\sigma\,e^{-2KR^2}\,e^{-2K|x|^2/(1+t)}}{K\,(1+t)^{N/2}} \tag{2.6}$$

for all $t\ge0$ and $x\in\mathbb{R}^N$.

We finally show that (2.1) holds for some $T>0$ large enough. Assume not. Then there exist a sequence $(T_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that $T_n\to+\infty$ as $n\to+\infty$ and $u(1+T_n,x_n)<u(1,x_n)$ for all $n\in\mathbb{N}$. Since $u(1,\cdot)\le\theta<1$ in $\mathbb{R}^N$ and $\min_{|x|\le ct}u(t,x)\to1$ as $t\to+\infty$ with $c>0$ by (1.11), it follows that $|x_n|\ge c(1+T_n)$ for n large enough, while $u(1+T_n,x_n)<u(1,x_n)$ and (2.5)-(2.6) yield

$$\frac{\sigma\,e^{-2KR^2}\,e^{-2K|x_n|^2/(1+T_n)}}{K\,(1+T_n)^{N/2}}<\omega\,e^{-\eta|x_n|^2/K}\ \text{ for all } n\in\mathbb{N},$$

whence

$$\begin{aligned}\sigma\,K^{-1}\,\omega^{-1}\,e^{-2KR^2}\,(1+T_n)^{-N/2}<e^{-\eta|x_n|^2/K+2K|x_n|^2/(1+T_n)} &\le e^{-\eta|x_n|^2/(2K)}\\ &\le e^{-\eta c^2(1+T_n)^2/(2K)}\end{aligned}$$

for all n large enough. This clearly leads to a contradiction. As a consequence, there is $T>0$ such that (2.1) holds.

Case 2: assumption (1.3). Since $0\le u_0\le1$ a.e. in $\mathbb{R}^N$, it follows from (1.3) that there is $\delta'>0$ such that $u_0(x)\le\delta'\,e^{-\lambda|x|}$ for a.e. $x\in\mathbb{R}^N$. Therefore, with the same notations as in case 1, one infers that

$$u(1,x)\le\delta'\,e^L\int_{\mathbb{R}^N}p(1,x;y)\,e^{-\lambda|y|}\,dy\le K\,\delta'\,e^L\int_{\mathbb{R}^N}e^{-|x-y|^2/K-\lambda|y|}\,dy\ \text{ for all } x\in\mathbb{R}^N.$$

Hence,

$$u(1,x) \le K\,\delta'\,e^L \int_{\mathbb{R}^N} e^{-|y|^2/K-\lambda|x-y|}\,dy \le K\,\delta'\,e^L\,e^{-\lambda|x|}\int_{\mathbb{R}^N} e^{-|y|^2/K+\lambda|y|}\,dy$$

for all $x \in \mathbb{R}^N$ and, since $u(1,\cdot)$ is continuous and less than 1 in $\mathbb{R}^N$, there are then $\theta' \in (0,1)$ and $\omega' > 0$ such that

$$u(1,x) \le \min\big(\theta', \omega'\,e^{-\lambda|x|}\big) \ \text{ for all } x \in \mathbb{R}^N. \tag{2.7}$$

On the other hand, assumption (1.3) yields the existence of $R > 0$ such that $u_0(x) \ge \gamma\,e^{-\lambda|x|}$ for all $|x| \ge R$. It follows then from (2.4) and the nonnegativity of f and u_0 that, for all $t \ge 0$ and $x \in \mathbb{R}^N$,

$$\begin{aligned} u(1+t,x) &\ge \int_{\mathbb{R}^N} p(1+t,x;y)\,u_0(y)\,dy \\ &\ge \frac{\gamma}{K\,(1+t)^{N/2}}\int_{\mathbb{R}^N\setminus B(0,R)} e^{-K|x-y|^2/(1+t)-\lambda|y|}\,dy \\ &= \frac{\gamma}{K}\int_{\{z\in\mathbb{R}^N;\,|x-\sqrt{1+t}\,z|\ge R\}} e^{-K|z|^2-\lambda|x-\sqrt{1+t}\,z|}\,dz. \end{aligned} \tag{2.8}$$

Assume now by contradiction that property (2.1) does not hold for any $T > 0$. Then there exist a sequence $(T_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that $T_n \to +\infty$ as $n \to +\infty$ and $u(1+T_n,x_n) < u(1,x_n)$ for all $n \in \mathbb{N}$. Since $u(1,\cdot) \le \theta' < 1$ in $\mathbb{R}^N$ and $\min_{|x|\le ct} u(t,x) \to 1$ as $t \to +\infty$ with $c > 0$ by (1.11), it follows that $|x_n| \ge c(1+T_n)$ for n large enough, while $u(1+T_n,x_n) < u(1,x_n)$ and (2.7)-(2.8) yield

$$\omega'\,e^{-\lambda|x_n|} > \frac{\gamma}{K}\int_{\{z\in\mathbb{R}^N;\,|x_n-\sqrt{1+T_n}\,z|\ge R\}} e^{-K|z|^2-\lambda|x_n-\sqrt{1+T_n}\,z|}\,dz \ \text{ for all } n \in \mathbb{N}.$$

Since $\liminf_{n\to+\infty}|x_n|/T_n \ge c > 0$, one has

$$B(x_n/|x_n|,1/2) \subset \{z\in\mathbb{R}^N;\,|x_n-\sqrt{1+T_n}\,z|\ge R\}$$

for n large enough, whence

$$\begin{aligned} \omega'\,e^{-\lambda|x_n|} &> \frac{\gamma}{K}\int_{B(x_n/|x_n|,1/2)} e^{-K|z|^2-\lambda|x_n-\sqrt{1+T_n}\,z|}\,dz \\ &\ge \frac{\gamma\,e^{-9K/4}}{K}\int_{B(0,1/2)} e^{-\lambda|x_n-\sqrt{1+T_n}\,(x_n/|x_n|+y)|}\,dy \end{aligned}$$

for n large enough. For n large enough so that $\sqrt{1+T_n} \le |x_n|$, it follows that, for all $y \in B(0,1/2)$,

$$\Big|x_n-\sqrt{1+T_n}\,\Big(\frac{x_n}{|x_n|}+y\Big)\Big| \le |x_n|\Big(1-\frac{\sqrt{1+T_n}}{|x_n|}\Big)+\frac{\sqrt{1+T_n}}{2} = |x_n|-\frac{\sqrt{1+T_n}}{2},$$

whence

$$\omega'\,e^{-\lambda|x_n|} > \frac{\gamma\,e^{-9K/4}\,e^{-\lambda|x_n|+\lambda\sqrt{1+T_n}/2}}{K}\int_{B(0,1/2)} dy$$

for n large enough. This leads to a contradiction since $T_n \to +\infty$ as $n \to +\infty$.

As a conclusion, (2.1) holds when (1.3) is fulfilled and the proof of Lemma 2.1 is thereby complete. □

From Lemma 2.1 and the maximum principle, the following corollary immediately holds.

Corollary 2.2. *For every $t \geq 1$, $T' \geq T$ and $x \in \mathbb{R}^N$, one has $u(t+T',x) \geq u(t,x)$.*

Remark 2.3. Notice from the proof of Lemma 2.1 that time 1 could be replaced by any positive time t_0 in the statement: namely, for any $t_0 > 0$, there exists $T_0 > 0$ such that $u(t_0+t,x) \geq u(t_0,x)$ for all $t \geq T_0$ and $x \in \mathbb{R}^N$. However, this property does not hold in general with $t_0 = 0$. Indeed, if $0 \leq u_0 \leq 1$ is continuous and $\max_{\mathbb{R}^N} u_0 = 1$, then u_0 can never be bounded from above in $\mathbb{R}^N$ by $u(t,\cdot)$ for any $t > 0$, since $u(t,x) < 1$ for all $t > 0$ and $x \in \mathbb{R}^N$ by the strong parabolic maximum principle.

Remark 2.4. The assumptions (1.2) or (1.3) were crucially used in the proof of Lemma 2.1, in order to trap $u(1,x)$ between two comparable functions as $|x| \to +\infty$, the lower one giving rise to a solution which, after some time, is above the upper one at time 1. The conclusion of Lemma 2.1 may not hold for more general initial conditions u_0, for instance if $\gamma\, e^{-\lambda_1|x|} \leq u_0(x) \leq \delta\, e^{-\lambda_2|x|}$ for $|x|$ large enough, with $\gamma,\, \delta > 0$, $0 < \lambda_2 < \lambda_1$, $\liminf_{|x|\to+\infty} u_0(x)\, e^{\lambda_1|x|} < +\infty$ and $\limsup_{|x|\to+\infty} u_0(x)\, e^{\lambda_2|x|} > 0$. For such initial conditions, more complex dynamics may occur in general, even for homogeneous one-dimensional equations, see e.g. [12, 18].

3. Improved monotonicity when $u(t,x)$ is away from 0 and 1

In this section, we improve the T-monotonicity result stated in Corollary 2.2, for the points (t,x) such that $0 < a \leq u(t,x) \leq b < 1$, where $0 < a \leq b < 1$ are given. To do so, let us first define

$$\tau_* = \inf\big\{\tau > 0;\ \exists\, t_0 \geq 0,\ \forall\, \tau' \geq \tau,\ \forall\, t \geq t_0,\ \forall\, x \in \mathbb{R}^N,\ u(t+\tau',x) \geq u(t,x)\big\}. \quad (3.1)$$

It follows from Corollary 2.2 that $0 \leq \tau_* \leq T < +\infty$. Our goal is to show that $\tau_* = 0$ (this goal will be achieved at the beginning of Section 7).

Lemma 3.1. *Let a and b be any two real numbers such that $0 < a \leq b < 1$ and let τ be any real number such that $\tau \geq \tau_*$ and $\tau > 0$. Then,*

$$\liminf_{t\to+\infty}\ \inf_{a\leq u(t,x)\leq b} \frac{u(t+\tau,x)}{u(t,x)} > 1,$$

that is, there exist $t_0 > 0$ and $\delta > 0$ such that, for all $(t,x) \in [t_0,+\infty) \times \mathbb{R}^N$ with $a \leq u(t,x) \leq b$, there holds $u(t+\tau,x) \geq (1+\delta)\, u(t,x)$.

Proof. The proof shall use the definition of τ_* and the positivity of τ together with the spreading properties of solutions of equations obtained as finite or infinite spatial shifts of (1.1). We argue by contradiction. So, assume that the conclusion of Lemma 3.1 does not hold. Then there are two sequences $(t_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that $\delta_n \to 0$ as $n \to +\infty$, $t_n \to +\infty$ as $n \to +\infty$ and

$$a \le u(t_n, x_n) \le b \ \text{ and } \ u(t_n + \tau, x_n) < (1 + \delta_n)\, u(t_n, x_n) \ \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Shift the origin at the points (t_n, x_n) and define

$$u_n(t, x) = u(t + t_n, x + x_n).$$

The functions u_n are classical solutions of

$$(u_n)_t = \operatorname{div}(A(x + x_n)\nabla u_n) + f(x + x_n, u_n), \ \ t > -t_n, \ x \in \mathbb{R}^N \tag{3.3}$$

with $0 < u_n(t, x) < 1$ for all $(t, x) \in (-t_n, +\infty) \times \mathbb{R}^N$. From Arzela-Ascoli theorem, up to extraction of a subsequence, the functions $\mathbb{R}^N \times [0, 1] \ni (x, s) \mapsto f(x + x_n, s)$ converge locally uniformly in $\mathbb{R}^N \times [0, 1]$ to a continuous function $f_\infty : \mathbb{R}^N \times [0, 1]$ which actually shares with f the following properties: $f_\infty(\cdot, 0) = f_\infty(\cdot, 1) = 0$, $f_\infty(x, 1 - u)/u$ is nonincreasing in $u \in (0, 1]$, and f_∞ satisfies (1.8), whence $\inf_{x\in\mathbb{R}^N} f_\infty(x, s) > 0$ for every $s \in (0, 1)$). Furthermore, up to extraction of another subsequence, the matrix fields $x \mapsto A(x + x_n)$ converge in $C^1_{loc}(\mathbb{R}^N)$ to a uniformly definite positive symmetric matrix field A_∞.[3] Lastly, from standard parabolic estimates, the functions u_n converge locally uniformly in $C^{1;2}_{t,x}(\mathbb{R} \times \mathbb{R}^N)$, up to extraction of another subsequence, to a classical solution u_∞ of

$$(u_\infty)_t = \operatorname{div}(A_\infty \nabla u_\infty) + f_\infty(x, u_\infty), \ \ t \in \mathbb{R}, \ x \in \mathbb{R}^N, \tag{3.4}$$

such that $0 \le u_\infty(t, x) \le 1$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

