# Asymptotic spreading in heterogeneous diffusive excitable media

Henri Berestycki [a], François Hamel [b] and Grégoire Nadin [c]

[a] EHESS, CAMS, 54 Boulevard Raspail, F-75006 Paris, France

[b] Université Aix-Marseille III, LATP, Faculté des Sciences et Techniques

Avenue Escadrille Normandie-Niemen, F-13397 Marseille Cedex 20, France

[c] ENS, DMA, 45 rue d'Ulm, F-75230 Paris cedex 05, France

*À Paul Malliavin,*
*en témoignage d'admiration*

**Abstract**

We establish propagation and spreading properties for nonnegative solutions of non homogeneous reaction-diffusion equations of the type:

$$\partial_t u - \nabla \cdot (A(t,x)\nabla u) + q(t,x) \cdot \nabla u = f(t,x,u)$$

with compactly supported initial conditions at $t = 0$. Here, $A, q, f$ have a general dependence in $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^N$. We establish properties of *families of propagation sets* which are defined as families of subsets $(S_t)_{t\geq 0}$ of $\mathbb{R}^N$ such that $\liminf_{t\to+\infty} \{\inf_{x\in S_t} u(t,x)\} > 0$. The aim is to characterize such families as sharply as possible. In particular, we give some conditions under which: 1) a given path $(\{\xi(t)\})_{t\geq 0}$, where $\xi(t) \in \mathbb{R}^N$, forms a family of propagation sets, or 2) $S_t \supset \{x \in \mathbb{R}^N,\ |x| \leq r(t)\}$ and $\lim_{t\to+\infty} r(t) = +\infty$. This second property is called here *complete spreading*. Furthermore, in the case $q \equiv 0$ and $\inf_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^N} f'_u(t,x,0) > 0$, as well as under some more general assumptions, we show that there is a positive spreading speed, that is, $r(t)$ can be chosen so that $\liminf_{t\to+\infty} r(t)/t > 0$. In the general case, we also show the existence of an explicit upper bound $C > 0$ such that $\limsup_{t\to+\infty} r(t)/t < C$. On the other hand, we provide explicit examples of reaction-diffusion equations such that for an arbitrary $\varepsilon > 0$, any family of propagation sets $(S_t)_{t\geq 0}$ has to satisfy $S_t \subset \{x \in \mathbb{R}^N,\ |x| \leq \varepsilon t\}$ for large $t$. In connection with spreading properties, we derive some new uniqueness results for the entire solutions of this type of equations. Lastly, in the case of space-time periodic media, we develop a new approach to characterize the largest propagation sets in terms of eigenvalues associated with the linearized equation in the neighborhood of zero.

**Key-words:** Propagation and spreading properties; Heterogeneous reaction-diffusion equations; Maximum principles; Principal eigenvalues.

## Contents



# 1 Introduction and main results

We are concerned here with qualitative properties of equations of the type:

$$\partial_t u - \nabla \cdot (A(t,x)\nabla u) + q(t,x) \cdot \nabla u = f(t,x,u), \tag{1.1}$$

and more specifically with large time behavior of the solutions of the associated Cauchy problem:

$$\begin{cases} \partial_t u - \nabla \cdot (A(t,x)\nabla u) + q(t,x) \cdot \nabla u = f(t,x,u) \text{ in } \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x) \quad \text{for all} \quad x \in \mathbb{R}^N \end{cases} \tag{1.2}$$

with initial data $u_0 \geq 0$. This equation arises in a wide variety of contexts such as phase transitions, combustion, ecology and many models of biology (for the original motivation in population genetics, see [1, 14, 18]).

The goal of this paper is to study *propagation* and *spreading properties* for this problem. That is, for some classes of initial data $u_0$, we want to characterize sets $S_t \subset \mathbb{R}^N$ such that

$$\liminf_{t\to+\infty} \Big\{ \inf_{x\in S_t} u(t,x) \Big\} \ > \ 0.$$

Such a family of sets will be termed a *family of propagation sets* (or *propagation sets* for short) in the space variables, and the family of their boundaries a *propagation surface*. We are interested in identifying - possibly in a sharp way - such propagation sets. As will be made more precise below, *spreading properties* refer to propagation sets of the form

$$S_t = \{x; x = re, e \in \mathbb{S}^{N-1}, 0 \le r \le r_e(t)\}$$

where $r_e(t)$ is a family of functions parameterized by $e \in \mathbb{S}^{N-1}$. In this case, we say that $r_e(t)$ is a *spreading radius in the direction* $e$. Naturally, the aim is to identify such functions as sharply as possible. We say that *complete spreading occurs* if such a family $(r_e(t))_e$ can be found such that $r_e(t) \to +\infty$ uniformly with respect to $e$ as $t \to +\infty$. This is equivalent to saying that $\liminf_{t\to+\infty} u(t,x) > 0$ locally uniformly in $x \in \mathbb{R}^N$.

## 1.1 Known results in the homogeneous and periodic cases

Before going any further on the precise statements, let us first recall some known results in the homogeneous and periodic cases. Equation (1.2) is indeed the generalization for heterogeneous media of the classical homogeneous equation

$$\partial_t u - \Delta u = f(u), \tag{1.3}$$

where $f(0) = f(1) = 0$ and $f(s) > 0$ if $s \in (0,1)$. This homogeneous equation has been widely studied. One of its main properties is that there exists a minimal speed $c^* > 0$ such that equation (1.3) admits travelling waves solutions, that is, solutions of the form

$$u(t,x) = U(x \cdot e - ct),$$

for all $c \ge c^*$. Here $e$ is the direction of propagation of the wave, $|e| = 1$ (we denote by $|\cdot|$ the Euclidian norm in $\mathbb{R}^N$), $c$ is its speed and $U(+\infty) = 0 < U < U(-\infty) = 1$. A classical result in the homogeneous framework is that the waves with minimal speed $c^*$ attract, in some sense, all the solutions of the Cauchy problem (1.2) with compactly supported nonnegative initial data $u_0 \not\equiv 0$ (see [30]). Furthermore, it was proved [1] that if $u$ is the solution of the Cauchy problem with a non-null compactly supported initial datum and if $\liminf_{s\to 0^+} f(s)/s^{1+2/N} > 0$, then

$$u(t, x + cte) \to 1 \text{ locally uniformly in } x \in \mathbb{R}^N \text{ as } t \to +\infty,$$

for all $0 \le c < c^*$. On the other hand, $u(t, x + cte) \to 0$ as $t \to +\infty$ if $c > c^*$. Thus, an observer who moves with speed $c \ge 0$ in direction $e$ will only see at large times the steady state 1 if $c < c^*$ and the steady state 0 if $c > c^*$. We refer to these results as *spreading properties*. They were first proved for the homogeneous equation (1.3) by Aronson and

Weinberger [1]. The minimal speed of travelling fronts $c^*$ may thus also be viewed as the *asymptotic directional spreading speed* in any direction $e$. Lastly, for KPP nonlinearity, that is a reaction term $f$ such that $f(s) \leq f'(0)s$ for all $s \geq 0$, it is well known that $c^* = 2\sqrt{f'(0)}$ (see [1, 18] for example).

Freidlin and Gärtner [16] in 1979 and Freidlin [15] in 1984 extended the spreading properties to space periodic media and to some classes of random media using probabilistic tools. Here and throughout the paper, when we say that the medium is homogeneous (respectively space periodic, space-time periodic or heterogeneous), we mean that the coefficients $(A, q, f)$ are homogeneous, i.e. do not depend on $(t, x)$ (respectively space periodic, space-time periodic or heterogeneous). Note that in the space periodic case, the coefficients do not depend on $t$. Periodicity is understood to mean the same period(s) for all the terms. If the reaction term $f$ is of KPP type, that is, if $f(x, s) \leq f'_u(x, 0)s$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}_+$ and under further assumptions that will be specified below, it has been proved that there exists an asymptotic directional spreading speed $w^*(e) > 0$ in each direction $e$, in the sense that

$$\begin{cases} \liminf_{t\to+\infty} u(t, x + cte) > 0 & \text{if } 0 \leq c < w^*(e), \\ \lim_{t\to+\infty} u(t, x + cte) = 0 & \text{if } c > w^*(e), \end{cases} \tag{1.4}$$

locally uniformly in $x \in \mathbb{R}^N$. Here, the initial data $u_0$ are supposed to have compact support and to satisfy $u_0 \not\equiv 0$, $u_0 \geq 0$. Furthermore, $w^*(e)$ is characterized by:

$$w^*(e) = \min_{e'\in\mathbb{S}^{N-1},\ e'\cdot e>0} \frac{c^*(e')}{e' \cdot e} \tag{1.5}$$

where the quantity $c^*(e')$ has later been identified in [7, 31] as the minimal speed of pulsating travelling fronts in direction $e'$.

In [31], H. Weinberger generalized the notion of waves to space-time periodic settings, using a rather elaborate discrete formalism. It enabled him to extend the spreading properties to these environments (see also [24] for related results in the case of media with a space-time periodic drift). Since this discrete formalism seems to only fit periodic frameworks, it is of interest to try to derive another approach to these properties, that relies on more general PDE tools and sheds light on more general classes of heterogeneous media. This is one of the goals of the present paper. In fact, in developing a new approach, we also obtain a new way to derive the results regarding the periodic case. This will be presented later on here, in section 4.

## 1.2 The general heterogeneous case and the scope of the paper

The investigation of the properties of solutions of reaction-diffusion equations in general unbounded media is more recent. Berestycki, Hamel and Rossi [9] and Berestycki and Rossi [11] established some existence and uniqueness results for the bounded entire solutions of equation (1.1) in time-independent media, with a general dependence of the coefficients of the equation on the space variable $x \in \mathbb{R}^N$. On the other hand, two definitions for fronts in non homogeneous media have been given by Berestycki and Hamel in [4, 5] and by Matano in [20].

Using Matano's definition, Shen proved the existence of generalized fronts in one-dimensional media with bistable nonlinearity in [29]. Using Berestycki and Hamel's definition, it has been proved by Nolen and Ryzhik in [25] and Mellet and Roquejoffre in [21] that such fronts exist in one-dimensional and time-independent media with ignition-type nonlinearity. The case of "random stationary" drifts has recently been investigated by Nolen and Xin in [26, 27, 28], where the existence of a deterministic speeds is proved (see also Nolen and Ryzhik in [25] for ignition-type nonlinearities).

Spreading properties in space general media have first been investigated by Berestycki, Hamel and Nadirashvili in [6, 8] in the case where the coefficients in the equation are homogeneous but the equation is set in a general unbounded domain which is neither the space, nor a periodic domain. In these articles, it was proved, among other things, that usual spreading properties may not hold for some particular unbounded domains. That is, for instance, the asymptotic spreading speed $w^*(e)$ in a direction $e$, as characterized by (1.4), may be equal to 0 or to $+\infty$. Actually, there are several natural notions of asymptotic directional spreading speeds, defined in $[0, +\infty]$. Their dependence on the geometry of the underlying domain is rather intricate and the analysis of their properties is the purpose of the paper [8].

The scope of the present paper is complementary to [8]. It is to study propagation sets and spreading properties for the solutions of the space-time heterogeneous Cauchy problem (1.2) set in all of space $\mathbb{R}^N$. Our purpose here is to go far beyond the space, time or space-time periodic cases. We give some conditions, depending on generalized eigenvalues or on the coefficients of (1.2), under which complete spreading occurs for the solutions $u$ with non-null initial conditions, in the sense that $\liminf_{t\to+\infty} \left\{\inf_{|x|\leq r(t)} u(t,x)\right\} > 0$, where $\lim_{t\to+\infty} r(t) = +\infty$ (see Theorem 1.3 and Corollary 1.4 below). We also get lower and upper bounds, which are optimal in the homogeneous case, for the quantities $r(t)/t$ as $t \to +\infty$ (see Theorems 1.5 and 1.10). In addition, we construct an explicit example for which any such spreading radius $r(t)$ satisfies $\lim_{t\to+\infty} r(t)/t = 0$ (see Theorem 1.11). The results generalize and go beyond the previous ones. One of the main outcomes here with respect to the homogeneous case is to show that certain conditions need only be imposed at infinity to derive the spreading properties.

We also get new types of results which deal with the more general notion of family of propagation sets $(S_t)_{t\geq 0}$, that is sets for which $\liminf_{t\to+\infty} \{\inf_{x\in S_t} u(t,x)\} > 0$. Furthermore, under some additional assumptions, we derive the existence and the uniqueness of the limit of $u(t,x)$ in the propagation sets as $t \to +\infty$ (see Theorem 1.6 and Propositions 1.7 and 1.8). It is important to have in mind that the sets $S_t$ may not be balls centered at the origin. As a matter of fact, we first give in Theorem 1.2 some sufficient conditions for a given family of singletons $(\{\xi(t)\})_{t\geq 0}$ to be propagation sets.

In the last part of the paper, we show how the ideas developed for general non homogeneous media yield a new approach to the precise description of directional asymptotic spreading speeds in space-time periodic media (see Theorem 1.13 and Corollary 1.15). These results had been established by Freidlin [15] for space periodic settings and by Weinberger [31] in space-time periodic media. Our approach relies on classical PDE techniques. In order to extend these properties from one-dimensional media to multidimensional media, we prove a new approximation result for the spreading speed, that also enables us to go back to the case of straight infinite cylinder with bounded cross section. Before the space-time periodic case, we will first present the argument in the space-periodic framework where our method provides a simplified and more transparent approach.

## 1.3 General hypotheses

Some regularity assumptions will be required on $f$, $A$, $q$ throughout the paper. For problem (1.2), these quantities only need to be defined for $t \geq 0$. But in some results we will consider solutions of (1.1) which are defined for all $t \in \mathbb{R}$. This is why, for the sake of simplicity, all functions $f$, $A$ and $q$ will from now on be defined for all $t \in \mathbb{R}$.

The function $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to be of class $C^{\frac{\delta}{2},\delta}$ in $(t,x)$, locally in $s$, for a given $0 < \delta < 1$. We also assume $f$ to be locally Lipschitz-continuous in $s$ and of class $C^1$ in $s$ for $s \in [0,\beta]$ with $\beta > 0$ uniformly with respect to $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Lastly, we assume that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, one has $f(t,x,0) = 0$.

The drift term $q : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is in the class $C^{\frac{\delta}{2},\delta}(\mathbb{R} \times \mathbb{R}^N) \cap L^\infty(\mathbb{R} \times \mathbb{R}^N)$. The matrix field $A : \mathbb{R} \times \mathbb{R}^N \to S_N(\mathbb{R})$ is of class $C^{\frac{\delta}{2},1+\delta}(\mathbb{R} \times \mathbb{R}^N)$. We also assume that $A$ is uniformly elliptic and continuous. There exist some positive constants $\gamma$ and $\Gamma$ such that for all $\xi \in \mathbb{R}^N$, $(t,x) \in \mathbb{R} \times \mathbb{R}^N$:

$$\begin{aligned} \gamma(t,x)|\xi|^2 \leq \textstyle\sum_{1\leq i,j\leq N} a_{i,j}(t,x)\xi_i\xi_j \leq \Gamma(t,x)|\xi|^2, \\ \text{and} \quad 0 < \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \gamma(t,x) \leq \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \Gamma(t,x) < \infty. \end{aligned} \tag{1.6}$$

From the parabolic maximum principle, it follows that, for any measurable non-null and nonnegative function $u_0 \in L^\infty(\mathbb{R}^N)$ *– from now on, we only consider such initial conditions –*, the solution $u$ of (1.2) is of class $\mathcal{C}^{1,2}((0,+\infty) \times \mathbb{R}^N)$ and it is nonnegative.

Throughout the paper, for any $s > 0$ and $y \in \mathbb{R}^N$, we denote by $B_s(y)$ the open euclidean ball of centre $y$ and radius $s$. We set $B_s = B_s(0)$.

## 1.4 Propagation sets and local propagation along a path

Our first goal is to find some paths $t \mapsto \xi(t) \in \mathbb{R}^N$ along which a solution of the Cauchy problem (1.2) does not converge to 0, that is the family of singletons $(\{\xi(t)\})_{t\geq 0}$ is a family of propagation sets. We call such a property local propagation along a path, as read in the definition below. We first define the general definition of propagation sets, and then we consider the particular case of local propagation along a path.

**Definition 1** *We say that a family $(S_t)_{t\geq 0}$ of subsets of $\mathbb{R}^N$ is a* (family of) propagation sets *for the solution $u$ of equation* (1.2) *if*

$$\liminf_{t\to+\infty} \Big\{ \inf_{x\in S_t} u(t,x) \Big\} \ > \ 0.$$

**Definition 2** 1) *We say that $t \in \mathbb{R}_+ \mapsto \xi(t) \in \mathbb{R}^N$ is an admissible path if $t \mapsto \xi(t) \in \mathcal{C}^{1+\delta/2}(\mathbb{R}_+;\mathbb{R}^N)$ and $\sup_{t\geq 0} |\xi'(t)| < +\infty$.*

2) *We say that there is* local propagation *of a solution $u$ of equation* (1.2) *along an admissible path $\xi$ if the family $(\{\xi(t)\})_{t\geq 0}$ is a family of propagation sets, that is*

$$\liminf_{t\to+\infty} u(t,\xi(t)) > 0. \tag{1.7}$$

The limit (1.7) is in fact locally uniform in $\mathbb{R}^N$ and thus, the path $\xi$ can be thought of as defined up to some bounded perturbation. This is made precise in the next statement.

**Lemma 1.1** *Assume that there is local propagation of a solution $u$ of* (1.2) *along an admissible path $\xi$, then for all admissible path $\widetilde{\xi}$ such that* $\sup_{t\geq 0}|\xi(t)-\widetilde{\xi}(t)| < +\infty$*, there is local propagation of the solution $u$ along the path $\widetilde{\xi}$. More generally, for all $R>0$ and $x_0\in\mathbb{R}^N$, the family $(B_R(\xi(t)+x_0))_{t\geq 0}$ is a family of propagation sets, that is:*

$$\liminf_{t\to+\infty}\Big\{\inf_{y\in B_R(x_0)} u(t,\xi(t)+y)\Big\} > 0.$$

**Proof.** Observe first that $u$ is a solution of a linear parabolic equations. Indeed, it suffices to write $f(x,u)=c(t,x)u$ where $c(t,x)=f(t,x,u(t,x))/u(t,x)$. Assume that $|\xi(t)-\widetilde{\xi}(t)|\leq R$ for all $t\geq 0$. As $\widetilde{\xi}'$ is uniformly bounded in $\mathbb{R}_+$, there exists a constant $C>0$ such that for all $t\geq 0$, $|\widetilde{\xi}(t+1)-\widetilde{\xi}(t)|\leq C$. From the Harnack inequality, there exists a positive constant $\alpha$ such that

$$\forall\, x\in B_{R+C},\ \forall\, t\geq 1,\ u(t+1,x+\xi(t))\geq \alpha\, u(t,\xi(t)).$$

Thus, as $\widetilde{\xi}(t+1)\in B_{R+C}(\xi(t))$ for all $t\geq 0$, one has

$$\liminf_{t\to+\infty} u(t+1,\widetilde{\xi}(t+1)) \geq \alpha \liminf_{t\to+\infty} u(t,\xi(t)) > 0,$$

which yields the result. □

When the path $t\mapsto\xi(t)$ along which propagation occurs is bounded (that is, when $\xi(t)$ can be assumed to be constant thanks to Lemma 1.1), then we say that there is *persistence* in the stationary frame. This is equivalent to saying that

$$\liminf_{t\to+\infty}\Big\{\inf_{|x|\leq R} u(t,x)\Big\} > 0 \quad \text{for all} \quad R>0.$$

Some conditions for persistence in the stationary frame are given in [9] for time-independent media. On the other hand, there may be propagation along an unbounded path but not persistence in the stationary frame (see [12, 13]).

Let us now look for conditions that guarantee propagation along a given path $t\mapsto\xi(t)$. In this paper, we will derive such conditions in two different contexts: that of general media with a positivity condition on the coefficients in the neighborhood of the path and that of space-time periodic coefficients.

**Theorem 1.2** *Assume that $t\mapsto\xi(t)$ is an admissible path such that:*

$$\liminf_{R\to+\infty}\left\{\liminf_{t\to+\infty}\left[\inf_{|x|\leq R}\left(4\gamma(t,x+\xi(t))f_u'(t,x+\xi(t),0)-|q(t,x+\xi(t))-\xi'(t)|^2\right)\right]\right\} > 0. \tag{1.8}$$

*Then, for any solution $u$ of the Cauchy problem* (1.2) *associated with an initial datum $u_0$, then there is local propagation of the solution $u$ along the path $\xi$.*

A few words of discussion of condition (1.8) may be useful to grasp its meaning. If the path $\xi$ is constant or is bounded, (1.8) means that there is $\delta > 0$ such that, for each $R > 0$, there holds $4\gamma(t,\cdot)f'_u(t,\cdot,0) - |q(t,\cdot)|^2 \geq \delta$ in $B_R$ for $t$ large enough. However, here, we are mostly interested in the general case where $|\xi(t)| \to +\infty$ as $t \to +\infty$, in which case (1.8) means that there is $\delta > 0$ such that, for each $R > 0$, there holds

$$4\gamma(t,\cdot+\xi(t))f'_u(t,\cdot+\xi(t),0) - |q(t,\cdot+\xi(t)) - \xi'(t)|^2 \geq \delta \text{ in } B_R$$

for $t$ large enough.