Now, for any $\varepsilon > 0$, it follows from $\tau \ge \tau_*$ and from the definition of τ_* in (3.1) that there is $T_0 > 0$ such that

$$u(t + \tau + \varepsilon, x) \ge u(t, x) \ \text{ for all } (t, x) \in [T_0, +\infty) \times \mathbb{R}^N$$

(actually, if $\tau > \tau_*$, then one can also take $\varepsilon = 0$). In particular, since $t_n \to +\infty$ as $n \to +\infty$, one infers that $u_\infty(t + \tau + \varepsilon, x) \ge u_\infty(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. Since $\varepsilon > 0$ can be arbitrary, one gets that

$$u_\infty(t + \tau, x) \ge u_\infty(t, x) \ \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

On the other hand, the inequalities (3.2) and $\lim_{n\to+\infty} \delta_n = 0$ imply that $a \le u_\infty(0, 0) \le b$ and $u_\infty(\tau, 0) \le u_\infty(0, 0)$, whence $u_\infty(\tau, 0) = u_\infty(0, 0)$. As a consequence, the bounded functions $u_\infty(\cdot + \tau, \cdot)$ and $u_\infty(\cdot, \cdot)$ are ordered in $\mathbb{R} \times \mathbb{R}^N$ and are equal at $(0, 0)$. It follows from the strong maximum principle that

$$u_\infty(t + \tau, x) = u_\infty(t, x)$$

for all $(t, x) \in (-\infty, 0] \times \mathbb{R}^N$, and then for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ from the uniqueness of the Cauchy problem associated with (3.4). Furthermore, $0 < a \le u_\infty(0, 0) \le b < 1$

[3]As a matter of fact, since $u(t_n, x_n) \le b < 1$ and $t_n \to +\infty$, then $|x_n| \to +\infty$ by (1.11), whence A_∞ is a constant matrix due to (1.5). However, the fact that A_∞ is constant is not used in the proof of the present lemma.

and $0 \le u_\infty \le 1$ in $\mathbb{R}\times\mathbb{R}^N$, whence $0 < u_\infty < 1$ in $\mathbb{R}\times\mathbb{R}^N$ from the strong maximum principle. Lastly, $u_\infty(t,x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$, as recalled in Section 1 for u and f, from the properties shared by f_∞ with f. Thus, the limit $\mathbb{N} \ni m \to +\infty$ in $u_\infty(m\tau, 0) = u_\infty(0,0) \le b < 1$ leads to a contradiction, since $\tau > 0$ by assumption. The proof of Lemma 3.1 is thereby complete. □

From Lemma 3.1 and the uniform continuity of u in, say, $[1,+\infty)\times\mathbb{R}^N$, the inequalities stated in Lemma 3.1 hold uniformly for some time-shifts in a neighborhood of τ_* if τ_* is positive, as the following corollary shows.

Corollary 3.2. *Let a and b be any two real numbers such that $0 < a \le b < 1$. If one assumes that $\tau^* > 0$, then there exist $t_0 > 0$, $\delta > 0$ and $0 < \underline{\tau} < \tau_* < \overline{\tau}$ such that, for all $\tau \in [\underline{\tau}, \overline{\tau}]$ and $(t,x) \in [t_0,+\infty)\times\mathbb{R}^N$ with $a \le u(t,x) \le b$, then $u(t+\tau,x) \ge (1+\delta)\, u(t,x)$.*

Proof. From Lemma 3.1 applied with $\tau = \tau_*$, there are $t_0 > 0$ and $\delta > 0$ such that $u(t+\tau_*,x) \ge (1+2\delta)\, u(t,x)$ for all $(t,x) \in [t_0,+\infty)\times\mathbb{R}^N$ with $a \le u(t,x) \le b$. Choose $\varepsilon \in (0,1)$ so that $(1-\varepsilon)(1+2\delta) \ge 1+\delta$. Since u is uniformly continuous in $[t_0,+\infty)\times\mathbb{R}^N$ from standard parabolic estimates, there exist some real numbers $\underline{\tau}$ and $\overline{\tau}$ such that $0 < \underline{\tau} < \tau_* < \overline{\tau}$ and

$$|u(t+\tau,x) - u(t+\tau_*,x)| \le \varepsilon\,(1+2\delta)\, a \ \text{ for all } \tau \in [\underline{\tau},\overline{\tau}] \text{ and for all } (t,x) \in [t_0,+\infty)\times\mathbb{R}^N.$$

Fix now any $\tau \in [\underline{\tau},\overline{\tau}]$ and any $(t,x) \in [t_0,+\infty)\times\mathbb{R}^N$ with $a \le u(t,x) \le b$. One has $u(t+\tau_*,x) \ge (1+2\delta)\, u(t,x) \ge (1+2\delta)\, a$, whence

$$\begin{aligned} u(t+\tau,x) \ge u(t+\tau_*,x) - \varepsilon\,(1+2\delta)\, a \ &\ge\ (1-\varepsilon)\, u(t+\tau_*,x) \\ &\ge\ (1-\varepsilon)\,(1+2\delta)\, u(t,x) \ge (1+\delta)\, u(t,x). \end{aligned}$$

This is the desired result and the proof is thereby complete. □

4. Improved monotonicity when $u(t,x)$ is away from 0

In this section, by using especially the fact that $f(x,1-u)/u$ is nonincreasing with respect to $u \in (0,1]$ for every $x \in \mathbb{R}^N$, we improve the τ-monotonicity of u (with $\tau > \tau_*$) in the region where $u(t,x)$ is close to 1 (we recall that $0 < u(t,x) < 1$ for all $(t,x) \in (0,+\infty)\times\mathbb{R}^N$). Namely, we will prove the following lemma.

Lemma 4.1. *Let a and τ be any real numbers such that $0 < a < 1$ and $\tau > \tau_*$. Then,*

$$\limsup_{t\to+\infty,\ u(t,x)\ge a} \frac{1-u(t+\tau,x)}{1-u(t,x)} < 1,$$

that is, there exist $t_0 > 0$ and $\delta > 0$ such that, for all $(t,x) \in [t_0,+\infty)\times\mathbb{R}^N$ with $u(t,x) \ge a$, there holds $1-u(t+\tau,x) \le (1-\delta)\,(1-u(t,x))$.

Proof. First of all, since $\tau > \tau_*$, it follows from the definition of τ_* that there is $T_0 > 0$ such that

$$u(t+\tau,x) \ge u(t,x) \text{ for all } (t,x) \in [T_0,+\infty)\times\mathbb{R}^N. \tag{4.1}$$

Notice that the strong maximum principle then yields $u(t+\tau,x) > u(t,x)$ in $(T_0,+\infty) \times \mathbb{R}^N$ (otherwise, one would have $u(t+\tau,x) = u(t,x)$ in $[T_0,T_1] \times \mathbb{R}^N$ with some $T_1 > T_0$, whence $u(t+\tau,x) = u(t,x)$ in $[T_0,+\infty) \times \mathbb{R}^N$ and $u(T_0+m\tau,0) = u(T_0,0) < 1$ for all $m \in \mathbb{N}$, whereas $u(t,0) \to 1$ as $t \to +\infty$. Even if it means increasing T_0, one can then assume without loss of generality that

$$u(t+\tau,x) > u(t,x) \text{ for all } (t,x) \in [T_0,+\infty) \times \mathbb{R}^N, \quad u(T_0,0) \geq a \;\text{ and }\; T_0 > \tau.$$

Define now, for every $k \in \mathbb{N} = \{0,1,2,\cdots\}$,

$$E_k = \big\{x \in \mathbb{R}^N;\, \exists\, t \in [T_0+k\tau, T_0+(k+1)\tau],\, u(t,x) \geq a\big\}.$$

The set E_0 is not empty since $u(T_0,0) \geq a$. As a consequence,

$$u(T_0+k\tau,0) \geq u(T_0+(k-1)\tau,0) \geq \cdots \geq u(T_0,0) \geq a,$$

whence $0 \in E_k$ for every $k \in \mathbb{N}$. Thanks to (4.1), the same argument implies that $E_k \subset E_{k+1}$ for every $k \in \mathbb{N}$. Furthermore, each set E_k is closed by continuity of u in $[T_0,+\infty) \times \mathbb{R}^N$. Lastly, as done for the proof of (2.5) and (2.7) in Lemma 2.1, one easily infers that $u(t,x) \to 0$ as $|x| \to +\infty$ locally uniformly in $t > 0$, whence each set E_k is bounded. Therefore, the sets E_k are a non-decreasing sequence of non-empty compact subsets of $\mathbb{R}^N$.

We are going to apply the maximum principle to the functions $1 - u(t+\tau,x)$ and $1-u(t,x)$ in the sets $[T_0+k\tau, T_0+(k+1)\tau] \times E_k$ by induction with respect to k, in order to improve quantitatively the inequality $1-u(t+\tau,x) \leq 1-u(t,x)$ in $[T_0+k\tau, T_0+(k+1)\tau] \times E_k$.

To do so, we first claim that the function u is bounded from below by a positive constant uniformly in the sets $[T_0+k\tau, T_0+(k+1)\tau] \times E_k$, that is, there is $\underline{a} \in (0,a]$ such that

$$\forall\, k \in \mathbb{N},\ \forall\, (t,x) \in [T_0+k\tau, T_0+(k+1)\tau] \times E_k,\ u(t,x) \geq \underline{a} > 0. \tag{4.2}$$

Indeed, otherwise, there exist a sequence $(k_n)_{n\in\mathbb{N}}$ of integers and, for each $n \in \mathbb{N}$, a time $t_n \in [T_0+k_n\tau, T_0+(k_n+1)\tau]$ and a point $x_n \in E_{k_n}$, with $u(t_n,x_n) \to 0$ as $n \to +\infty$. For each $n \in \mathbb{N}$, since $x_n \in E_{k_n}$, there is a time $t'_n \in [T_0+k_n\tau, T_0+(k_n+1)\tau]$ such that $u(t'_n,x_n) \geq a$. Consider the functions

$$(t,x) \mapsto u_n(t,x) = u(t+t_n, x+x_n),$$

which are defined in $(-t_n,+\infty) \times \mathbb{R}^N \supset (-T_0,+\infty) \times \mathbb{R}^N$ and solve (3.3), together with $0 \leq u_n \leq 1$. From Arzela-Ascoli theorem and standard parabolic estimates, up to extraction of a subsequence, these functions u_n converge locally uniformly in $C^{1,2}_{t,x}((-T_0,+\infty) \times \mathbb{R}^N)$ to a solution $0 \leq u_\infty \leq 1$ of an equation of the type (3.4) in $(-T_0,+\infty) \times \mathbb{R}^N$ (notice that the sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ may not be unbounded and the limiting equation satisfied by u_∞ may just be a finite spatial shift of (1.1)). Anyway, $u_n(0,0) = u(t_n,x_n) \to 0$ as $n \to +\infty$, whence $u_\infty(0,0) = 0$. Therefore, $u_\infty = 0$ in $(-T_0,0] \times \mathbb{R}^N$ from the strong maximum principle, and $u_\infty = 0$ in $(-T_0,+\infty) \times \mathbb{R}^N$ from the uniqueness of the Cauchy problem associated with (3.4). On the other hand, $|t'_n - t_n| \leq \tau < T_0$ for every $n \in \mathbb{N}$. Up to extraction of another subsequence, one can assume that $t'_n - t_n \to t'_\infty > -T_0$ as $n \to +\infty$. Since $u_n(t'_n - t_n, 0) = u(t'_n, x_n) \geq a$, one gets $u_\infty(t'_\infty, 0) \geq a > 0$, which leads to a contradiction. As a consequence, the claim (4.2) is proved.

The second claim is concerned with an upper bound of the values of u on the boundaries ∂E_k of the sets E_k, on the time intervals $[T_0 + k\tau, T_0 + (k+1)\tau]$. Namely, we claim that there is a real number $b \in (0,1)$ such that

$$\forall\, k \in \mathbb{N},\ \forall\,(t,x) \in [T_0 + k\tau, T_0 + (k+1)\tau] \times \partial E_k,\ u(t,x) \leq b < 1. \tag{4.3}$$

Assume not. Then, there exist a sequence $(k_n)_{n\in\mathbb{N}}$ of integers and, for each $n \in \mathbb{N}$, a time $t_n \in [T_0 + k_n\tau, T_0 + (k_n+1)\tau]$ and a point $x_n \in \partial E_{k_n}$, with $u(t_n, x_n) \to 1$ as $n \to +\infty$. For each $n \in \mathbb{N}$, since $x_n \in \partial E_{k_n} \subset E_{k_n}$ and since $u(\cdot, x_n)$ is continuous on $[T_0 + k_n\tau, T_0 + (k_n+1)\tau]$, the definition of E_{k_n} yields

$$\max_{[T_0+k_n\tau, T_0+(k_n+1)\tau]} u(\cdot, x_n) \geq a.$$

Furthermore, if $\min_{[T_0+k_n\tau, T_0+(k_n+1)\tau]} u(\cdot, x_n) > a$, then by uniform continuity of u in $[T_0, +\infty) \times \mathbb{R}^N$ one would have $\min_{[T_0+k_n\tau, T_0+(k_n+1)\tau]} u(\cdot, x) > a$ for all x in a neighborhood of x_n and x_n would then be an interior point of E_{k_n}. Therefore,

$$\min_{[T_0+k_n\tau, T_0+(k_n+1)\tau]} u(\cdot, x_n) \leq a$$

and there is a time $t'_n \in [T_0 + k_n\tau, T_0 + (k_n+1)\tau]$ such that

$$u(t'_n, x_n) = a.$$

Now, as in the previous paragraph, the functions $(t,x) \mapsto u_n(t,x) = u(t + t_n, x + x_n)$ converge, up to extraction of a subsequence, locally uniformly in $C^{1,2}_{t,x}((-T_0, +\infty) \times \mathbb{R}^N)$ to a solution $0 \leq u_\infty \leq 1$ of an equation of the type (3.4) in $(-T_0, +\infty) \times \mathbb{R}^N$. One has $u_\infty(0,0) = 1$, whence $u_\infty = 1$ in $(-T_0, 0] \times \mathbb{R}^N$ and then in $(-T_0, +\infty) \times \mathbb{R}^N$. On the other hand, up to extraction of another subsequence, there holds $\lim_{n\to+\infty}(t'_n - t_n) = t'_\infty \in [-\tau, \tau] \subset (-T_0, +\infty)$ and $u_\infty(t'_\infty, 0) = a < 1$. One has reached a contradiction, and the claim (4.3) follows.

Similarly, we claim that there is a real number $\bar{b} \in (0,1)$ such that

$$\forall\, k \in \mathbb{N},\ \forall\, x \in E_{k+1} \backslash E_k,\ u(T_0 + (k+1)\tau, x) \leq \bar{b} < 1. \tag{4.4}$$

Otherwise, there exist a sequence $(k_n)_{n\in\mathbb{N}}$ of integers and, for each $n \in \mathbb{N}$, a point $x_n \in E_{k_n+1} \backslash E_{k_n}$, with $u(T_0 + (k_n+1)\tau, x_n) \to 1$ as $n \to +\infty$. For each $n \in \mathbb{N}$, since $x_n \in E_{k_n+1} \backslash E_{k_n}$, there holds

$$\max_{[T_0+(k_n+1)\tau, T_0+(k_n+2)\tau]} u(\cdot, x_n) \geq a \ \text{ and } \ \max_{[T_0+k_n\tau, T_0+(k_n+1)\tau]} u(\cdot, x_n) < a,$$

whence there is a time $t_n \in [T_0 + (k_n+1)\tau, T_0 + (k_n+2)\tau]$ such that $u(t_n, x_n) = a$. Up to extraction of a subsequence, the functions

$$(t,x) \mapsto u_n(t,x) = u(t + T_0 + (k_n+1)\tau, x + x_n)$$

converge locally uniformly in $C^{1,2}_{t,x}((-T_0-\tau,+\infty)\times\mathbb{R}^N)$ to a solution $0\le u_\infty\le 1$ of an equation of the type (3.4) in $(-T_0-\tau,+\infty)\times\mathbb{R}^N$. One has $u_\infty(0,0)=1$, whence $u_\infty=1$ in $(-T_0-\tau,0]\times\mathbb{R}^N$ and then in $(-T_0-\tau,+\infty)\times\mathbb{R}^N$. On the other hand, up to extraction of another subsequence, there holds $\lim_{n\to+\infty}(t_n-(T_0+(k_n+1)\tau))=t_\infty\in[0,\tau]\subset(-T_0-\tau,+\infty)$ and $u_\infty(t_\infty,0)=a<1$. One has reached a contradiction, and the claim (4.4) is proved.

Putting together (4.2), (4.3) and (4.4), one gets that

$$\left\{\begin{array}{ll}\forall\,k\in\mathbb{N},\ \forall\,(t,x)\in[T_0+k\tau,T_0+(k+1)\tau]\times\partial E_k, & 0<\underline{a}\le u(t,x)\le b<1,\\ \forall\,k\in\mathbb{N},\ \forall\,x\in E_{k+1}\backslash E_k, & 0<\underline{a}\le u(T_0+(k+1)\tau,x)\le\overline{b}<1.\end{array}\right.$$

It follows then from Lemma 3.1 applied once with $(\underline{a},b,\tau)$ and another time with $(\underline{a},\overline{b},\tau)$ (notice that $\tau\ge\tau_*$ and $\tau>0$ since here $\tau>\tau_*$) that there are $k_0\in\mathbb{N}$ and $\delta_0\in(0,+\infty)$ such that, for all $k\ge k_0$,

$$\left\{\begin{array}{ll}\forall\,(t,x)\in[T_0+k\tau,T_0+(k+1)\tau]\times\partial E_k, & u(t+\tau,x)\ge(1+\delta_0)\,u(t,x),\\ \forall\,x\in E_{k+1}\backslash E_k, & u(T_0+(k+2)\tau,x)\ge(1+\delta_0)\,u(T_0+(k+1)\tau,x).\end{array}\right.$$

Define

$$\delta=\delta_0\underline{a}>0.$$

One infers that

$$\begin{array}{l}\forall\,k\ge k_0,\ \forall\,(t,x)\in[T_0+k\tau,T_0+(k+1)\tau]\times\partial E_k,\\ \quad 1-u(t+\tau,x)\le 1-(1+\delta_0)\,u(t,x)\ \le\ 1-u(t,x)-\delta_0\underline{a}\\ \qquad\qquad\qquad\qquad\qquad\qquad\quad =\ 1-u(t,x)-\delta\le(1-\delta)\,(1-u(t,x)),\end{array}\tag{4.5}$$

and, by arguing similarly with $x\in E_{k+1}\backslash E_k$, that

$$\forall\,k\ge k_0,\ \forall\,x\in E_{k+1}\backslash E_k,\ \ 1-u(T_0+(k+2)\tau,x)\le(1-\delta)\,(1-u(T_0+(k+1)\tau,x)).\tag{4.6}$$

On the other hand, since

$$1>u(t+\tau,x)>u(t,x)>0$$

for all $(t,x)\in[T_0+k_0\tau,T_0+(k_0+1)\tau]\times E_{k_0}$ and since both functions $u(\cdot+\tau,\cdot)$ and u are continuous on this compact set $[T_0+k_0\tau,T_0+(k_0+1)\tau]\times E_{k_0}$, if follows that, even if it means decreasing $\delta>0$,

$$\forall\,(t,x)\in[T_0+k_0\tau,T_0+(k_0+1)\tau]\times E_{k_0},\ \ 1-u(t+\tau,x)\le(1-\delta)\,(1-u(t,x)).\tag{4.7}$$

Finally, we claim by induction on k that

$$\forall\,k\ge k_0,\ \forall\,(t,x)\in[T_0+k\tau,T_0+(k+1)\tau]\times E_k,\ \ 1-u(t+\tau,x)\le(1-\delta)\,(1-u(t,x)).\tag{4.8}$$

First of all, the property is true at $k=k_0$, by (4.7). Assume now that the property is satisfied for some $k\in\mathbb{N}$ with $k\ge k_0$. In particular, by choosing $t=T_0+(k+1)\tau$, there holds

$$\forall\,x\in E_k,\ \ 1-u(t_0+(k+2)\tau,x)\le(1-\delta)\,(1-u(t_0+(k+1)\tau,x)).$$

This last inequality also holds for all $x \in E_{k+1} \backslash E_k$, by (4.6). Therefore,

$$\forall\, x \in E_{k+1}, \quad 1 - u(t_0 + (k+2)\tau, x) \le (1-\delta)\,(1 - u(t_0 + (k+1)\tau, x)). \tag{4.9}$$

Furthermore, property (4.5) yields

$$\forall\, (t,x) \in [t_0+(k+1)\tau, t_0+(k+2)\tau] \times \partial E_{k+1}, \quad 1-u(t+\tau,x) \le (1-\delta)\,(1-u(t,x)). \tag{4.10}$$

Consider the functions

$$v(t,x) = 1 - u(t+\tau, x) \ \text{ and } \ \overline{v}(t,x) = (1-\delta)\,(1-u(t,x))$$

in the compact set

$$Q_k = [t_0 + (k+1)\tau, t_0 + (k+2)\tau] \times E_{k+1}.$$

The inequalities (4.9) and (4.10) mean that

$$v(t,x) \le \overline{v}(t,x) \ \text{ for all } (t,x) \in \{t_0+(k+1)\tau\} \times E_{k+1} \cup [t_0+(k+1)\tau, t_0+(k+2)\tau] \times \partial E_{k+1},$$

namely $v \le \overline{v}$ on the parabolic boundary of Q_k. Let us now check that $\overline{v}$ is a supersolution of the equation satisfied by v. On the one hand, the function v satisfies $0 \le v \le 1$ and obeys

$$v_t = \mathrm{div}(A(x)\nabla v) + g(x,v) \ \text{ in } Q_k,$$

where g is defined by $g(x,s) = -f(x, 1-s)$ for all $(x,s) \in \mathbb{R}^N \times [0,1]$. On the other hand, the function $\overline{v}$ satisfies $0 \le \overline{v} \le 1$ in Q_k and

$$\begin{array}{rcl} \overline{v}_t - \mathrm{div}(A(x)\nabla\overline{v}) - g(x,\overline{v}) & = & -(1-\delta)\,u_t + (1-\delta)\,\mathrm{div}(A(x)\nabla u) - g(x,\overline{v}) \\ & = & -(1-\delta)\,f(x,u) - g(x,\overline{v}) \\ & = & (1-\delta)\,g(x,1-u) - g(x,(1-\delta)\,(1-u)). \end{array}$$

But the function $g(x,s)/s$ is nondecreasing with respect to $s \in (0,1]$, since by assumption the function $f(x,1-s)/s$ is nonincreasing with respect to $s \in (0,1]$. Hence,

$$g(x,(1-\delta)\,(1-u(t,x))) \le (1-\delta)\,g(x,1-u(t,x)) \ \text{ in } Q_k$$

and

$$\overline{v}_t - \mathrm{div}(A(x)\nabla\overline{v}) - g(x,\overline{v}) \ge 0 \ \text{ in } Q_k.$$

The parabolic maximum principle then implies that $v \le \overline{v}$ in Q_k. This means that property (4.8) is satisfied with $k+1$ and finally that it holds by induction for all $k \ge k_0$.

As a conclusion, set $t_0 = T_0 + k_0\tau$ and consider any $(t,x) \in [t_0, +\infty) \times \mathbb{R}^N$ such that $u(t,x) \ge a$. Let $k \in \mathbb{N}$, $k \ge k_0$ be such that $T_0 + k\tau \le t \le T_0 + (k+1)\tau$. Thus, $x \in E_k$ and property (4.8) yields

$$1 - u(t+\tau, x) \le (1-\delta)\,(1-u(t,x)).$$

The proof of Lemma 4.1 is thereby complete. □

5. Monotonicity in time when $u(t,x)$ is close to 1

In this section, based on Lemma 4.1, we will show that u is actually increasing in time at large time when it is close to 1.