Actually, in Theorem 1.2, it is not possible to go further and to prove that the function $t \mapsto u(t, x+\xi(t))$ converges in general as $t \to +\infty$ for a given $x \in \mathbb{R}^N$. It has been proved recently in [2] that such a function may oscillate between two travelling fronts in homogeneous media.

## 1.5 Spreading properties

We now discuss *spreading properties*. We first define the notion of complete spreading.

**Definition 3** *We say that* complete spreading occurs *for a solution $u$ of* (1.2) *if there is a function $t \mapsto r(t) > 0$ such that $r(t) \to +\infty$ as $t \to +\infty$ and the family $(B_{r(t)})_{t\geq 0}$ is a family of propagation sets for $u$, that is*

$$\liminf_{t\to+\infty} \Big\{ \inf_{x\in B_{r(t)}} u(t,x) \Big\} > 0.$$

This definition corresponds to the natural notion of uniform spreading in all directions from the origin (or equivalently from any point in $\mathbb{R}^N$). However, it is of interest to introduce a more precise notion of spreading radius along a given direction $e$.

**Definition 4** *Let $e \in \mathbb{S}^{N-1}$ be given. We say that a family $(r_e(t))_{t\geq 0}$ of nonnegative real numbers is* a family of asymptotic spreading radii in the direction $e$ *for a solution $u$ of* (1.2) *if the family of segments $([0, r_e(t)e])_{t\geq 0}$ is a family of propagation sets for $u$, that is*

$$\liminf_{t\to+\infty} \Big\{ \inf_{0\leq s\leq r_e(t)} u(t,se) \Big\} > 0.$$

Lastly, we define the class of admissible radii $(r_e)_{e\in\mathbb{S}^{N-1}}$ that will be considered in the sequel.

**Definition 5** *We say that a family $(r_e)_{e\in\mathbb{S}^{N-1}}$ is a family of admissible radii if $r_e \in \mathcal{C}^{1+\frac{\delta}{2}}(\mathbb{R}_+,\mathbb{R}_+)$ for all $e \in \mathbb{S}^{N-1}$ and*

$$\begin{cases} \displaystyle\sup_{e\in\mathbb{S}^{N-1},\ t\in\mathbb{R}_+} |r'_e(t)| < +\infty, \\ \forall\,(e,e') \in \mathbb{S}^{N-1}\times\mathbb{S}^{N-1},\ \forall\, t\in\mathbb{R}_+, \quad r_e(t)e\cdot\xi \leq r_{e'}(t). \end{cases}$$

*For all $t \in \mathbb{R}_+$, we define the set associated with these radii as*

$$S_t = \{x \in \mathbb{R}^N;\ x\cdot e \leq r_e(t) \text{ for all } e \in \mathbb{S}^{N-1}\}.$$

The hypothesis $r_e(t)e \cdot e' \leq r_{e'}(t)$ is equivalent to $r_e(t)e \in S_t$ for all $e$ and $t$. Thus this hypothesis guarantees that $\partial S_t = \{r_e(t)e, e \in \mathbb{S}^{N-1}\}$. We can remark that if $r_e(t)$ does not depend on $e$, then this hypothesis is always true, but we will see in the sequel that it may be relevant to consider radii that depend on the direction $e$.

We shall use here the *generalized principal eigenvalue* associated with the linearization in the neighborhood of 0 of equation (1.1). This generalized principal eigenvalue is defined as

$$\lambda_1' = \inf\{\lambda \in \mathbb{R},\ \exists\ \phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N),\ \inf_{\mathbb{R}\times\mathbb{R}^N} \phi > 0,\ \mathcal{L}\phi \leq \lambda\phi\}, \qquad (1.9)$$

where for all $\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$:

$$\mathcal{L}\phi = \partial_t\phi - \nabla\cdot(A(t,x)\nabla\phi) + q(t,x)\cdot\nabla\phi - f_u'(t,x,0)\phi.$$

Related notions were defined in [9, 10, 11] for time-independent problems.

With the assumption $\lambda_1' < 0$, that is, in a sense, that the equilibrium 0 is unstable, our first main result in general media is the following one:

**Theorem 1.3** *Let $u$ be the solution of the Cauchy problem* (1.2) *associated with an initial datum $u_0$. Assume that $\lambda_1' < 0$ and that there exists a family $(r_e)_e$ of admissible radii such that:*

$$\liminf_{t\to+\infty} \big\{ \inf_{e\in\mathbb{S}^{N-1}} u(t, r_e(t)e)\big\} > 0. \qquad (1.10)$$

*Then*

$$\liminf_{t\to+\infty} \big\{ \inf_{x\in S_t} u(t,x)\big\} > 0, \qquad (1.11)$$

*where $S_t$ is the set associated with the family $(r_e)_e$ as in Definition* 5*. In other words, the family $(S_t)_{t\geq 0}$ is a family of propagation sets for the solution $u$.*

Gathering Theorems 1.2 and 1.3, one immediately gets:

**Corollary 1.4** *Assume that $\lambda_1' < 0$ and that there exists a family of admissible radii $(r_e)_e$ such that*

$$\liminf_{R\to+\infty}\Big\{\liminf_{t\to+\infty}\Big[\inf_{x\in B_R}\inf_{e\in\mathbb{S}^{N-1}}\big(4\gamma(t,x+r_e(t)e)f_u'(t,x+r_e(t)e,0) - |q(t,x+r_e(t)e) - r_e'(t)e|^2\big)\Big]\Big\} > 0.$$

*Let $u$ be the solution of the Cauchy problem* (1.2) *associated with an initial datum $u_0$. Then, property* (1.11) *holds for $u$, where $S_t$ is the set associated with the family $(r_e)_e$ as in Definition* 5*.*

The difficulty with this corollary is that its hypothesis depends on the family of admissible vectors $r_e(t)e$ along which propagation occurs. Under an appropriate hypothesis of the coefficients at infinity, we get our second main result on spreading properties in general media:

**Theorem 1.5** *Assume that $\lambda_1' < 0$ and that*

$$\liminf_{|x|\to+\infty} \Big\{ \inf_{t\in\mathbb{R}_+} \big(4\gamma(t,x) f_u'(t,x,0). - |q(t,x)|^2\big)\Big\} > 0, \tag{1.12}$$

*Set*

$$c^* = \liminf_{|x|\to+\infty} \Big\{ \inf_{t\in\mathbb{R}_+} \big(2\sqrt{\gamma(t,x) f_u'(t,x,0)} - |q(t,x)|\big)\Big\} > 0.$$

*Then for all speed $0 \le c < c^*$ and for any solution $u$ of the Cauchy problem* (1.2) *associated with an initial datum $u_0$, the family $(B_{ct})_{t\ge 0}$ is a family of propagation sets, that is*

$$\liminf_{t\to+\infty} \Big\{ \inf_{|x|\le ct} u(t,x)\Big\} > 0.$$

Note that hypothesis (1.12) is checked when $q \equiv 0$ and

$$\liminf_{|x|\to+\infty} \Big\{ \inf_{t\in\mathbb{R}_+} f_u'(t,x,0)\Big\} > 0.$$

In homogeneous media, if $q \equiv 0$ and $A = \gamma I_N$ (where $I_N$ is the identity matrix), the previous theorem yields the speed $c^* = 2\sqrt{\gamma f'(0)}$, which is the minimal speed of existence of planar travelling waves for KPP nonlinearities. This speed is optimal for KPP nonlinearities, but not for other nonlinearities. Neither is it optimal in space periodic media, as it has been observed in [7]: there exist some speeds $c > 2\sqrt{\gamma \inf_{x\in\mathbb{R}^N} f_u'(x,0)}$ such that $\liminf_{t\to+\infty} \sup_{|x|\le ct} u(t,x) > 0$, even for KPP nonlinearities.

## 1.6 Convergence to an entire solution

The next result state that if equation (1.1) admits a unique (in some sense) uniformly positive entire solution, then the spreading properties not only imply the instability of 0 along a given surface, but also the convergence to this entire solution. We thus assume that there exists an entire solution $p \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ of equation (1.1) which is uniformly positive, that is, that $\inf_{\mathbb{R}\times\mathbb{R}^N} p > 0$. We will need some uniqueness hypotheses for the entire solutions of all the translations of equation (1.1).

**Hypothesis 1** *Consider any coefficients $(B, r, g)$ such that there exists some sequence $(t_n, x_n) \in \mathbb{R}\times\mathbb{R}^N$ such that for some $0 < \delta' < \delta$, one has:*

$$\begin{array}{rclcccl} A(t+t_n, x+x_n) & \to & B(t,x) & \textit{as} & n\to+\infty & \textit{in} & \mathcal{C}_{loc}^{\frac{\delta'}{2},1+\delta'}(\mathbb{R}\times\mathbb{R}^N), \\ q(t+t_n, x+x_n) & \to & r(t,x) & \textit{as} & n\to+\infty & \textit{in} & \mathcal{C}_{loc}^{\frac{\delta'}{2},\delta'}(\mathbb{R}\times\mathbb{R}^N), \\ f(t+t_n, x+x_n, s) & \to & g(t,x,s) & \textit{as} & n\to+\infty & \textit{in} & \mathcal{C}_{loc}^{\frac{\delta'}{2},\delta',0}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}_+), \end{array} \tag{1.13}$$

*locally uniformly with respect to $s$. Consider any positive entire solution $v \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ of*

$$\partial_t v - \nabla\cdot(B\nabla v) + r\cdot\nabla v = g(t,x,v) \textit{ in } \mathbb{R}\times\mathbb{R}^N,$$

*such that:*

$$\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} v(t,x) > 0.$$

*We assume that for all such $(B, r, g)$ and $v$, one has*

$$p(t+t_n, x+x_n) \to v(t,x) \text{ as } n \to +\infty \quad \text{in } \mathcal{C}^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^N).$$

Under this hypothesis, we are able to improve the result of Theorem 1.2:

**Theorem 1.6** *Assume that there exists an entire solution $p \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ of equation* (1.1) *which is uniformly positive and satisfies Hypothesis* 1*. Let $u$ be the solution of the Cauchy problem* (1.2) *associated with an initial datum $u_0$. Assume that $\lambda_1' < 0$, where $\lambda_1'$ is defined by* (1.9)*, and that there exists a family of admissible radii $(r_e)_e$ as in Definition* 5 *such that*

$$\liminf_{t\to+\infty}\Big\{\inf_{e\in\mathbb{S}^{N-1}} r_e(t)\Big\} = +\infty$$

*and*

$$\liminf_{t\to+\infty}\Big\{\inf_{x\in S_t} u(t,x)\Big\} > 0.$$

*Then, for all $\varepsilon \in (0,1)$, there holds:*

$$\lim_{t\to+\infty}\Big\{\sup_{x\in\mathcal{S}_\varepsilon(t)} |u(t,x)-p(t,x)|\Big\} = 0,$$

*where $\mathcal{S}_\varepsilon(t) = \{x\in\mathbb{R}^N,\ x\cdot e \le (1-\varepsilon)r_e(t) \text{ for all } e\in\mathbb{S}^{N-1}\}$.*

If Hypothesis 1 is not satisfied, then this theorem is not true anymore. The translated function $(t,x)\mapsto u(t, x+(1-\varepsilon)r_e(t)e)$ may a priori oscillate between two entire solutions. In subsection 1.7, we give some conditions which guarantee that Hypothesis 1 is satisfied.