Lemma 5.1. *There exist $b \in (0,1)$ and $\widetilde{T} > 0$ such that, for all $(t,x) \in [\widetilde{T}, +\infty) \times \mathbb{R}^N$ with $u(t,x) \geq b$, there holds $u_t(t,x) > 0$.*

Proof. As in the proof of Lemma 4.1, denote $v = 1 - u$. The function v satisfies $0 < v < 1$ in $(0,+\infty) \times \mathbb{R}^N$ and

$$v_t = \operatorname{div}(A(x)\nabla v) + g(x,v), \quad t > 0, \ x \in \mathbb{R}^N$$

with $g(x,s) = -f(x, 1-s)$. Furthermore, by choosing, say, $\tau = \tau_* + 1$, it follows from definition (3.1) that there is $t_0 > 1$ such that, for all $(t,x) \in [t_0, +\infty) \times \mathbb{R}^N$, $1 > u(t+\tau, x) \geq u(t,x)$, that is,

$$0 < v(t+\tau, x) \leq v(t,x) \ \text{ in } [t_0, +\infty) \times \mathbb{R}^N. \tag{5.1}$$

From standard parabolic estimates and Harnack inequality, there are some positive constants C_1 and C_2 such that

$$\forall\, (t,x) \in [t_0,+\infty) \times \mathbb{R}^N, \ |v_t(t,x)| + |\nabla v(t,x)| \leq C_1 \max_{[t-1,t]\times \overline{B(x,1)}} v \leq C_2\, v(t+\tau, x).$$

Together with (5.1), it follows that the fields v_t/v and $\nabla v/v$ are bounded in $[t_0,+\infty)\times\mathbb{R}^N$. Define now

$$M = \limsup_{t\to+\infty,\, v(t,x)\to 0} \frac{v_t(t,x)}{v(t,x)}. \tag{5.2}$$

From the previous observations and the fact that $v(t,x) = 1 - u(t,x) \to 0$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$, one infers that M is a real number. To complete the proof of Lemma 5.1, it will actually be sufficient to show that $M < 0$.

To do so, owing to the definition of M, pick a sequence of points $(t_n, x_n)_{n\in\mathbb{N}}$ in $[t_0, +\infty) \times \mathbb{R}^N$ such that

$$t_n \to +\infty, \quad v(t_n, x_n) \to 0 \ \text{ and } \ \frac{v_t(t_n,x_n)}{v(t_n,x_n)} \to M \ \text{ as } n \to +\infty.$$

Define

$$v_n(t,x) = \frac{v(t+t_n, x+x_n)}{v(t_n,x_n)} > 0 \ \text{ in } (-t_n, +\infty) \times \mathbb{R}^N.$$

Since the fields v_t/v and $\nabla v/v$ are bounded in $[t_0,+\infty)\times\mathbb{R}^N$, one infers that the functions v_n are bounded locally in $\mathbb{R}\times\mathbb{R}^N$, in the sense that, for any compact subset K of $\mathbb{R}\times\mathbb{R}^N$, there is $n_K \in \mathbb{N}$ such that v_n is well defined in K for every $n \geq n_K$ and $\sup_{n\geq n_K} \|v_n\|_{L^\infty(K)} < +\infty$. Furthermore, the functions v_n obey

$$(v_n)_t(t,x) = \operatorname{div}(A(x+x_n)\nabla v_n(t,x)) + \frac{g(x+x_n, v(t_n,x_n)\, v_n(t,x))}{v(t_n,x_n)} \tag{5.3}$$

for $(t,x)\in(-t_n,+\infty)\times\mathbb{R}^N$. Remember now that $f(\cdot,1)=0$ in $\mathbb{R}^N$, that the function $(x,s)\mapsto f(x,s)$ is Lipschitz continuous with respect to s uniformly in $x\in\mathbb{R}^N$, of class C^1 with respect to s in $\mathbb{R}^N\times[s_1,1]$ for some $s_1\in(0,1)$, and that f_s is uniformly continuous in $\mathbb{R}^N\times[s_1,1]$ and of class $C^{0,\alpha}$ with respect to x uniformly in $s\in[s_1,1]$. Therefore, the function g satisfies the same properties in $\mathbb{R}^N\times[0,1-s_1]$. In particular, the functions

$$(t,x)\mapsto h_n(t,x):=\frac{g(x+x_n,v(t_n,x_n)\,v_n(t,x))}{v(t_n,x_n)}$$

are bounded locally in $\mathbb{R}\times\mathbb{R}^N$ and $\|h_n-g_s(x+x_n,0)\,v_n\|_{L^\infty(K)}\to 0$ as $n\to+\infty$ for any compact set $K\subset\mathbb{R}\times\mathbb{R}^N$, from the mean value theorem. From standard parabolic estimates and Sobolev estimates, it follows that the functions v_n are bounded locally in $W^{1,2,p}_{t,x}(\mathbb{R}\times\mathbb{R}^N)$ and are therefore bounded locally in $C^{0,\alpha}(\mathbb{R}\times\mathbb{R}^N)$. It is then straightforward to check that the functions h_n are actually bounded locally in $C^{0,\alpha}(\mathbb{R}\times\mathbb{R}^N)$. Notice also that, up to extraction of a subsequence, the functions $g_s(\cdot+x_n,0)$ converge locally uniformly in $\mathbb{R}^N$ to a function $a\in C^{0,\alpha}(\mathbb{R}^N)$ and that the matrix fields $A(\cdot+x_n)$ converge locally uniformly in $\mathbb{R}^N$ to a uniformly definite positive symmetric matrix field $A_\infty\in C^{1,\alpha}(\mathbb{R}^N)$. As a consequence, again by standard parabolic estimates, the functions v_n converge, up to extraction of a subsequence, locally uniformly in $C^{1,2}_{t,x}(\mathbb{R}\times\mathbb{R}^N)$, to a nonnegative classical solution v_∞ of

$$(v_\infty)_t=\mathrm{div}(A_\infty(x)\nabla v_\infty)+a(x)\,v_\infty,\quad t\in\mathbb{R},\ x\in\mathbb{R}^N. \tag{5.4}$$

On the other hand, $v_n(0,0)=1$, whence $v_\infty(0,0)=1$ and $v_\infty>0$ in $\mathbb{R}\times\mathbb{R}^N$ from the strong maximum principle. Hence, the functions $(v_n)_t/v_n$ and $\nabla v_n/v_n$ converge locally uniformly in $\mathbb{R}\times\mathbb{R}^N$ to $(v_\infty)_t/v_\infty$ and $\nabla v_\infty/v_\infty$. In particular,

$$\frac{(v_\infty)_t(0,0)}{v_\infty(0,0)}=(v_\infty)_t(0,0)=M.$$

Moreover, since the fields v_t/t and $\nabla v/v$ are bounded in $[t_0,+\infty)\times\mathbb{R}^N$ together with $\lim_{n\to+\infty}t_n=+\infty$ and $\lim_{n\to+\infty}v(t_n,x_n)=0$, it follows that the fields $(v_\infty)_t/v_\infty$ and $\nabla v_\infty/v_\infty$ are bounded in $\mathbb{R}\times\mathbb{R}^N$ and that $v(t+t_n,x+x_n)\to 0$ as $n\to+\infty$ locally uniformly in $\mathbb{R}\times\mathbb{R}^N$. Hence, owing to the definition of M in (5.2), one infers that

$$\frac{(v_\infty)_t(t,x)}{v_\infty(t,x)}\le M\quad\text{for all }(t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

Denote $z=(v_\infty)_t/v_\infty$. Since the coefficients of (5.4) do not depend on t, it follows from standard parabolic estimates and the differentiation of (5.4) with respect to t that z is a classical solution of

$$z_t=\mathrm{div}(A_\infty\nabla z)+2\frac{\nabla v_\infty}{v_\infty}\cdot A_\infty\nabla z \tag{5.5}$$

in $\mathbb{R}\times\mathbb{R}^N$. Furthermore, z and $\nabla v_\infty/v_\infty$ are bounded in $\mathbb{R}\times\mathbb{R}^N$ and $z\le M$ in $\mathbb{R}\times\mathbb{R}^N$ with $z(0,0)=M$. The strong parabolic maximum principle then implies that $z=M$ in $(-\infty,0]\times\mathbb{R}^N$, and hence in $\mathbb{R}\times\mathbb{R}^N$ by uniqueness of the Cauchy problem associated with (5.5). In other words, $(v_\infty)_t/v_\infty=M$ in $\mathbb{R}\times\mathbb{R}^N$. In particular, since $v_\infty(0,0)=1$, one gets that $v_\infty(\tau,0)=e^{M\tau}$ (we recall that $\tau=\tau_*+1$).

Lastly, by Lemma 4.1 applied with $\tau = \tau_* + 1$ and $a = 1/2$, there are $T_0 > 0$ and $\delta > 0$ such that

$$1 - u(t+\tau, x) \le (1-\delta)\,(1 - u(t,x)) \ \text{ for all } (t,x) \in [T_0, +\infty) \times \mathbb{R}^N \text{ with } u(t,x) \ge \frac{1}{2}.$$

Thus, for any given $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, since $t + t_n \ge T_0$ and

$$u(t+t_n, x+x_n) = 1 - v(t+t_n, x+x_n) \ge \frac{1}{2}$$

for all n large enough, one infers that

$$1 - u(t+t_n+\tau, x+x_n) \le (1-\delta)\,(1 - u(t+t_n, x+x_n)),$$

whence $v_n(t+\tau, x) \le (1-\delta)\, v_n(t,x)$ for all n large enough. Thus, $v_\infty(t+\tau, x) \le (1-\delta)\, v_\infty(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Consequently, $e^{M\tau} = v_\infty(\tau, 0) \le (1-\delta) v_\infty(0,0) = 1 - \delta < 1$ and $M < 0$.

As a conclusion, owing to the definition of M in (5.2) and since $v = 1 - u$, the conclusion of Lemma 5.1 follows. □

6. τ-monotonicity in time when $u(t,x)$ is close to 0

In this section, for any arbitrary $\tau > 0$, we show the τ-monotonicity in time at large time in the region where $u(t,x)$ is close to 0. We shall use in particular the assumptions (1.5) and (1.9) on asymptotic homogeneity of the coefficients A and $f_u(\cdot, 0)$. The key step will be the following proposition, which is of independent interest.

Proposition 6.1. *Let $\underline{\nu}$ and $\overline{\nu}$ be two fixed positive real numbers such that $0 < \underline{\nu} \le \overline{\nu}$ and let $\sigma \in (0,1)$ be fixed. Then, there exist $\tau > 0$ and $\eta > 0$ such that, for every $C^1(\mathbb{R}^N)$ symmetric matrix field $a = (a_{ij})_{1 \le i,j \le N} : \mathbb{R}^N \to \mathbb{S}_N(\mathbb{R})$ with $\underline{\nu} I \le a \le \overline{\nu} I$ and $|\nabla a| \le \eta$ in $\mathbb{R}^N$ (where $|\nabla a(x)| = \max_{1 \le i,j \le N} |\nabla a_{ij}(x)|$), the fundamental solution $p(t,x;y)$ of*

$$\begin{cases} p_t(t,x;y) &= \operatorname{div}(a(x) \nabla p(t,x;y)), \quad t > 0, \ x \in \mathbb{R}^N, \\ p(0,\cdot;y) &= \delta_y \end{cases} \tag{6.1}$$

satisfies

$$p(\tau+1, x; 0) \ge \sigma\, p(\tau, x; 0) \quad \text{for all } x \in \mathbb{R}^N. \tag{6.2}$$

Let us postpone the proof of this proposition to Section 8. We continue in this section the proof of Theorem 1.1 for the solution u of (1.1). The main result proved in this section is the following lemma.