It is straightforward to see that, even under Hypothesis 1, the convergence of $u$ to $p$ as $t\to+\infty$ may not hold in the whole set $S_t$. For example, if $N=1$, $A=1$, $q=0$ and $f(u)=u(1-u)$, for all initial datum $u_0$ with compact support, the solution $u$ converges to 1 locally in $x$ as $t\to+\infty$, while $u(t,\pm\infty)=0$ for each time $t\ge 0$. Set

$$r(t) = \inf\{x\in\mathbb{R},\ \forall\, y\ge x,\ u(t,y)\le 1/2\}.$$

Then $r(t)\to+\infty$ as $t\to+\infty$ and there is local propagation of the solution $u$ along the path $t\mapsto r(t)$ since $u(t,r(t))=1/2$ for all $t\ge 0$. Furthermore, the unique entire solution $p$ of (1.1) which is uniformly positive is identically equal to 1 (this will be stated in Subsection 1.7). But $u(t,r(t))\not\to 1$ as $t\to+\infty$.

## 1.7 Uniqueness of the entire solutions

Hypothesis 1 is far from being easy to check. We now give two sets of easily checkable hypotheses that guarantee these existence and uniqueness hypotheses. Our first set of hypotheses is:

$$\forall\, s_2 > s_1 > 0, \quad \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N}\left(\frac{f(t,x,s_1)}{s_1} - \frac{f(t,x,s_2)}{s_2}\right) > 0, \tag{1.14}$$

$$\exists\, M>0,\ \forall\,(t,x)\in\mathbb{R}\times\mathbb{R}^N,\ \forall\, s\ge M,\quad f(t,x,s)\le 0. \tag{1.15}$$

These hypotheses are relevant for biological models. The first hypothesis means that the intrinsic growth rate uniformly decreases when the population density increases. This is the result of the intraspecific competition for resources. The second hypothesis means that there is a saturation effect: when the population is very important, the mortality rate is higher than the birth rate and the population decreases.

Under these two hypotheses, the following existence and uniqueness results hold:

**Proposition 1.7** 1) *Assume that $\lambda_1' < 0$, where $\lambda_1'$ is defined by* (1.9), *and that* (1.15) *is satisfied, then there exists at least one positive bounded entire solution $p \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ of* (1.1) *such that*

$$\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} p(t,x) > 0. \tag{1.16}$$

2) *If* (1.14) *holds, there exists at most one nonnegative bounded entire solution $p \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ of* (1.1) *satisfying* (1.16).

3) *If* (1.14) *and* (1.15) *hold and if $\lambda_1' < 0$, then Hypothesis* 1 *is satisfied.*

Our second set of hypotheses is relevant for combustion models. Namely, we assume that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$,

$$\begin{cases} f(t,x,1) &= 0, \\ \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} f(t,x,s) &> 0 \text{ if } 0 < s < 1, \\ \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} f(t,x,s) &< 0 \text{ if } s > 1. \end{cases} \tag{1.17}$$

**Proposition 1.8** *Assume that* (1.17) *is satisfied. If $p \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ is a nonnegative bounded entire solution of* (1.1) *such that $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} p(t,x) > 0$, then $p \equiv 1$. Furthermore, Hypothesis* 1 *is satisfied.*

## 1.8 Upper bounds for the spreading speeds

We now give some upper bounds for the asymptotic directional spreading speeds, that is, for the ratios $\limsup_{t\to+\infty} r_e(t)/t$, where $e$ is any given direction and $t \mapsto r_e(t)e$ is any path of local propagation of a solution $u$ of (1.2) with compactly supported initial condition. We assume here that:

$$\sup_{(t,x,s)\in\mathbb{R}\times\mathbb{R}^N\times(0,+\infty)} \frac{f(t,x,s)}{s} < \infty. \tag{1.18}$$

Set

$$\eta(t,x) = \sup_{s>0} \frac{f(t,x,s)}{s} \tag{1.19}$$

and $P_\lambda\phi = e^{\lambda\cdot x}\mathcal{P}(e^{-\lambda\cdot x}\phi)$ for any $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ and $\lambda \in \mathbb{R}^N$, where

$$\mathcal{P}\psi = \partial_t\psi - \nabla\cdot(A(t,x)\nabla\psi) + q(t,x)\cdot\nabla\psi - \eta(t,x)\psi.$$

In this subsection, we use the following notion of generalized principal eigenvalue for the operator $P_\lambda$, which is slightly different from (1.9). Namely, we set

$$k_\lambda(\eta) = \sup\{k > 0,\ \exists\, \phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R} \times \mathbb{R}^N),\ \inf_{\mathbb{R}\times\mathbb{R}^N} \phi > 0,\ P_\lambda\phi \geq k\phi\}. \tag{1.20}$$

As we shall see in Subsection 1.10, in the space-time periodic case, it is possible to identify this eigenvalue as the space-time periodic principal eigenvalue associated with the coefficients $(A, q, \eta)$. However, in the general case, the following results still hold:

**Proposition 1.9** *Assume that* (1.18) *holds. Then, for each $\lambda \in \mathbb{R}^N$, the quantity $k_\lambda(\eta)$ is a real number.*

**Theorem 1.10** *Assume that* (1.18) *holds and that $k_\lambda(\eta) < 0$ for all $\lambda \in \mathbb{R}^N$. Set*

$$w^{**}(e) = \inf_{\lambda \in \mathbb{R}^N,\ \lambda \cdot e > 0} \frac{-k_\lambda(\eta)}{\lambda \cdot e}$$

*Then, for all solution $u$ of* (1.2) *with a compactly supported initial condition $u_0$, and for all $w > w^{**}(e)$, there holds:*

$$\lim_{t \to +\infty} u(t, wte) = 0.$$

## 1.9 Complete spreading in sublinearly growing balls

In this subsection, we give an example of an equation for which the spreading speed is 0 in all directions but for which complete spreading occurs. More precisely, we prove that for all map $t \mapsto r(t)$ such that $\lim_{t \to +\infty} r'(t) = 0$, there exist some equations for which the balls $(B_{(1-\varepsilon)r(t)})_{t \geq 0}$ are a family of propagation sets for all $0 < \varepsilon \leq 1$, while the solutions $u$ converge to 0 as $t \to +\infty$ outside the balls $B_{ct}$ for all $c > 0$.

**Theorem 1.11** *Assume that $N = 1$, $A \equiv 1$, $q \equiv 0$ and $f(t, x, s) = g(r(t) - |x|)s(1 - s)$, where $g$ is a Hölder-continuous function such that $g(-\infty) < 0$, $g(+\infty) > 0$ and $r$ is a Lipschitz-continuous function such that $\lim_{t \to +\infty} r(t) = +\infty$ and $\lim_{t \to +\infty} r'(t) = 0$. Let $u$ be the solution of* (1.2) *with a compactly supported initial datum $u_0$ such that $0 \leq u_0 \leq 1$. Then*

$$\limsup_{t \to +\infty} \Big\{ \sup_{|x| \geq ct} |u(t, x)| \Big\} = 0 \quad \textit{for all } c > 0$$

*and*

$$\liminf_{t \to +\infty} \Big\{ \inf_{|x| \leq (1-\varepsilon) r(t)} u(t, x) \Big\} > 0 \quad \textit{for all } 0 < \varepsilon \leq 1.$$

## 1.10 The periodic case

In *space periodic* and *space-time periodic* media, more precise estimates on the propagation sets and asympotic spreading speeds of the solutions $u$ of (1.2) are available.

We first define what we mean by periodicity. We say that $f$, $A$ and $q$ are space-time periodic if there exist a positive constant $T$ and some vectors $L_1, ..., L_N$, where $L_i \neq 0$ is colinear to the axis of coordinates $e_i$, such that for all $i \in [1, N]$, for all $(t, x, u) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}_+$, one has:

$$\left\{ \begin{array}{ccccc} A(t, x + L_i) & = & A(t + T, x) & = & A(t, x), \\ f(t, x + L_i, u) & = & f(t + T, x, u) & = & f(t, x, u), \\ q(t, x + L_i) & = & q(t + T, x) & = & q(t, x). \end{array} \right. \tag{1.21}$$

In the sequel, space-time periodicity will always refer to these fixed space-time periods. We will say that the coefficients $(A, q, f)$ are space periodic if they satisfy (1.21) and if they do not depend on the time variable $t$. We set

$$\mu(t,x) = f'_u(t,x,0)$$

and, for all $\lambda \in \mathbb{R}^N$ and $\psi \in \mathcal{C}^{1,2}_{per}(\mathbb{R}\times\mathbb{R}^N)$, we define:

$$L_\lambda \psi = \partial_t \psi - \nabla\cdot(A\nabla\psi) + 2\lambda A\nabla\psi + q\cdot\nabla\psi - (\lambda A\lambda - \nabla\cdot(A\lambda) + \mu + q\cdot\lambda)\psi. \tag{1.22}$$

**Definition 6** *A* space-time periodic principal eigenfunction *of the operator $L_\lambda$ is a function $\psi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ such that there exists $k \in \mathbb{R}$ with:*

$$\begin{cases} L_\lambda\psi &= k\psi \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \psi &> 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \psi & \text{is space-time periodic.} \end{cases} \tag{1.23}$$

*Such a real number $k$ is called a principal eigenvalue.*

This family of eigenvalues has been investigated in [23], where it is proved that there exists a couple $(k, \psi)$ that satisfies (1.23). Furthermore, $k$ is unique and $\psi$ is unique up to multiplication by a positive constant and we define $k_\lambda(\mu) = k$ the space-time periodic principal eigenvalue associated with $L_\lambda$. Thus, if the coefficients are only space periodic, the principal eigenfunction $\psi$ does not depend on $t$ and is only space periodic.