Lemma 6.2. *Let θ and θ' be any two real numbers such that $0 < \theta \le \theta'$. Then there exist $T_0 > 0$ and $\varepsilon > 0$ such that*

$$\forall\, \tau \in [\theta, \theta'], \ \forall\, (t,x) \in [T_0, +\infty) \times \mathbb{R}^N, \quad \big(u(t,x) \le \varepsilon\big) \Longrightarrow \big(u(t+\tau, x) > u(t,x)\big).$$

Proof. Let us argue by way of contradiction. So, assume that there exist a sequence $(\tau_n)_{n\in\mathbb{N}}$ in $[\theta,\theta']$, a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that

$$t_n \underset{n\to+\infty}{\longrightarrow} +\infty,\ u(t_n,x_n) \underset{n\to+\infty}{\longrightarrow} 0 \text{ and } u(t_n+\tau_n,x_n)\le u(t_n,x_n) \text{ for all } n\in\mathbb{N}. \tag{6.3}$$

Up to extraction of a subsequence, one can assume without loss of generality that

$$\tau_n \to \tau_\infty \in [\theta,\theta'] \subset (0,+\infty) \ \text{ as } n\to+\infty.$$

Notice that (1.11) and (6.3) yield $\lim_{n\to+\infty}|x_n|=+\infty$ and even $\liminf_{n\to+\infty}|x_n|/t_n \ge c>0$. Therefore, up to extraction of another subsequence, the matrix fields $x\mapsto A(x+x_n)$ converge in $C^1_{loc}(\mathbb{R}^N)$ to a definite positive symmetric matrix field A_∞, which turns out to be a constant matrix due to (1.5). Observe also that (6.3) and the nonnegativity of u imply that $u(t_n+\tau_n,x_n)\to 0$ as $n\to+\infty$.

Define the functions

$$u_n(t,x)=\frac{u(t+t_n+\tau_n,x+x_n)}{u(t_n+\tau_n,x_n)},\ \ (t,x)\in(-t_n-\tau_n,+\infty)\times\mathbb{R}^N.$$

Each function u_n obeys

$$(u_n)_t(t,x)=\mathrm{div}(A(x+x_n)\nabla u_n(t,x))+\frac{f(x+x_n,u(t_n+\tau_n,x_n)\,u_n(t,x))}{u(t_n+\tau_n,x_n)}.$$

For each compact set $K\subset(-\infty,0)\times\mathbb{R}^N$, there is $n_K\in\mathbb{N}$ such that $K\subset(-t_n-\tau_n+1,+\infty)\times\mathbb{R}^N$ for all $n\ge n_K$ and it follows from Harnack inequality applied to u that $\sup_{n\ge n_K}\|u_n\|_{L^\infty(K)}<+\infty$. Remember that $f(\cdot,0)=0$ in $\mathbb{R}^N$, that the function $(x,s)\mapsto f(x,s)$ is Lipschitz continuous with respect to s uniformly in $x\in\mathbb{R}^N$, of class C^1 with respect to s in $\mathbb{R}^N\times[0,s_0]$ with $s_0\in(0,1)$, and that f_s is uniformly continuous in $\mathbb{R}^N\times[0,s_0]$ and of class $C^{0,\alpha}$ with respect to x uniformly in $s\in[0,s_0]$. Furthermore, up to extraction of another subsequence, the functions $x\mapsto f_s(x+x_n,0)$ converge locally uniformly in $\mathbb{R}^N$ to a function $r\in C^{0,\alpha}(\mathbb{R}^N)$, which is actually a constant such that $r\ge\mu>0$ by (1.8) and (1.9). Therefore, as we did in the proof of Lemma 5.1 for the functions v_n satisfying (5.3), we get that, up to extraction of a subsequence, the positive functions u_n converge locally uniformly in $C^{1,2}_{t,x}((-\infty,0)\times\mathbb{R}^N)$ to a nonnegative solution u_∞ of

$$(u_\infty)_t=\mathrm{div}(A_\infty\nabla u_\infty)+r\,u_\infty \ \text{ in } (-\infty,0)\times\mathbb{R}^N.$$

Since $u_n(-\tau_n,0)\ge 1$ by (6.3) and $\tau_n\to\tau_\infty>0$ as $n\to+\infty$, we get that $u_\infty(-\tau_\infty,0)\ge 1$, whence $u_\infty>0$ in $(-\infty,0)\times\mathbb{R}^N$ from the strong maximum principle and the uniqueness of the Cauchy problem associated with that equation. Since A_∞ is a constant symmetric definite positive matrix, there is an invertible matrix M such that the function $\widetilde{u}_\infty$ defined in $(-\infty,0)\times\mathbb{R}^N$ by $\widetilde{u}_\infty(t,x)=u_\infty(t,Mx)$ satisfies $(\widetilde{u}_\infty)_t=\Delta\widetilde{u}_\infty+r\,\widetilde{u}_\infty$ in $(-\infty,0)\times\mathbb{R}^N$. In other words, the function $(t,x)\mapsto e^{-rt}\widetilde{u}_\infty(t,x)$ is a positive solution of the heat equation in $(-\infty,0)\times\mathbb{R}^N$. Thus, by [17], it is nondecreasing with respect to t in $(-\infty,0)\times\mathbb{R}^N$. Therefore, the function $(t,x)\mapsto e^{-rt}u_\infty(t,x)$ is nondecreasing with respect to t in $(-\infty,0)\times\mathbb{R}^N$.

Remember now that $\mu > 0$ and that $\theta > 0$ is given in the statement of Lemma 6.2. Fix a real number $\sigma \in (0,1)$ close enough to 1 so that

$$\sigma\, e^{\mu\,\theta/2} > 1,$$

and let

$$\tau > 0 \text{ and } \eta > 0$$

be as in Proposition 6.1 applied with this real number σ and with $\underline{\nu} = \nu^{-1}$ and $\overline{\nu} = \nu$ (remember that $\nu^{-1} I \le A \le \nu I$ in $\mathbb{R}^N$ with $\nu \ge 1$). Finally, let $L > 0$ be such that (2.2) holds and let us fix $\varepsilon > 0$ small enough so that

$$0 < \varepsilon \le \frac{\theta}{4}, \quad \sqrt{\varepsilon}\,|\nabla A| \le \eta \text{ in } \mathbb{R}^N \quad \text{and} \quad e^{-\varepsilon L\tau}\,\sigma\, e^{\mu\,\theta/2} > 1. \tag{6.4}$$

Let us finally complete the proof of Lemma 6.2 by reaching a contradiction. Since the function $t \mapsto e^{-rt} u_\infty(t,x)$ is nondecreasing in $(-\infty, 0)$ for each given $x \in \mathbb{R}^N$ and since $0 < \varepsilon < 2\varepsilon < \theta \le \tau_\infty$, one has $e^{r\varepsilon} u_\infty(-\varepsilon, 0) \ge e^{r\tau_\infty} u_\infty(-\tau_\infty, 0)$. But $u_n \to u_\infty > 0$ locally uniformly in $(-\infty, 0) \times \mathbb{R}^N$ and $\tau_n \to \tau_\infty$ as $n \to +\infty$. As a consequence,

$$e^{r\varepsilon} u_n(-\varepsilon, 0) \ge e^{r(\tau_n - \varepsilon)} u_n(-\tau_n, 0) \ \text{ for all } n \text{ large enough,}$$

whence

$$u(-\varepsilon + t_n + \tau_n, x_n) \ge e^{r(\tau_n - 2\varepsilon)} u(t_n, x_n) \ge e^{\mu\,\theta/2} u(t_n, x_n) \ge e^{\mu\,\theta/2} u(t_n + \tau_n, x_n)$$

for all n large enough, since $r \ge \mu > 0$, $\tau_n - 2\varepsilon \ge \theta - 2\varepsilon \ge \theta/2 > 0$ and $u(t_n, x_n) \ge u(t_n + \tau_n, x_n) > 0$ by (6.3). Lastly, consider the parabolically rescaled functions

$$v_n(t,x) = u(\varepsilon\, t + t_n + \tau_n, \sqrt{\varepsilon}\, x + x_n), \quad (t,x) \in (-\varepsilon^{-1}(t_n + \tau_n), +\infty) \times \mathbb{R}^N$$

and observe that

$$v_n(-1, 0) \ge e^{\mu\,\theta/2} v_n(0,0) \ \text{ for all } n \text{ large enough.} \tag{6.5}$$

Furthermore, the functions v_n obey

$$(v_n)_t = \operatorname{div}(A_n(x)\nabla v_n) + \varepsilon\, f(\sqrt{\varepsilon}\, x + x_n, v_n) \ \text{ in } (-\varepsilon^{-1}(t_n + \tau_n), +\infty) \times \mathbb{R}^N, \tag{6.6}$$

with $A_n(x) = A(\sqrt{\varepsilon}\, x + x_n)$, and they are positive in $(-\varepsilon^{-1}(t_n + \tau_n), +\infty) \times \mathbb{R}^N$. For each $n \in \mathbb{N}$ and $y \in \mathbb{R}^N$, call p_n and $p_{n,y}$ the fundamental solutions of (6.1) with diffusion matrix fields $a = A_n$ and $a = A_n(\cdot + y)$, respectively. Remember that $\tau > 0$ is given above from Proposition 6.1 and choose $n \in \mathbb{N}$ large enough so that $-\varepsilon^{-1}(t_n + \tau_n) < -\tau - 1$ and (6.5) holds. Since the function f is such that $0 \le f(x,s) \le Ls$ for all $(x,s) \in \mathbb{R}^N \times [0,1]$, it follows from (6.6) that

$$\begin{aligned} v_n(0,0) &\ge \int_{\mathbb{R}^N} p_n(\tau+1, 0; y)\, v_n(-\tau+1, y)\, dy \\ &= \int_{\mathbb{R}^N} p_{n,y}(\tau+1, -y; 0)\, v_n(-\tau+1, y)\, dy \end{aligned} \tag{6.7}$$

and

$$\begin{array}{rcl} v_n(-1,0) & \le & e^{\varepsilon L\tau} \displaystyle\int_{\mathbb{R}^N} p_n(\tau,0;y)\, v_n(-\tau+1,y)\, dy \\ & = & e^{\varepsilon L\tau} \int_{\mathbb{R}^N} p_{n,y}(\tau,-y;0)\, v_n(-\tau+1,y)\, dy. \end{array} \tag{6.8}$$

On the other hand, for every $y \in \mathbb{R}^N$, the matrix field $A_{n,y} := A_n(\cdot+y)$ is of class $C^1(\mathbb{R}^N)$ it satisfies $\nu^{-1}I \le A_{n,y} \le \nu I$ in $\mathbb{R}^N$ and $|\nabla A_{n,y}(x)| = \sqrt{\varepsilon}\,|\nabla A(\sqrt{\varepsilon}\,(x+y)+x_n)| \le \eta$ for all $x \in \mathbb{R}^N$ by (6.4), where $\eta > 0$ is given above from Proposition 6.1. It follows then from the conclusion of Proposition 6.1 that

$$p_{n,y}(\tau+1,-y;0) \ge \sigma\, p_{n,y}(\tau,-y;0) \ \text{ for all } y \in \mathbb{R}^N,$$

whence $v_n(0,0) \ge e^{-\varepsilon L\tau}\sigma\, v_n(-1,0)$ by (6.7) and (6.8), and finally

$$v_n(0,0) \ge e^{-\varepsilon L\tau}\sigma\, e^{\mu\,\theta/2} v_n(0,0)$$

by (6.5). The positivity of $v_n(0,0) = u(t_n+\tau_n,x_n)$ (since $t_n+\tau_n > 0$) contradicts the last property in (6.4). The proof of Lemma 6.2 is thereby complete. □

7. Conclusion of the proof of Theorem 1.1

Remember the definition of τ_* in (3.1) and remember that $0 \le \tau^* < +\infty$. Before completing the proof of Theorem 1.1, we first show that $\tau^* = 0$.

Lemma 7.1. *There holds*

$$\tau_* = 0.$$

Proof. Assume by contradiction that $\tau_* > 0$. It follows from the definition (3.1) of τ_* that there exist two sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that

$$\lim_{n\to+\infty} t_n = +\infty, \ \ \liminf_{n\to+\infty} \tau_n \ge \tau_* \ \text{ and } \ u(t_n+\tau_n,x_n) < u(t_n,x_n) \text{ for all } n \in \mathbb{N}. \tag{7.1}$$

As a matter of fact, $\tau_n \to \tau_*$ as $n \to +\infty$, otherwise the assumption $\limsup_{n\to+\infty} \tau_n > \tau_*$ together with $\lim_{n\to+\infty} t_n = +\infty$ would contradict the definition of τ_*. The real numbers $u(t_n,x_n)$ take values in $(0,1)$. Thus, up to extraction of a subsequence, three cases can occur.

Case 1: $u(t_n,x_n) \to m \in (0,1)$ as $n \to +\infty$. There are then two real numbers a and b such that $0 < a \le u(t_n,x_n) \le b < 1$ for all $n \in \mathbb{N}$. Since $t_n \to +\infty$ and $\tau_n \to \tau_* > 0$ as $n \to +\infty$, Corollary 3.2 yields the existence of $\delta > 0$ such that $u(t_n+\tau_n,x_n) \ge (1+\delta)\, u(t_n,x_n)$ for all n large enough. This is impossible by (7.1).

Case 2: $u(t_n,x_n) \to 1$ as $n \to +\infty$. Therefore, $u(t_n,x_n) \ge b$ and $t_n \ge \widetilde{T}$ for all n large enough, where $b \in (0,1)$ and $\widetilde{T}$ are given as in Lemma 5.1. One infers then from Lemma 5.1 that, for all n large enough, $u_t(t_n,x_n) > 0$ and thus even that $u_t(t,x_n) > 0$ for all $t \ge t_n$. In particular, $u(t_n+\tau_n,x_n) > u(t_n,x_n)$ for all n large enough, contradicting (7.1). Thus, Case 2 is ruled out too.

Case 3: $u(t_n,x_n) \to 0$ as $n \to +\infty$. Since $\tau_n \to \tau_* > 0$ and each τ_n is positive, there are two real numbers $0 < \theta \le \theta'$ such that $\theta \le \tau_n \le \theta'$ for all $n \in \mathbb{N}$. Let then $T_0 > 0$

and $\varepsilon > 0$ be as in Lemma 6.2. For all n large enough, one has $t_n \geq T_0$ and $u(t_n, x_n) \leq \varepsilon$, whence $u(t_n + \tau_n, x_n) > u(t_n, x_n)$. This contradicts (7.1).

As a conclusion, all three cases are impossible. Finally, $\tau_* = 0$ and the proof of Lemma 7.1 is complete. □

Based on the previous lemma, we can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We first notice that, for any $t > 0$,

$$u_t(t, x) \to 0 \text{ as } |x| \to +\infty. \tag{7.2}$$

Indeed, on the one hand, as in the proof of Lemma 3.1, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ with $\lim_{n\to+\infty} |x_n| = +\infty$, the functions $u_n : (t, x) \mapsto u(t, x + x_n)$ converge locally uniformly in $C^{1,2}_{t,x}((0, +\infty) \times \mathbb{R}^N)$, up to extraction of a subsequence, to a solution $0 \leq u_\infty \leq 1$ of an equation of the type

$$(u_\infty)_t = \mathrm{div}(A_\infty \nabla u_\infty) + f_\infty(x, u_\infty) \text{ in } (0, +\infty) \times \mathbb{R}^N$$

for some constant symmetric definite positive matrix A_∞ and for some function f_∞ satisfying the same properties as f. On the other hand, as already emphasized from the proof of Lemma 2.1, $u(t, x) \to 0$ as $|x| \to +\infty$, for every $t > 0$. Therefore, $u_\infty = 0$ in $(0, +\infty) \times \mathbb{R}^N$. Hence, by uniqueness the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to 0 locally uniformly in $C^{1,2}_{t,x}((0, +\infty) \times \mathbb{R}^N)$ and $u_t(t, x + x_n) \to 0$ as $n \to +\infty$ for every $(t, x) \in (0, +\infty) \times \mathbb{R}^N$. As a consequence, (7.2) holds.

In particular, it follows that $\inf_{\mathbb{R}^N} u_t(t, \cdot) \leq 0$ for all $t > 0$. We now prove (1.12), and then (1.13).[4] Assume now by contradiction that $\inf_{\mathbb{R}^N} u_t(t, \cdot) \not\to 0$ as $t \to +\infty$. Since u_t is bounded in $(1, +\infty) \times \mathbb{R}^N$, it follows then that there are a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that $t_n \to +\infty$ and $\limsup_{n\to+\infty} u_t(t_n, x_n) \in (-\infty, 0)$. As done in the previous paragraph or in the proof of Lemma 3.1, the functions

$$(t, x) \mapsto u(t + t_n, x + x_n)$$

converge, up to extraction of a subsequence, locally uniformly in $C^{1,2}_{t,x}(\mathbb{R} \times \mathbb{R}^N)$ to a solution $0 \leq u_\infty \leq 1$ of an equation of the type (3.4) with $(u_\infty)_t(0, 0) < 0$. However, for any $\tau > 0$, it follows from the definition (3.1) of τ_* together with $\lim_{n\to+\infty} t_n = +\infty$ and Lemma 7.1 ($\tau_* = 0$) that, for any given $(t, x) \in \mathbb{R}^N$, there holds

$$u(t + \tau + t_n, x + x_n) \geq u(t + t_n, x + x_n) \text{ for all } n \text{ large enough,}$$

whence $u_\infty(t+\tau, x) \geq u_\infty(t, x)$. Therefore, since $\tau > 0$ and $(t, x) \in \mathbb{R}\times\mathbb{R}^N$ are arbitrary, one gets that $(u_\infty)_t \geq 0$ in $\mathbb{R} \times \mathbb{R}^N$, which yields a contradiction. In other words, one has shown that $\inf_{\mathbb{R}^N} u_t(t, \cdot) \to 0$ as $t \to +\infty$, that is, (1.12).

Finally, let $\varepsilon \in (0, 1)$ be given and let us show (1.13), that is, the existence of $T_\varepsilon > 0$ such that $u_t(t, x) > 0$ for every $(t, x) \in [T_\varepsilon, +\infty) \times \mathbb{R}^N$ with $u(t, x) \geq \varepsilon$. Assume not. Then there are a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$ of

[4]We could also view (1.12) as a consequence of (1.13) by observing that $u_t(t, x) \to 0$ as $u(t, x) \to 0$. But since the proof of (1.12) is easy even without (1.13), we choose to carry it out before (1.13).

points in $\mathbb{R}^N$ such that $t_n \to +\infty$ as $n \to +\infty$, while $u(t_n, x_n) \geq \varepsilon$ and $u_t(t_n, x_n) \leq 0$ for all $n \in \mathbb{N}$. It follows then from Lemma 5.1 that 1 is not a limiting value of the sequence $(u(t_n, x_n))_{n\in\mathbb{N}}$. Therefore, up to extraction of a sequence, one can assume without loss of generality that

$$u(t_n, x_n) \to m \in (0,1) \ \text{ as } n \to +\infty.$$

As done in the previous paragraph, one infers that the functions $(t, x) \mapsto u(t+t_n, x+x_n)$ converge, up to extraction of a subsequence, locally uniformly in $C^{1,2}_{t,x}(\mathbb{R} \times \mathbb{R}^N)$ to a solution $0 \leq u_\infty \leq 1$ of an equation of the type

$$(u_\infty)_t = \operatorname{div}(A_\infty \nabla u_\infty) + f_\infty(x, u_\infty), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^N$$

with $(u_\infty)_t(t, x) \geq 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, whereas $u_\infty(0, 0) = m \in (0, 1)$ and $(u_\infty)_t(0, 0) \leq 0$. It follows from the strong maximum principle applied to the function $(u_\infty)_t$ that $(u_\infty)_t = 0$ in $(-\infty, 0] \times \mathbb{R}^N$ and then in $\mathbb{R} \times \mathbb{R}^N$. Furthermore, the strong maximum principle applied to u_∞ also implies that $0 < u_\infty < 1$ in $\mathbb{R}\times\mathbb{R}^N$. As recalled in Section 1, one infers then from the properties shared by f_∞ with f that $u_\infty(t, x) \to 1$ as $t \to +\infty$ for every $x \in \mathbb{R}^N$. This leads to a contradiction, since $(u_\infty)_t = 0$ in $\mathbb{R}\times\mathbb{R}^N$ and $u_\infty(0, 0) = m < 1$. Therefore, (1.13) is shown and the proof of Theorem 1.1 is thereby complete. □

8. Proof of Proposition 6.1

Let $0 < \underline{\nu} \leq \overline{\nu}$ and $0 < \sigma < 1$ be fixed. When $a = D\, I$ with a real number $D \in [\underline{\nu}, \overline{\nu}]$, then the conclusion (6.2) holds immediately for $\tau > 0$ large enough, from the explicit formula

$$p(t, x; 0) = \frac{e^{-|x|^2/(4Dt)}}{(4\pi t)^{N/2}}.$$

In the general case where a may not be constant, we will get the estimates (6.2) by using uniform Gaussian estimates for large x and small t, and by approximating locally, when a is nearly locally constant, the solutions p of (6.1) by explicit fundamental solutions of parabolic equations with constant coefficients.

More precisely, choose first any real number $\tau > 0$ large enough so that

$$\Big(\frac{\tau}{\tau+1}\Big)^{N/2} > \sigma. \tag{8.1}$$

We shall prove the conclusion of Proposition 6.1 is fulfilled with any such real number τ.

First of all, it follows from the Gaussian upper and lower bounds of the fundamental solutions of (6.1) in [15] that there exist a constant $K \geq 1$ such that, for every $L^\infty(\mathbb{R}^N)$ matrix field $a : \mathbb{R}^N \to \mathbb{S}_N(\mathbb{R})$ with $\underline{\nu} I \leq a \leq \overline{\nu} I$ a.e. in $\mathbb{R}^N$, the fundamental solution p of (6.1) satisfies

$$\frac{e^{-K\,|x|^2/t}}{K\, t^{N/2}} \leq p(t, x; 0) \leq \frac{K\, e^{-|x|^2/(Kt)}}{t^{N/2}} \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R}^N. \tag{8.2}$$

Therefore, we claim that there exists $\tau_0 \in (0, \tau)$ small enough such that, for every $L^\infty(\mathbb{R}^N)$ matrix field $a : \mathbb{R}^N \to \mathbb{S}_N(\mathbb{R})$ with $\underline{\nu} I \leq a \leq \overline{\nu} I$ a.e. in $\mathbb{R}^N$, there holds

$$p(\tau_0 + 1, x; 0) \geq \sigma\, p(\tau_0, x; 0) \ \text{ for all } x \in \mathbb{R}^N \text{ with } |x| \geq 1. \tag{8.3}$$

Indeed, otherwise, by picking any sequence $(\tau_n)_{n\in\mathbb{N}}$ in $(0,\tau)$ such that $\lim_{n\to+\infty}\tau_n=0$, there would exist a sequence $(a_n)_{n\in\mathbb{N}}$ of bounded symmetric matrix fields with $\underline{\nu}I\le a_n\le\overline{\nu}I$ a.e. in $\mathbb{R}^N$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that

$$|x_n|\ge 1 \ \text{ and } \ p_n(\tau_n+1,x_n;0)<\sigma\,p_n(\tau_n,x_n;0) \ \text{ for all } n\in\mathbb{N},$$

where p_n denotes the fundamental solution of (6.1) with $a=a_n$. It would then follow from (8.2) that

$$\frac{e^{-K|x_n|^2/(\tau_n+1)}}{K\,(\tau_n+1)^{N/2}}<\frac{\sigma\,K\,e^{-|x_n|^2/(K\tau_n)}}{\tau_n^{N/2}}$$

for all $n\in\mathbb{N}$, that is,

$$e^{|x_n|^2(1/(K\tau_n)-K/(\tau_n+1))}<\sigma\,K^2\Big(\frac{\tau_n+1}{\tau_n}\Big)^{N/2}.$$

But $|x_n|\ge 1$ for all $n\in\mathbb{N}$ and $1/(K\tau_n)-K/(\tau_n+1)\ge 1/(2K\tau_n)$ for all n large enough, since $\tau_n\to 0^+$ as $n\to+\infty$. Therefore, $e^{1/(2K\tau_n)}<\sigma\,K^2\,(1+1/\tau_n)^{N/2}$ for all n large enough, which leads to a contradiction by passing to the limit as $n\to+\infty$. As a consequence, there is $\tau_0\in(0,\tau)$ such that (8.3) holds for every $L^\infty(\mathbb{R}^N)$ matrix field $a:\mathbb{R}^N\to\mathbb{S}_N(\mathbb{R})$ with $\underline{\nu}I\le a\le\overline{\nu}I$ a.e. in $\mathbb{R}^N$.