In Subsection 1.8, we defined the notion of generalized principal eigenvalue $k_\lambda(\eta)$ for some operators $P_\lambda$, as defined by (1.20) – with $\eta$ instead of $\mu$. Actually, in the space-time periodic case, these generalized eigenvalues coincide with the space-time periodic principal eigenvalues:

**Proposition 1.12** *If $A$, $q$ and $\mu$ are space-time periodic, then, for each $\lambda \in \mathbb{R}^N$, the space-time periodic principal eigenvalue $k_\lambda(\mu)$ defined by formula* (1.23) *is equal to the generalized principal eigenvalue defined in* (1.20) *– with $\mu$ instead of $\eta$.*

The next result gives an estimate of the asymptotic spreading radii in all directions $e$ in the space-time periodic case.

**Theorem 1.13** *Assume that $A$, $q$ and $\mu$ are space-time periodic and that $k_\lambda(\mu) < 0$ for all $\lambda \in \mathbb{R}^N$. Define*

$$w^*(e) = \inf_{\lambda\in\mathbb{R}^N,\ \lambda\cdot e>0} \frac{-k_\lambda(\mu)}{\lambda\cdot e} \tag{1.24}$$

*for all $e \in \mathbb{S}^{N-1}$, and*

$$\mathcal{S} = \{x \in \mathbb{R}^N,\ e\cdot x < w^*(e) \text{ for all } e \in \mathbb{S}^{N-1}\}.$$

*Then the infimum in* (1.24) *is reached and $w^*(e) > 0$ for all $e \in \mathbb{S}^{N-1}$. Furthemore, for any solution $u$ of the Cauchy problem* (1.2) *and for any compact subset $K$ of $\mathcal{S}$, the family $(tK)_{t\geq 0}$ is a family of propagation sets for $u$, that is*

$$\liminf_{t\to+\infty} \big\{ \inf_{x\in tK} u(t,x) \big\} > 0. \tag{1.25}$$

This theorem has first been proved in space periodic media by Freidlin and Gärtner [16] and Freidlin [15]. It has been extended to space-time periodic media by Weinberger in [31]. The set $\mathcal{S}$ describes the shape of the invasion. We give here two alternative proofs of this result. We first prove this result for space periodic media. The method that is used may be extended to space-time periodic media but we choose to give still another proof for such media, that includes auxiliary results of independent interest.

Theorem 1.13 implies that, for each solution $u$ of (1.2), for each direction $e$ and for each $w \in \mathbb{R}$ such that $0 \le w < w^*(e)$, the family $(wt)_{t\ge 0}$ is a family of asymptotic spreading radii in the direction $e$, in the sense of Definition 4. In particular, there is local propagation of the solution $u$ along the path $t \mapsto wte$ as soon as $0 \le w < w^*(e)$, that is $\liminf_{t\to+\infty} u(t, wte) > 0$.

We now assume that $N = 1$. In this case, the notion of spreading speeds can still be defined, as stated in the result below, provided that $k_0(\mu) < 0$, but the speeds may not be positive. We can thus weaken the hypotheses of the previous theorem.

**Theorem 1.14** *Assume that $N = 1$, that $A$, $q$ and $\mu$ are space-time periodic and that $k_0(\mu) < 0$. Then, for any solution $u$ of the Cauchy problem* (1.2) *and for any compact subset $K \subset (-w^*(-e_1), w^*(e_1))$, property* (1.25) *holds.*

The main difference here is that one can have $k_\lambda(\mu) > 0$ for some $\lambda \in \mathbb{R}$. Assume that such a $\lambda$ is positive, then one gets $w^*(e_1) < 0$. Furthermore, for KPP nonlinearities such that $f(t, x, s) \le f'_u(t, x, 0)s$, it easily follows, as in the proof of Theorem 1.10, that $u(t, x) \to 0$ as $t \to +\infty$ locally in $x \in \mathbb{R}^N$. This means that the population is blown away. In other words, a standing observer sees the extinction of the population and one really has to follow the population in order to see the growing effect.

If $N \ge 2$ and $k_{\alpha e}(\mu) > 0$ for some $\alpha > 0$ and $e \in \mathbb{S}^{N-1}$, then the quantity $w^*(e')$ is not a real number for each direction $e'$ such that $e' \cdot e = 0$. Indeed, take a sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^{N-1}$ such that $e_n \to e$ as $n \to +\infty$ and $e_n \cdot e' > 0$ for all $n$. For $n$ large enough, one has $k_{\alpha e_n}(\mu) > k_{\alpha e}(\mu)/2 > 0$, whence

$$w^*(e') \le \frac{-k_{\alpha e_n}(\mu)}{\alpha\, e_n \cdot e'} \to -\infty.$$

Lastly, in the KPP case, as a corollary of Proposition 1.12 and Theorems 1.10 and 1.13, we get that the speed $w^*(e)$ is the optimal asymptotic spreading speed in the direction $e$, in the sense that:

**Corollary 1.15** *If $A$, $q$ and $\mu$ are space-time periodic, if $f$ is of KPP type, that is, $f(t, x, s) \le f'_u(t, x, 0)s$ for all $(t, x, s) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+$, and if $k_\lambda(\mu) < 0$ for all $\lambda \in \mathbb{R}^N$, then*

$$w^*(e) = w^{**}(e) = \min_{\lambda\in\mathbb{R}^N,\ \lambda\cdot e>0} \frac{-k_\lambda(\mu)}{\lambda \cdot e}$$

*is such that, for any solution $u$ of* (1.2) *with a compactly supported initial condition $u_0$, there holds: $\liminf_{t\to+\infty} u(t, wte) > 0$ for all $0 \le w < w^*(e)$, and $\lim_{t\to+\infty} u(t, wte) = 0$ for all $w > w^*(e)$. In other words, $w^*(e)$ is the optimal asymptotic spreading speed in the direction $e$.*

# 2 Description of the method and general results

The proof of propagation relies on the construction of subsolutions of the evolution equation that can initially be made arbitrarily small on compact sets and that remain bounded away from zero at later times. Indeed, as will be seen here, then, after some initial time, the solution will lie above such a subsolution. Hence by the comparison principle, it will stay bounded away form zero. To illustrate this approach, we write these results in a separate subsection. Then, we prove Theorems 1.3 and 1.6.

## 2.1 The method

The method we use to prove the propagation of a solution along a path relies on the next proposition:

**Proposition 2.1** *Consider some admissible path $t \mapsto \xi(t)$. Assume that there exists a function $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N) \cap L^\infty(\mathbb{R} \times \mathbb{R}^N)$, two radii $R > r > 0$ and a constant $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0]$:*

$$\begin{cases} \phi(t,x) = 0 \text{ for all } t \in \mathbb{R} \text{ if } |x| \geq R, \\ \inf_{(t,x)\in\mathbb{R}\times B_r} \phi(t,x) > 0, \\ \partial_t \kappa\phi - \nabla \cdot (A(t, x + \xi(t))\nabla\kappa\phi) + (q(t, x + \xi(t)) - \xi'(t)) \cdot \nabla(\kappa\phi) \leq f(t, x, \kappa\phi) \text{ in } \mathbb{R} \times B_R. \end{cases} \tag{2.26}$$

*Then for all bounded measurable nonnegative non-null initial datum $u_0$, there is propagation of the solution $u$ of the Cauchy problem* (1.2) *along the path $t \mapsto \xi(t)$.*

**Proof.** Up to some shift in time, we can assume that the initial datum $u_0$ is continuous and positive. Set $\kappa_1 = \min\{\kappa_0, \inf_{x\in B_r} \frac{u(0,x)}{\phi(0,x)}\}$ so that $0 < \kappa_1 \leq \kappa_0$. Next, define $v(t,x) = u(t, x + \xi(t))$. This function satisfies:

$$\begin{cases} \partial_t v - \nabla \cdot (A(t, x + \xi(t))\nabla v) + (q(t, x + \xi(t)) - \xi'(t)) \cdot \nabla v = f(t, x, v) \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ v(0, x) = u(0, x) \text{ in } \mathbb{R}^N. \end{cases} \tag{2.27}$$

As $\kappa_1\phi$ is a subsolution, in the generalized sense, of the Cauchy problem (2.27), we infer from the weak maximum principle that

$$v(t,x) \geq \kappa_1\phi(t,x).$$

Thus

$$\liminf_{t\to+\infty} u(t, \xi(t)) = \liminf_{t\to+\infty} v(t, 0) \geq \kappa_1 \inf_{t\in\mathbb{R}} \phi(t, 0) > 0.$$

□

In view of this proposition, we only need to search for a function satisfying the inequalities of (2.26) in order to get the propagation of a solution. In fact, there lies the main difficulty.

## 2.2 Proof of the general propagation and convergence results

**Proof of Theorem 1.3.** From the hypothesis, there exists some $\kappa_1 > 0$ and $t_1 > 0$ such that for all $e \in \mathbb{S}^{N-1}$:

$$\inf_{t \geq t_1} u(t, r_e(t)e) \geq \kappa_1.$$

Even if it means decreasing $\kappa_1$, we can assume that $\inf_{x \in S(t_1)} u(t_1, x) \geq \kappa_1$ since $S(t_1)$ is bounded. We know from Definition 5 that $r_e(t)e \in \partial S_t$ for all $e$ and $t$ and thus

$$\partial S_t = \{r_e(t)e,\ e \in \mathbb{S}^{N-1}\}.$$

Define now $Q = \{(t, x) \in [t_1, +\infty) \times \mathbb{R}^N,\ x \in S_t\}$. One has

$$\inf_{(t,x) \in \partial Q} u(t, x) \geq \kappa_1.$$

We need a modified maximum principle in order to get an estimate in the whole set $Q$. As $Q$ is not a cylinder, we cannot apply the classical weak maximum principle. In fact, it is possible to extend this maximum principle to the set $Q$ and there is no particular issue but, for the sake of completeness, we prove that this extension works well.