Now, we fix a real number $R\ge 1$ large enough such that

$$\sigma\Big(1+\frac{1}{\tau_0}\Big)^{N/2}e^{-R^2/(4\overline{\nu}\tau(\tau+1))}<1. \tag{8.4}$$

Given this choice of $R\ge 1$, we claim that there exists a real number $\eta>0$ such that, for every $a\in C^1(\mathbb{R}^N;\mathbb{S}_N(\mathbb{R}))$ with $\underline{\nu}I\le a\le\overline{\nu}I$ and $|\nabla a|\le\eta$ in $\mathbb{R}^N$, the fundamental solution $p(t,x;y)$ of (6.1) satisfies

$$\left\{\begin{array}{ll} p(\tau+1,x;0)\ge\sigma\,p(\tau,x;0) & \text{for all } |x|\le R,\\ p(t+1,x;0)\ge\sigma\,p(t,x;0) & \text{for all } t\in[\tau_0,\tau] \text{ and } |x|=R.\end{array}\right. \tag{8.5}$$

Indeed, otherwise, there exist a sequence $(a_n)_{n\in\mathbb{N}}$ in $C^1(\mathbb{R}^N;\mathbb{S}_N(\mathbb{R}))$ with $\underline{\nu}I\le a_n\le\overline{\nu}I$ in $\mathbb{R}^N$ and $\lim_{n\to+\infty}\|\,|\nabla a_n|\,\|_{L^\infty(\mathbb{R}^N)}=0$, as well as a sequence of points $(t_n,x_n)_{n\in\mathbb{N}}$ in $(0,+\infty)\times\mathbb{R}^N$ such that

$$(t_n,x_n)\in\{\tau\}\times\overline{B(0,R)}\cup[\tau_0,\tau]\times\partial B(0,R) \ \text{ and } \ p_n(t_n+1,x_n;0)<\sigma\,p_n(t_n,x_n;0) \tag{8.6}$$

for all $n\in\mathbb{N}$, where p_n denotes the fundamental solution of (6.1) with $a=a_n$. Up to extraction of a subsequence, the matrix fields a_n converge locally uniformly in $\mathbb{R}^N$ to a constant symmetric definite positive matrix a_∞ such that $\underline{\nu}I\le a_\infty\le\overline{\nu}I$. Furthermore, the functions $(p_n)_{n\in\mathbb{N}}$ are bounded locally in $(0,+\infty)\times\mathbb{R}^N$ from the bounds (8.2). From standard parabolic estimates, the functions $p_n(\cdot,\cdot;0)$ converge then locally uniformly in $(0,+\infty)\times\mathbb{R}^N$ to a classical solution p_∞ of

$$(p_\infty)_t=\mathrm{div}(a_\infty\nabla p_\infty) \ \text{ in } (0,+\infty)\times\mathbb{R}^N$$

such that

$$K^{-1}t^{-N/2}e^{-K|x|^2/t}\le p_\infty(t,x)\le K\,t^{-N/2}e^{-|x|^2/(Kt)} \ \text{ for all } (t,x)\in(0,+\infty)\times\mathbb{R}^N. \tag{8.7}$$

Moreover, it follows from (8.6) that there exists a point (t_∞, x_∞) such that

$$(t_\infty, x_\infty) \in \{\tau\}\times\overline{B(0,R)} \cup [\tau_0,\tau]\times\partial B(0,R) \ \text{ and } \ p_\infty(t_\infty+1, x_\infty) \le \sigma\, p_\infty(t_\infty, x_\infty). \quad (8.8)$$

From the aforementioned Gaussian estimates, the function p_∞ is therefore positive in $(0,+\infty)\times\mathbb{R}^N$ and since $a_\infty \in \mathbb{S}_N(\mathbb{R}^N)$ satisfies $\underline{\nu} I \le a_\infty \le \overline{\nu} I$, there is an orthogonal linear map $M : x \mapsto y$ such that the function $(t,y)\mapsto q(t,y) = p_\infty(t, M^{-1}y) = p_\infty(t,x)$ is a positive solution of

$$q_t = \sum_{1\le i\le N} \lambda_i \frac{\partial^2 q}{\partial y_i^2} \ \text{ in } (0,+\infty)\times\mathbb{R}^N$$

for some real numbers $\lambda_1,\dots,\lambda_N \in [\underline{\nu}, \overline{\nu}]$. Hence, there is a nonnegative Radon measure λ such that

$$p_\infty(t,x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\sum_{1\le i\le N} |y_i/\sqrt{\lambda_i} - z_i|^2/(4t)}\, d\lambda(z) \ \text{ for all } (t,x)\in(0,+\infty)\times\mathbb{R}^N.$$

Since, by (8.7), $\int_{B(x_0,r)} p_\infty(t,x)\,dx \to 0^+$ as $t\to 0^+$ for every $x_0\in\mathbb{R}^N$ and $r>0$ such that $0 \notin \overline{B(x_0,r)}$, it follows easily that λ is supported on the singleton $\{0\}$ and that there is $\rho>0$ such that

$$p_\infty(t,x) = \frac{\rho\, e^{-\sum_{1\le i\le N} |y_i|^2/(4\lambda_i t)}}{(4\pi t)^{N/2}} \ \text{ for all } (t,x)\in(0,+\infty)\times\mathbb{R}^N. \quad (8.9)$$

Remember now (8.8). On the one hand, if $t_\infty = \tau$ (and $|x_\infty|\le R$), then (8.8) and (8.9) imply that

$$\frac{\rho\, e^{-\sum_{1\le i\le N} |y_{\infty,i}|^2/(4\lambda_i(\tau+1))}}{(4\pi(\tau+1))^{N/2}} \le \sigma \times \frac{\rho\, e^{-\sum_{1\le i\le N} |y_{\infty,i}|^2/(4\lambda_i\tau)}}{(4\pi\tau)^{N/2}},$$

where $y_\infty = M\,x_\infty$. Since $\rho>0$, $\lambda_i>0$ for all $1\le i\le N$ and $0<\tau<\tau+1$, one gets that $\tau^{N/2} \le \sigma(\tau+1)^{N/2}$, which contradicts (8.1). On the other hand, if $|x_\infty| = R$ (and $t_\infty\in[\tau_0,\tau]$), then (8.8) and (8.9) yield, with the same notations as above,

$$\frac{\rho\, e^{-\sum_{1\le i\le N} |y_{\infty,i}|^2/(4\lambda_i(t_\infty+1))}}{(4\pi(t_\infty+1))^{N/2}} \le \sigma \times \frac{\rho\, e^{-\sum_{1\le i\le N} |y_{\infty,i}|^2/(4\lambda_i t_\infty)}}{(4\pi t_\infty)^{N/2}},$$

whence

$$1 \le \sigma \Big(1+\frac{1}{t_\infty}\Big)^{N/2} e^{-\sum_{1\le i\le N} |y_{\infty,i}|^2/(4\lambda_i t_\infty(t_\infty+1))}.$$

Since $0<\tau_0\le t_\infty\le\tau$, $0<\underline{\nu}\le\lambda_i\le\overline{\nu}$ for all $1\le i\le N$ and $|y_\infty| = |M\,x_\infty| = |x_\infty| = R$, one infers that

$$1 \le \sigma \Big(1+\frac{1}{\tau_0}\Big)^{N/2} e^{-R^2/(4\overline{\nu}\tau(\tau+1))},$$

which contradicts (8.4).

As a consequence, there is $\eta>0$ such that (8.5) holds for every $a\in C^1(\mathbb{R}^N;\mathbb{S}_N(\mathbb{R}))$ with $\underline{\nu} I \le a \le \overline{\nu} I$ and $|\nabla a|\le\eta$ in $\mathbb{R}^N$. Consider finally any such matrix field a and let us show that the conclusion (6.2) holds, that is $p(\tau+1,\cdot;0) \geq \sigma\, p(\tau,\cdot;0)$ in $\mathbb{R}^N$, where p solves (6.1). First of all, it follows from (8.5) that

$$p(\tau+1,\cdot;0) \geq \sigma\, p(\tau,\cdot;0) \ \text{ in } \overline{B(0,R)}. \tag{8.10}$$

On the other hand,

$$p(\tau_0+1,\cdot;0) \geq \sigma\, p(\tau_0,\cdot;0) \ \text{ in } \mathbb{R}^N \backslash B(0,R) \subset \mathbb{R}^N \backslash B(0,1)$$

by (8.3) and $R \geq 1$. Lastly,

$$p(t+1,\cdot;0) \geq \sigma\, p(t,\cdot;0) \ \text{ on } \partial B(0,R) \ \text{ for all } t \in [\tau_0,\tau]$$

by (8.5). Therefore, since $p(\cdot,\cdot;0)$ and $p(\cdot+1,\cdot;0)$ are two positive bounded solutions of the same linear parabolic equation in (at least) $[\tau_0,\tau] \times (\mathbb{R}^N \backslash B(0,R))$, it follows from the parabolic maximum principle that

$$p(\tau+1,\cdot;0) \geq \sigma\, p(\tau,\cdot;0) \ \text{ in } \mathbb{R}^N \backslash B(0,R).$$

Together with (8.10), one concludes that $p(\tau+1,\cdot;0) \geq \sigma\, p(\tau,\cdot;0)$ in $\mathbb{R}^N$ and the proof of Proposition 6.1 is thereby complete. □

9. Proof of Theorem 1.2

The key-point in the proof of Theorem 1.2 is the following result of independent interest on some monotonicity properties of the solutions of a boundary value problem in a half-line for a homogeneous linear one-dimensional reaction-diffusion.

Proposition 9.1. *Let a and λ be two positive real numbers and let u be the solution of*

$$v_t = a\, v_{xx} + \lambda\, v, \quad t > 0, \ x > 0, \tag{9.1}$$

with boundary condition

$$v(t,0) = g(t), \quad t > 0, \tag{9.2}$$

and initial datum $v_0 \in L^\infty(0,+\infty)\backslash\{0\}$. Assume that $v_0(x) \geq 0$ for a.e. $x > 0$ and that g is continuous, nonnegative and nondecreasing on $(0,+\infty)$. Then

$$v_t(t,x) > 0$$

provided $t \geq t_0$ and $x \geq \sqrt{8at}$, where $t_0 = (2\lambda)^{-1} + e(e-1)^{-1}\lambda^{-1} > 0$ is a positive constant depending only on λ.

Proof. Observe that v can be written as

$$v = w + z,$$

where w and z solve (9.1), with $w(0,x) = v_0(x)$ and $z(0,x) = 0$ for a.e. $x > 0$, while $w(t,0) = 0$, $z(t,0) = g(t)$ for all $t > 0$.

Let us first consider the solution w of the homogeneous Dirichlet boundary condition. There holds, for all $t > 0$ and $x > 0$,

$$w(t,x) = \int_0^{+\infty} G(t,x,y)\, v_0(y)\, dy,$$

where the Green function G is given by

$$G(t,x,y) = \frac{e^{\lambda t}}{\sqrt{4\pi a t}} \Big[e^{-|x-y|^2/(4at)} - e^{-|x+y|^2/(4at)} \Big]$$

for $t > 0$, $x > 0$ and $y > 0$. The time derivative of G satisfies

$$\begin{aligned} G_t(t,x,y) \quad &= \quad \frac{e^{\lambda t}}{\sqrt{4\pi a t^3}} e^{-|x-y|^2/(4at)} \Big[\lambda t - \frac{1}{2} + \frac{|x-y|^2}{4at} \Big] \\ &\quad - \frac{e^{\lambda t}}{\sqrt{4\pi a t^3}} e^{-|x+y|^2/(4at)} \Big[\lambda t - \frac{1}{2} + \frac{|x+y|^2}{4at} \Big]. \end{aligned}$$

Let us also introduce the function $\phi : [0,+\infty) \to [0,+\infty)$ defined by

$$\phi(x) = x\, e^{-x}.$$

This function ϕ has a maximum at $x = 1$ and it is increasing in $[0,1]$ and decreasing in $[1,+\infty)$.