**Lemma 2.2** *Assume that $z$ satisfies:*

$$\begin{cases} \partial_t z - \nabla \cdot (A \nabla z) + q \cdot z + bz & \geq 0 \text{ in } Q \\ z & \geq 0 \text{ in } \partial Q \end{cases}$$

*where $b$ is a bounded continuous function. Then one has $z \geq 0$ in $Q$.*

**Proof.** Assume first that $b > 0$. Set $Q_\tau = Q \cap \{t \leq \tau\}$ and assume that there exists $(t, x) \in \overline{Q_\tau}$ such that $z(t, x) < 0$. Take $(t_0, x_0) \in \overline{Q_\tau}$ such that $z(t_0, x_0) = \min_{(t,x) \in \overline{Q_\tau}} z(t, x) < 0$. One necessarily has $(t_0, x_0) \in Q_\tau$ and thus:

$$\nabla z(t_0, x_0) = 0, \nabla \cdot (A \nabla z)(t_0, x_0) \geq 0, b(t_0, x_0) z(t_0, x_0) < 0.$$

This leads to:

$$\partial_t z(t_0, x_0) > 0.$$

But the definition of the minimum yields that for all $0 \leq t \leq t_0$, if $(t, x_0) \in Q$, one has $z(t, x_0) \geq z(t_0, x_0)$. As $t_0 > 0$, for $\varepsilon$ small enough, one has $(t_0 - \varepsilon, x_0) \in Q$. Thus, it is possible to differentiate the inequality to get $\partial_t z(t_0, x_0) \leq 0$. This is a contradiction. Thus for all $\tau > 0$, one has $\min_{Q_\tau} z \geq 0$ and then $z \geq 0$ in $Q$.

If $b$ is not positive, set $z_1(t, x) = e^{-(\|b\|_\infty + 1)t} z(t, x)$ for all $(t, x) \in Q$. This function satisfies:

$$\partial_t z_1 - \nabla \cdot (A \nabla z_1) + q \cdot z_1 + (b + \|b\|_\infty + 1) z_1 = (\partial_t z - \nabla \cdot (A \nabla z) + q \cdot z + bz) e^{-(\|b\|_\infty + 1)t} \geq 0,$$

and for all $(t, x) \in \partial Q$, one has $z_1(t, x) \geq 0$. As $b + \|b\|_\infty + 1 > 0$, the first case yields that $z_1 \geq 0$ and then $z \geq 0$. $\square$

We can now finish the proof of Theorem 1.3. As $\lambda_1' < 0$, we know that there exists some $\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)$ such that $\inf_{\mathbb{R}\times\mathbb{R}^N} \phi > 0$ and

$$\partial_t \phi - \nabla\cdot(A(t,x)\nabla\phi) + q(t,x)\cdot\nabla\phi \leq (f_u'(t,x,0) + \frac{\lambda_1'}{2})\phi \text{ in } \mathbb{R}\times\mathbb{R}^N.$$

We can assume that $\sup_{\mathbb{R}\times\mathbb{R}^N} \phi = 1$.

As $f$ is of class $\mathcal{C}^1$ in $s$ in the neighborhood of 0, there exists some positive $\kappa_0 \leq \kappa_1$ such that

$$\forall 0 < \kappa \leq \kappa_0, \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ f(t,x,\kappa) \geq (f_u'(t,x,0) + \frac{\lambda_1'}{2})\kappa.$$

Fix $0 < \kappa \leq \kappa_0$ such that $u(t_1,x) \geq \kappa\phi(t_1,x)$ for all $x \in S(t_1)$. Then $\kappa\phi$ is a subsolution of equation (1.1) in $\mathbb{R}\times\mathbb{R}^N$.

We apply the modified maximum principle to the function $z = u - \kappa\phi$. This shows that $u \geq \kappa\phi$ over $Q$, which means that

$$\inf_{t\geq t_1}\inf_{x\in S_t} u(t,x) \geq \kappa \inf_{\mathbb{R}\times\mathbb{R}^N} \phi > 0.$$

□

**Proof of Theorem 1.6.** Take a sequence $t_n \to +\infty$ and $x_n \in \mathcal{S}_\varepsilon(t_n)$ such that

$$|p(t_n,x_n) - u(t_n,x_n)| \to \limsup_{t\to+\infty}\sup_{x\in\mathcal{S}_\varepsilon(t)} |p(t,x) - u(t,x)|$$

We know that there exist some $t_1$ and $\kappa_1 > 0$ such that

$$\inf_{t\geq t_1}\inf_{x\in S_t} u(t,x) \geq \kappa_1 > 0. \tag{2.28}$$

Set $u_n(t,x) = u(t+t_n, x+x_n)$. This function satisfies:

$$\begin{array}{l}\partial_t u_n - \nabla\cdot(A(t+t_n,x+x_n)\nabla u_n) + q(t+t_n,x+x_n)\cdot\nabla u_n \\ = f(t+t_n,x+x_n,u_n) \text{ in } (-t_n,+\infty)\times\mathbb{R}^N.\end{array} \tag{2.29}$$

Up to some extraction in Hölder spaces, one may assume that there exists some function $(B,r,g)$ such that $A(t+t_n,x+x_n) \to B(t,x)$ in $\mathcal{C}_{loc}^{\delta'/2,1+\delta'}(\mathbb{R}\times\mathbb{R}^N)$, $q(t+t_n,x+x_n) \to r(t,x)$ in $\mathcal{C}_{loc}^{\delta'/2,\delta'}(\mathbb{R}\times\mathbb{R}^N)$ and $f(t+t_n,x+x_n,s) \to g(t,x,s)$ in $\mathcal{C}_{loc}^{\delta'/2,\delta,0}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}_+)$ for all $0 \leq \delta' < \delta$.

Next, the Schauder parabolic regularity estimates yield that the sequence $(u_n)_n$ converges, up to some extraction, to some function $u_\infty$ in $\mathcal{C}_{loc}^{1+\delta'/2,2+\delta'}(\mathbb{R}\times\mathbb{R}^N)$ for all $0 \leq \delta' < \delta$. This function satisfies:

$$\partial_t u_\infty - \nabla\cdot(B(t,x)\nabla u_\infty) + r(t,x)\cdot\nabla u_\infty = g(t,x,u_\infty) \text{ in } \mathbb{R}\times\mathbb{R}^N. \tag{2.30}$$

For all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, consider some $n_0$ such that

$$|x| + (1-\varepsilon)\sup_{e\in\mathbb{S}^{N-1}} \|r_e'\|_\infty |t| \leq \varepsilon \inf_{e\in\mathbb{S}^{N-1}} r_e(t+t_n) \text{ for all } n \geq n_0.$$

For all $e$, one can compute:

$$\begin{array}{rcl}(x+x_n)\cdot e & \leq & (1-\varepsilon)r_e(t_n)+|x| \\ & \leq & (1-\varepsilon)r_e(t+t_n)+(1-\varepsilon)\|r_e'\|_\infty|t|+|x| \\ & \leq & r_e(t+t_n).\end{array} \tag{2.31}$$

Hence, $x+x_n \in S(t+t_n)$ and then $u_n(t,x) \geq \kappa_1 > 0$ for all $n \geq n_0$. Thus,

$$\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} u_\infty(t,x) \geq \kappa_1 > 0.$$

Hypothesis 1 implies $p(t+t_n, x+x_n) \to u_\infty(t,x)$ in $\mathcal{C}^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^N)$, which can also be written as

$$|p(t+t_n,x+x_n)-u(t+t_n,x+x_n)| \to 0 \text{ in } \mathcal{C}^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^N),$$

whence

$$\limsup_{t\to+\infty} \sup_{x\in\mathcal{S}_\varepsilon(t)} |p(t,x)-u(t,x)| = 0.$$

□

# 3 The case of general media

The proofs rest on the use of some subsolutions. The construction of the subsolutions that we use here rests on an idea introduced in [9]. In this section, we show how to adapt the methods of [9] and then prove Theorem 1.2, Corollary 1.4 and Theorem 1.5.

## 3.1 The key lemma

We first recall the following result, which has been proved by Berestycki, Hamel and Rossi:

**Lemma 3.1** [9] *Let $\beta, \eta$ and $\varepsilon$ be three arbitrary positive numbers. Then there exists a nonnegative function $h \in C^2(\mathbb{R})$ and a positive number $\theta$ such that:*

$$\left\{\begin{array}{lll} h(\rho)=0 & \text{for} & \rho\leq 0, \\ h(\rho)>0 & \text{for} & \theta>\rho\geq 0, \\ h(\rho)=1 & \text{for} & \rho\geq\theta, \end{array}\right. \tag{3.32}$$

*and*

$$\forall(\rho,X)\in(0,\theta]\times\mathcal{A},\ -\mathcal{L}_{a,Q,C}h(\rho) = -a(X)h''(\rho)+Q(X)h'(\rho)-C(X)h(\rho)<0$$

*for any set $\mathcal{A}$ and any nonnegative functions $a, Q, C$ defined on $\mathcal{A}$ and verifying*

$$\forall X\in\mathcal{A},\ a(X)\leq\beta,\ Q(X)\leq\eta \text{ and } 4a(X)C(X)-Q^2(X)\geq\varepsilon.$$

We next apply this result to prove our key lemma:

**Lemma 3.2** *Let $\beta, \eta$ and $\nu$ be three arbitrary positive numbers and $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}^N$. Then there exist a positive constant $r$ and a function $\psi \in \mathcal{C}^2(\mathbb{R}^N)$, both depending on $\beta, \eta$, $\nu$ and on the dimension $N$, such that for all $(t_0, x_0)$ that satisfies $(t_0, +\infty) \times B_r(x_0) \subset \mathcal{A}$, one has:*

$$\begin{cases} \psi(x) > 0 & \text{in} \quad B_r(x_0), \\ \psi(x) = 0 & \text{in} \quad \mathbb{R}^N \backslash B_r(x_0), \\ -\mathcal{L}\psi(x) < 0 & \text{in} \quad (t_0, +\infty) \times B_r(x_0), \end{cases} \tag{3.33}$$

*where*

$$\mathcal{L} = \nabla \cdot (A(t,x)\nabla) - q(t,x) \cdot \nabla + c(t,x),$$

*for any coefficients $A, q, c, \gamma$ verifying for all $(t,x) \in \mathcal{A}$:*

$$0 \leq \gamma(t,x) I_N \leq A(t,x) \leq \beta I_N, \;\; q(t,x) \leq \eta \text{ and } 4\gamma(t,x)c(t,x) - |q|^2(t,x) \geq \nu,$$

*where the inequality holds in the sense of positive matrix.*

**Proof.** Let choose some positive $s$ large enough so that:

$$\varepsilon = \inf_{(t,x)\in\mathcal{A}} \big(4\gamma(t,x)c(t,x) - (|q(t,x)| + \frac{\beta N}{s})^2\big) > 0.$$

Set $\eta' = \sup_{(t,x)\in\mathcal{A}}(|q(t,x)| + \frac{N\beta}{s}) < \infty$. The previous lemma yields some $h$ and $\theta$ associated with the positive constants $\varepsilon, \beta$ and $\eta'$. Set $r = s + \theta$ and define the function $\psi(x) = h(r - |x|)$. Consider some $x_0 \in \mathcal{A}$ such that $(t_0, +\infty) \times B_r(x_0) \subset \mathcal{A}$. A straightforward computation (see [9]) shows that:

$$\begin{aligned} -\mathcal{L}\psi(x - x_0) \quad \leq \quad & -\frac{(x-x_0)A(t,x)(x-x_0)}{|x-x_0|^2} h''(r - |x - x_0|) \\ & + (|q(t,x)| + \tfrac{N\beta}{s}) h'(r - |x - x_0|) \\ & - c(t,x) h(r - |x - x_0|). \end{aligned}$$

Next, denote for all $(t,x) \in \mathcal{A}$:

$$a(t,x) = \begin{cases} \frac{(x-x_0)A(t,x)(x-x_0)}{|x-x_0|^2} & \text{if } x \neq x_0, \\ \gamma(t,x_0) & \text{if } x = x_0, \end{cases} \quad Q(t,x) = |q(t,x)| + \frac{N\beta}{s}, \;\; C(t,x) = c(t,x).$$

The choice of $s$ yields $Q(t,x) \leq \eta$ for all $(t,x) \in \mathcal{A}$. As $\gamma(t,x) \leq a(t,x) \leq \beta$, one gets

$$4a(t,x)C(t,x) - Q^2(t,x) \geq 4\gamma(t,x)c(t,x) - Q^2(t,x) \geq \varepsilon.$$

Thus the functions $a, Q, C$ satisfy the hypotheses of lemma 3.1 and it follows that for all $(t,x) \in (t_0, +\infty) \times B_r(x_0)$

$$-\mathcal{L}\psi(x - x_0) < 0.$$

Thus $\psi$ satisfies the properties of Lemma 3.2. $\square$

### 3.2 Propagation along a path

**Proof of Theorem 1.2.** We have to prove that there exist some $t_0$ and $\kappa_1 > 0$ such that

$$\inf_{t \geq t_0} u(t, \xi(t)) \geq \kappa_1 > 0.$$

We consider some $t_0 > 0$ and $R_0 > 0$, which are as large as needed, for which there exist some $\nu > 0, \delta > 0$ such that for all $t \geq t_0$ , one has

$$\inf_{|x| \leq R_0} (4\gamma(t, x + \xi(t))(f'_u(t, x + \xi(t), 0) - \delta) - |q(t, x + \xi(t)) - \xi'(t)|^2) \geq \nu. \tag{3.34}$$

We apply Lemma 3.2 to $\beta = \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} \Gamma(t, x)$, where $\Gamma$ is defined by (1.6), $\eta = \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} |q(t, x)|$ and $\nu$ defined by (3.34). This gives us some radius $\rho$ and some function $\psi$. We can assume that $R_0$ is large enough so that $R_0 \geq \rho$. Recalling the properties of the function $\psi$ given by lemma 3.2, we know that:

$$-\nabla \cdot [A(t, x + \xi(t))\nabla\psi] + [q(t, x + \xi(t)) - \xi'(t)] \cdot \nabla\psi < \big(f'_u(t, x + \xi(t)) - \delta\big)\psi$$

in $(t_0, +\infty) \times B_\rho$ and that $\psi$ is compactly supported in $B_\rho$.

As $f$ is a uniformly $\mathcal{C}^1$ function in the neighborhood of 0, there exists some $\kappa_0 > 0$ which does not depend on $e$ such that for all $0 < s \leq \kappa_0$ and for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, one has:

$$f(t, x, s) \geq (f'_u(t, x, 0) - \delta)s.$$

We know from the construction of $\psi$ that $\|\psi\|_\infty = 1$. Set $\phi(t, x) = \psi(x - \xi(t))$, for all $0 < \kappa \leq \kappa_0$, one has:

$$\partial_t \kappa\phi - \nabla \cdot (A(t, x)\nabla\kappa\phi) + q(t, x) \cdot \nabla\kappa\phi < (f'_u(t, x, 0) - \delta)\kappa\phi \leq f(t, x, \kappa\phi),$$

as soon as $t \geq t_0$ and $|x - \xi(t)| \leq \rho$.

Take now any nonnegative and non-null initial datum $u_0$ and $u$ the associated solution of the Cauchy problem (1.2). Set

$$Q = \{(t, x) \in (t_0, +\infty) \times \mathbb{R}^N, |x - \xi(t)| \leq \rho\}.$$

We know that $\kappa\phi(t, x) = 0$ if $|x - \xi(t)| \geq \rho$. Next, even if it means decreasing $\kappa_0 > 0$, we can assume that

$$u(t_0, x + x(t_0)) \geq \kappa_0 \,\text{for all}\, x \in \mathbb{R}^N,\ |x| \leq \rho$$

since $u(t_0, \cdot + x(t_0))$ is continuous and positive for $t_0 > 0$. This implies that

$$u \geq \kappa_0\phi \text{ in } \partial Q.$$

Thus, we infer from the modified weak maximum principle of Lemma 2.2 in $Q$ that

$$u \geq \kappa_0\phi \text{ in } Q.$$

Thus:

$$\inf_{t \geq t_0} u(t, \xi(t)) \geq \kappa_0\psi(0) > 0. \tag{3.35}$$

□

## 3.3 Lower estimates of the spreading radii

**Proof of Corollary 1.4.** First, one can easily check that, as the assumption made in Corollary 1.4 is uniform with respect to $e$, the proof of Theorem 1.2 gives a uniform lower bound on $u$. Namely, observe first that there exist $t_0 > 0$, $R_0 > 0$ and $\delta > 0$ such that

$$\nu = \inf_{|x|\le R_0} \inf_{t\ge t_0} \inf_{e\in\mathbb{S}^{N-1}} (4\gamma(t, x + r_e(t)e)(f'_u(t, x + r_e(t)e, 0) - \delta) - |q(t, x + r_e(t)e) - r'_e(t)e|^2) > 0 \tag{3.36}$$

Setting $\psi$ as in the proof of Theorem 1.2, one gets some $\kappa_0 > 0$ such that for all $e \in \mathbb{S}^{N-1}$:

$$\inf_{t\ge t_0} u(t, r_e(t)e) \ge \kappa_0\psi(0) > 0. \tag{3.37}$$

Thus the hypotheses of Theorem 1.3 are satisfied and one gets the conclusion. □

**Proof of Theorem 1.5.** Fix some $c \in (0, c^*)$ and set for all $e \in \mathbb{S}^{N-1}$, $r_e(t) = ct$. Using the definition of $c^*$, we get the existence of some $r > 0$ and $\delta > 0$ such that

$$\inf_{e\in\mathbb{S}^{N-1}} \inf_{|x|\ge r} \inf_{t\in\mathbb{R}} (4\gamma(t, x)f'_u(t, x, 0) - |q(t, x) - r'_e(t)e|^2) \ge \delta > 0.$$

Thus for all $R_0 > 0$, taking some $t_0$ such that $ct_0 \ge R_0 + r$, we get

$$\inf_{e\in\mathbb{S}^{N-1}} \inf_{t\ge t_0} \inf_{|x-cte|\le R_0} (4\gamma(t, x)f'_u(t, x, 0) - |q(t, x) - r'_e(t)e|^2) \ge \delta > 0,$$

since $|x| \ge ct - |x - cte| \ge R_0 + r - R_0 = r$. Corollary 1.4 then implies

$$\liminf_{t\to+\infty} \inf_{|x|\le ct} u(t, x) > 0,$$

which concludes the proof. □

## 3.4 Uniqueness results

In this subsection, we prove that our uniqueness hypothesis 1 is satisfied in some important cases. This kind of results has been proved in time independent media in [9] and in space-time periodic media in [22]. We will follow the same sketch of proof as in [9]. In order to extend these results to time dependent media, we first require the following technical result, which has been proved by Berestycki, Hamel and Rossi:

**Lemma 3.3** [9] *Let $u_1,\ u_2 \in C^0(\mathbb{R} \times \mathbb{R}^N)$ be two positive bounded functions satisfying:*

$$\inf_{\mathbb{R}\times\mathbb{R}^N} u_1 > 0, \quad \inf_{\mathbb{R}\times\mathbb{R}^N} (u_2 - u_1) > 0.$$

*If* (1.14) *holds, there exists $\varepsilon > 0$ such that*

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \frac{u_2(t, x)}{u_1(t, x)} f(t, x, u_1(t, x)) \ge f(t, x, u_2(t, x)) + \varepsilon. \tag{3.38}$$

The next lemma is the extension of a result of [9] to time heterogeneous media:

**Lemma 3.4** *Consider a nonnegative function* $z \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ *such that*

$$\mathcal{P}z = \partial_t z - \nabla\cdot(A(t,x)\nabla z) + q(t,x)\cdot\nabla z - c(t,x)z \geq \varepsilon,$$

*where* $A$ *and* $q$ *satisfy the same hypothesis as in section 1,* $c \in L^\infty(\mathbb{R}\times\mathbb{R}^N)$ *and* $\varepsilon > 0$. *Then* $\inf_{\mathbb{R}\times\mathbb{R}^N} z > 0$.

**Proof.** Consider a nonnegative function $\theta \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R}^N)$ that satisfies:

$$\theta(0,0) = 0, \lim_{|t|+|x|\to+\infty}\theta(t,x) = 1, \|\theta\|_{\mathcal{C}^{1,2}} < \infty.$$

There exists $\kappa > 0$ sufficiently large such that:

$$\forall (s,y)\in\mathbb{R}\times\mathbb{R}^N, \mathcal{P}(\tau_{s,y}\theta) > -\kappa\varepsilon/2,$$

where we denote $\tau_{s,y}\theta = \theta(.-s,.-y)$.