We are going to prove that $G_t(t,x,y) > 0$ for all $y \in (0,+\infty)$ provided $x \geq \sqrt{8at}$ and $t \geq t_0$, where $t_0 > 0$ is given in the statement of Proposition 9.2. In this paragraph, we fix any $t \geq t_0$, $x \geq \sqrt{8at}$ and $y > 0$, and we first observe that $|x+y|^2/(4at) \geq 2$. Two cases can then appear, depending on whether $|x-y|^2/(4at)$ is larger or smaller than 1. In the first case, namely if $|x-y|^2/(4at) \geq 1$, then $1 \leq |x-y|^2/(4at) < |x+y|^2/(4at)$ and

$$\phi\Big(\frac{|x-y|^2}{4at}\Big) > \phi\Big(\frac{|x+y|^2}{4at}\Big),$$

hence we get $G_t(t,x,y) > 0$ (we also use the fact that $t \geq t_0 \geq (2\lambda)^{-1}$). In the second case, one has $0 \leq |x-y|^2/(4at) < 1$ and

$$\big(\lambda t - \frac{1}{2}\big) \big[e^{-|x-y|^2/(4at)} - e^{-|x+y|^2/(4at)} \big] \geq \big(\lambda t - \frac{1}{2}\big)(e^{-1} - e^{-2}) \geq e^{-1}$$

since $t \geq t_0 = (2\lambda)^{-1} + e(e-1)^{-1}\lambda^{-1} \geq (2\lambda)^{-1}$, whereas

$$\frac{|x+y|^2}{4at} e^{-|x+y|^2/(4at)} < \phi(1) = e^{-1}.$$

Hence, we also have $G_t(t,x,y) > 0$ in the second case. As a consequence, $G_t(t,x,\cdot) > 0$ in $(0,+\infty)$ for all $t \geq t_0$ and $x \geq \sqrt{8at}$, hence $w_t(t,x) > 0$ for all $t \geq t_0$ and $x \geq \sqrt{8at}$, since v_0 is nonnegative and nontrivial in $(0,+\infty)$.

Let us now turn to the solution z of equation (9.1) with vanishing initial condition and with boundary condition (9.2). Since g is nonnegative and bounded in any interval $(0,T)$ with $T \in (0,+\infty)$, it follows from the maximum principle that z is nonnegative and bounded in $(0,T)\times(0,+\infty)$ too. Furthermore, for any $h>0$ and $t>0$, there holds $z(h,\cdot)\ge 0=z(0,\cdot)$ in $(0,+\infty)$ and $z(t+h,0)=g(t+h)\ge g(t)=z(t,0)$. Hence, the maximum principle yields $z(t+h,x)\ge z(t,x)$ for all $t>0$ and $x>0$. In other words, the function z is nondecreasing with respect to t.

As a conclusion, the solution v of (9.1) with boundary condition (9.2) satisfies $v_t(t,x)>0$ provided $t\ge t_0$ and $x\ge\sqrt{8at}$. □

From the previous proposition and a change of variable $x\to -x$, the following result immediately follows.

Proposition 9.2. *Let a and λ be two positive real numbers and let u be the solution of*

$$v_t = a\,v_{xx}+\lambda\,v,\quad t>0,\ x<0, \tag{9.3}$$

with boundary condition (9.2) and initial datum $v_0\in L^\infty(-\infty,0)\backslash\{0\}$. Assume that $v_0(x)\ge 0$ for a.e. $x<0$ and that g is continuous, nonnegative and nondecreasing on $(0,+\infty)$. Then $v_t(t,x)>0$ provided $t\ge t_0$ and $x\le-\sqrt{8at}$, where $t_0=(2\lambda)^{-1}+e(e-1)^{-1}\lambda^{-1}>0$.

With these two propositions in hand, let us now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let A, f, $f^\pm$, $\lambda^\pm>0$, $\theta\in(0,1)$ and u_0 be as in the statement and let $R>0$ and $a^\pm\in(0,+\infty)$ be such that

$$f(x,\cdot)=f^\pm \text{ in } [0,1] \text{ and } A(x)=a^\pm \text{ for all } |x|\ge R \text{ with } \pm x>0. \tag{9.4}$$

Let us call

$$T_0=\max\big((2\lambda^-)^{-1}+e(e-1)^{-1}(\lambda^-)^{-1},(2\lambda^+)^{-1}+e(e-1)^{-1}(\lambda^+)^{-1}\big)>0. \tag{9.5}$$

Firstly, we claim that there exists $\eta\in(0,\theta)$ such that

$$\forall\, t\ge 1,\ \forall\, x\in\mathbb{R},\quad \big(u(t,x)\le\eta\big)\implies\big(u(s,x)\le\theta \text{ for all } s\in[t,t+T_0]\big). \tag{9.6}$$

Assume not. Then there are a sequence $(t_n)_{n\in\mathbb{N}}$ in $[1,+\infty)$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$u(t_n,x_n)\to 0 \text{ as } n\to+\infty \text{ and } \max_{[t_n,t_n+T_0]}u(\cdot,x_n)>\theta \text{ for all } n\in\mathbb{N}.$$

If the sequence $(x_n)_{n\in\mathbb{N}}$ were bounded, then the sequence $(t_n)_{n\in\mathbb{N}}$ would be bounded too due to (1.11). Hence, up to extraction of a subsequence, (t_n,x_n) would converge to a point (t,x) in $[1,+\infty)\times\mathbb{R}$ with $u(t,x)=0$, which is impossible due to (1.10). Therefore, the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded. Up to extraction of a subsequence, let us then assume without loss of generality that $x_n\to+\infty$ as $n\to+\infty$ (the case $\lim_{n\to+\infty}x_n=-\infty$, up to extraction of a subsequence, can be handled similarly). From standard parabolic estimates, up to extraction of a subsequence, the functions $u_n:(t,x)\mapsto u(t+t_n,x+x_n)$ converge in $C^{1,2}_{t,x}$ locally in (at least) $(-1,+\infty)\times\mathbb{R}$ to a solution v of

$$v_t=a^+v_{xx}+f^+(v) \text{ in } (-1,+\infty)\times\mathbb{R}$$

such that $v(0,0)=0$ and $\max_{[0,T_0]} v(\cdot,0) \geq \theta$. Furthermore, $0 \leq v \leq 1$ in $(-1,+\infty)\times\mathbb{R}$ by (1.10). Hence, $v=0$ in $(-1,0]\times\mathbb{R}$ from the strong maximum principle, and $v=0$ in $[0,+\infty)\times\mathbb{R}$ by the uniqueness of the solutions of the associated Cauchy problem. This contradicts the property $\max_{[0,T_0]} v(\cdot,0) \geq \theta\ (>0)$. Finally, the claim (9.6) has been proved.

Secondly, since the function u is positive in $(0,+\infty)\times\mathbb{R}$ and the function f is Lipschitz continuous with respect to $u \in [0,1]$ uniformly in $x \in \mathbb{R}$, it follows from Harnack inequality that there is a constant $C \in (0,1)$ such that, for all $(t,x) \in [1,+\infty)\times\mathbb{R}$,

$$u(t+T_0, x \pm \sqrt{8a^{\pm}T_0}) \geq C\,u(t,x). \tag{9.7}$$

Let us denote

$$\varepsilon = C\,\eta \tag{9.8}$$

and notice that $0<\varepsilon<\eta<\theta<1$. From Theorem 1.1, there is $T_\varepsilon>0$ such that (1.13) holds, that is,

$$\forall\,(t,x)\in[T_\varepsilon,+\infty)\times\mathbb{R},\quad u(t,x)\geq\varepsilon \implies u_t(t,x)>0. \tag{9.9}$$

From (1.11), since $\eta<\theta<1$, one can also assume without loss of generality that $T_\varepsilon \geq 1$ and that

$$\min_{[-R,R]} u(T_\varepsilon,\cdot) \geq \eta,$$

where $R>0$ is given in (9.4). Since $\eta>0$ and $u(T_\varepsilon,\pm\infty)=0$ by (1.14), it follows from the continuity of $u(T_\varepsilon,\cdot)$ that there are some real numbers $x^{\pm}$ such that

$$x^- \leq -R < R \leq x^+,\quad u(T_\varepsilon,x^{\pm})=\eta \ \text{ and } \ u(T_\varepsilon,\cdot)\leq\eta \text{ in } (-\infty,x^-]\cup[x^+,+\infty).$$

Let now v be the solution of (9.1) with $a=a^+$, $\lambda=\lambda^+$ and initial and boundary conditions given by

$$v_0 = u(T_\varepsilon,\cdot+x^+)\,(>0) \text{ in } (0,+\infty) \ \text{ and } \ v(t,0)=u(t+T_\varepsilon,x^+)\,(>0) \text{ for all } t>0.$$

Since $u(T_\varepsilon,x^+)=\eta>\varepsilon$, one infers from (9.9) that the continuous nonnegative function $t\mapsto u(t+T_\varepsilon,x^+)$ is actually increasing in $[0,+\infty)$. Therefore, as $T_0 \geq (2\lambda^+)^{-1}+e(e-1)^{-1}(\lambda^+)^{-1}$ by (9.5), it follows from Proposition 9.1 that, in particular,

$$v_t(T_0,x)>0 \text{ for all } x\geq\sqrt{8a^+T_0}.$$

On the other hand, since $T_\varepsilon\geq 1$ and $u(T_\varepsilon,\cdot)\leq\eta$ in $[x^+,+\infty)$, it follows from (9.6) that $u\leq\theta$ in $[T_\varepsilon,T_\varepsilon+T_0]\times[x^+,+\infty)$. In that set, since $x^+\geq R$, there holds $f(u)=\lambda^+u$ (and $A(x)=a^+$), by (9.4) and the assumption on f^+. Therefore, $u(\cdot+T_\varepsilon,\cdot+x^+)$ satisfies the same linear equation as v in the set $[0,T_0]\times[0,+\infty)$, with the same initial and boundary conditions on $\{0\}\times[0,+\infty)$ and $[0,T_0]\times\{0\}$. Thus,

$$u(t+T_\varepsilon,x+x^+)=v(t,x) \text{ for all } (t,x)\in[0,T_0]\times[0,+\infty)$$

and

$$u_t(T_0+T_\varepsilon,x)>0 \text{ for all } x\geq x^+ + \sqrt{8a^+T_0}. \tag{9.10}$$

Furthermore, since $T_\varepsilon \geq 1$ and $u(T_\varepsilon, x^+) = \eta$, it follows from (9.7) and (9.8) that

$$u(T_\varepsilon + T_0, x^+ + \sqrt{8a^+ T_0}) \geq C\,\eta = \varepsilon.$$

Hence, $u_t(t, x^+ + \sqrt{8a^+ T_0}) > 0$ for all $t \geq T_\varepsilon + T_0$ by (9.9). Together with (9.10) and the fact that the equation (1.1) does not depend on time, one concludes from the maximum principle applied to u_t that

$$u_t(t,x) > 0 \text{ for all } (t,x) \in [T_\varepsilon + T_0, +\infty) \times [x^+ + \sqrt{8a^+ T_0}, +\infty). \tag{9.11}$$

Similarly, by using Proposition 9.2 among other things, one can show that

$$u_t(t,x) > 0 \text{ for all } (t,x) \in [T_\varepsilon + T_0, +\infty) \times (-\infty, x^- - \sqrt{8a^- T_0}]. \tag{9.12}$$

Finally, due to (1.11), there is $\tau \geq T_\varepsilon + T_0$ such that $u(\tau, \cdot) \geq \varepsilon$ in $[x^- - \sqrt{8a^- T_0}, x^+ + \sqrt{8a^+ T_0}]$. Hence, by (9.9), there holds

$$u_t(t,x) > 0 \text{ for all } (t,x) \in [\tau, +\infty) \times [x^- - \sqrt{8a^- T_0}, x^+ + \sqrt{8a^+ T_0}].$$

Together with (9.11) and (9.12), one concludes that

$$u_t(t,x) > 0 \text{ for all } (t,x) \in [\tau, +\infty) \times \mathbb{R}$$

and the proof of Theorem 1.2 is thereby complete. □

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