Assume that $\inf_{\mathbb{R}\times\mathbb{R}^N} z = 0$. Then one can find some $(t_0,x_0)\in\mathbb{R}\times\mathbb{R}^N$ such that:

$$z(t_0,x_0) < \min\{\frac{1}{\kappa}, \frac{\varepsilon}{2\|c\|_\infty}\}$$

where $\frac{1}{\|c\|_\infty} = +\infty$ if $c\equiv 0$. Since $\lim_{|t|+|x|\to+\infty}\theta(t,x) = 1$, there exists a positive constant $R$ such that $\tau_{t_0,x_0}\theta(t,x)/\kappa > z(t_0,x_0)$ if $|t-t_0|+|x-x_0|\geq R$. Consequently, setting $\widetilde{z} = z + \tau_{t_0,x_0}\theta(t,x)/\kappa$, one finds for all $|t-t_0|+|x-x_0|\geq R$, that:

$$\widetilde{z}(t,x) \geq \tau_{t_0,x_0}\theta(t,x)/\kappa > z(t_0,x_0) = \widetilde{z}(t_0,x_0).$$

Hence, if $\alpha = \inf_{\mathbb{R}\times\mathbb{R}^N}\widetilde{z}$, this infimum is reached in

$$B_R(t_0,x_0) = \{(t,x)\in\mathbb{R}\times\mathbb{R}^N,\ |t-t_0|+|x-x_0| < R\}.$$

Moreover:

$$\alpha \leq \widetilde{z}(t_0,x_0) = z(t_0,x_0) < \frac{\varepsilon}{2\|c\|_\infty}.$$

One can compute:

$$\begin{array}{rcl}\mathcal{P}(\widetilde{z}-\alpha) & = & \mathcal{P}(z) + \frac{1}{\kappa}\mathcal{P}(\tau_{t_0,x_0}\theta(t,x)) + c(t,x)\alpha \\ & > & \varepsilon - \frac{\varepsilon}{2} - \|c\|_\infty\alpha \\ & > & 0\end{array}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Thus, the strong maximum principle yields that $\widetilde{z}(t,x) = \alpha$ for all $t\leq t_0$ and $x\in\mathbb{R}^N$, which contradicts $\mathcal{P}(\widetilde{z}-\alpha) > 0$. This shows that $\inf_{\mathbb{R}\times\mathbb{R}^N} z > 0$. $\Box$

**Proof of Proposition 1.7.** 1) As $\lambda_1' < 0$, we know that there exists some $-\lambda_1' > \mu > 0$ and some $\psi\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)$ such that $\inf_{\mathbb{R}\times\mathbb{R}^N}\psi > 0$ and

$$\partial_t\psi - \nabla\cdot(A(t,x)\nabla\psi) + q(t,x)\cdot\nabla\psi \leq (f_u'(t,x,0)-\mu)\psi \text{ in } \mathbb{R}\times\mathbb{R}^N.$$

As $f$ is of class $\mathcal{C}^1$ with respect to $s$ uniformly in $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ in the neighborhood of zero, we know that there exists some $\kappa > 0$ and $\kappa \leq M$ such that:

$$\forall s \in (0,\kappa),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ f(t,x,s) \geq (f'_u(t,x,0) - \mu)s.$$

Up to some multiplication by a positive constant, we can assume that $\psi(t,x) < \kappa$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

Next, as $\psi$ is a subsolution and $M$ is such that $f(t,x,M) \leq 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, an iteration method produces a solution $p \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ of equation (1.1) that satisfies

$$\psi(t,x) \leq p(t,x) \leq M, \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

This solution is clearly bounded, nonnegative and satisfies

$$\inf_{\mathbb{R} \times \mathbb{R}^N} p \geq \inf_{\mathbb{R} \times \mathbb{R}^N} \psi > 0.$$

2) Assume that $u$ and $p$ are two positive bounded entire solutions of equation (1.1) such that $\inf_{\mathbb{R} \times \mathbb{R}^N} u > 0$ and $\inf_{\mathbb{R} \times \mathbb{R}^N} p > 0$. Thus we can define:

$$\kappa^* = \inf\{\kappa > 0,\ \kappa u \geq p \text{ in } \mathbb{R} \times \mathbb{R}^N\} > 0.$$

We will now assume that $\kappa^* > 1$ and get a contradiction.

As $\inf_{\mathbb{R} \times \mathbb{R}^N} u > 0$ and $\inf_{\mathbb{R} \times \mathbb{R}^N} (\kappa^* u - u) > 0$, one knows from lemma 3.3 that there exists $\varepsilon > 0$ such that

$$\forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \frac{\kappa^* u(t,x)}{u(t,x)} f(t,x,u(t,x)) \geq f(t,x,\kappa^* u(t,x)) + \varepsilon. \tag{3.39}$$

Set $z = \kappa^* u - p$, this function is nonnegative, satisfies $\inf_{\mathbb{R} \times \mathbb{R}^N} z = 0$ and

$$\begin{array}{rcl} \partial_t z - \nabla \cdot (A \nabla z) + q \cdot \nabla z &=& \kappa^* f(t,x,u) - f(t,x,p) \\ &\geq& f(t,x,\kappa^* u) + \varepsilon - f(t,x,p). \end{array}$$

Set

$$g(t,x) = \left\{ \begin{array}{l} \frac{f(t,x,\kappa^* u(t,x)) - f(t,x,p(t,x))}{\kappa^* u(t,x) - p(t,x)} \text{ if } \kappa^* u(t,x) \neq p(t,x), \\ 0 \text{ if } \kappa^* u(t,x) = p(t,x). \end{array} \right.$$

As $f$ is Lipschitz-continuous, this function lies in $L^\infty(\mathbb{R} \times \mathbb{R}^N)$. One has

$$\partial_t z - \nabla \cdot (A(t,x) \nabla z) + q(t,x) \cdot \nabla z - g(t,x) z \geq \varepsilon.$$

Lemma 3.4 then yields $\inf_{\mathbb{R} \times \mathbb{R}^N} z > 0$ which is a contradiction.

Thus $\kappa^* \leq 1$ and $p \leq u$. As $u$ and $p$ play a symmetric role, one has $u \equiv p$.

Lastly, consider some coefficients $(B, r, g)$ as in Hypothesis 1, that is, there exist some sequences $(t_n)_n$ and $(x_n)_n$ such that

$$\begin{array}{rclcclcl} A(t+t_n, x+x_n) &\to& B(t,x) & \text{as} & n \to +\infty & \text{in} & \mathcal{C}^{\frac{\delta'}{2}, 1+\delta'}_{loc}(\mathbb{R} \times \mathbb{R}^N), \\ q(t+t_n, x+x_n) &\to& r(t,x) & \text{as} & n \to +\infty & \text{in} & \mathcal{C}^{\frac{\delta'}{2}, \delta'}_{loc}(\mathbb{R} \times \mathbb{R}^N), \\ f(t+t_n, x+x_n, s) &\to& g(t,x,s) & \text{as} & n \to +\infty & \text{in} & \mathcal{C}^{\frac{\delta'}{2}, \delta', 0}_{loc}(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}_+), \end{array} \tag{3.40}$$

and assume that $u$ is a bounded entire function such that $\inf_{\mathbb{R}\times\mathbb{R}^N} u > 0$ and

$$\partial_t u - \nabla\cdot(B(t,x)\nabla u) + r(t,x)\cdot\nabla u = g(t,x,u) \text{ in } \mathbb{R}\times\mathbb{R}^N. \tag{3.41}$$

As (1.14) is uniform with respect to $(t,x)$, this decreasing property also holds for $g$. Thus $u$ is the unique entire solution of (3.41) such that $\inf_{\mathbb{R}\times\mathbb{R}^N} u > 0$.

On the other hand, we know from (1.15) that there exists a solution $p$ of (1.1) associated with the coefficients $(A,q,f)$ such that $\inf_{\mathbb{R}\times\mathbb{R}^N} p > 0$. Set $p_n(t,x) = p(t+t_n, x+x_n)$, this function satisfies

$$\partial_t p_n - \nabla\cdot(A(t+t_n,x+x_n)\nabla p_n) + q(t+t_n,x+x_n)\cdot\nabla p_n = f(t+t_n,x+x_n,p_n) \text{ in } \mathbb{R}\times\mathbb{R}^N. \tag{3.42}$$

The Schauder parabolic estimates yield that there exists a function $p_\infty$ such that $p_n(t,x) \to p_\infty(t,x)$ in $\mathcal{C}^{\frac{\delta'}{2},\delta',0}_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}_+)$ for all $0<\delta'<\delta$. The function $p_\infty$ is a solution of (3.41) and $\inf_{\mathbb{R}\times\mathbb{R}^N} p_\infty > 0$. Thus $p_\infty \equiv u$, which can be written $p(t+t_n, x+x_n) \to u(t,x)$ as $n\to+\infty$ in $\mathcal{C}^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^N)$. This ends the proof. □

**Proof of Proposition 1.8.** Assume that $p$ is a uniformly positive continuous entire solution of equation (1.1) such that $m = \inf_{\mathbb{R}\times\mathbb{R}^N} p > 0$. Assume that $m < 1$. Consider a sequence $(t_n,x_n)\in\mathbb{R}\times\mathbb{R}^N$ such that $p(t_n,x_n)\to m$.

Set $p_n(t,x) = p(t+t_n,x+x_n)$, $A_n(t,x) = A(t+t_n,x+x_n)$, $q_n(t,x) = q(t+t_n,x+x_n)$, $f_n(t,x,s) = f(t+t_n,x+x_n,s)$ for all $(t,x,s)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}_+$. As $(A_n,q_n,f_n)_n$ is bounded in some Hölder space, it is possible to assume, up to extraction, that this sequence converges to some limit $(A_\infty,q_\infty,f_\infty)$ in some Hölder space with a lower rate. Thus, the Schauder parabolic estimates yield that the sequence $(p_n)_n$ converges to some function $p_\infty$ in $\mathcal{C}^{\frac{\delta'}{2},1+\delta'}(\mathbb{R}\times\mathbb{R}^N)$ for all $0<\delta'<\delta$. Hence, this function is a solution of:

$$\partial_t p_\infty - \nabla\cdot(A_\infty(t,x)\nabla p_\infty) + q_\infty(t,x)\cdot\nabla p_\infty = f_\infty(t,x,p_\infty). \tag{3.43}$$

Moreover, one has $p_\infty \geq m$ and $p(0,0) = m$. If $m<1$, then $f_\infty(t,x,m) > 0$ by 1.17 and the strong parabolic maximum principle implies $p_\infty(t,x) = m$ for all $t\geq 0$ and $x\in\mathbb{R}^N$, which yields a contradiction. Thus $m\geq 1$.

Similarly, one can prove that $\sup_{\mathbb{R}\times\mathbb{R}^N} p \leq 1$ since $f(t,x,s)<0$ if $s>1$. Thus $p\equiv 1$.

Lastly, it is possible to prove that Hypothesis 1 is satisfied as in the proof of Proposition 1.7. □

Lastly, we give the proof of a result of independent interest about the uniform positivity of the entire solutions of (1.1) that are uniformly positive with respect to time at a given point $x_0$, under some positivity hypothesis at infinity:

**Proposition 3.5** *Assume that*

$$\liminf_{|x|\to+\infty} \left\{ \inf_{t\in\mathbb{R}} \left[4\gamma(t,x) f'_u(t,x,0) - |q(t,x)|^2\right] \right\} > 0\,. \tag{3.44}$$

*and that $p\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ is a nonnegative bounded entire solution of* (1.1) *such that there exists some $x_0\in\mathbb{R}^N$ for which*

$$\inf_{t\in\mathbb{R}} p(t,x_0) > 0.$$

*Then one has $\inf_{\mathbb{R}\times\mathbb{R}^N} p > 0$.*

**Proof.** We first prove that for all compact subset $K \subset \mathbb{R}^N$, one has $\inf_{\mathbb{R}\times K} p > 0$. Assume that there exists $t_n \in \mathbb{R}, x_n \in K$ such that $u(t_n, x_n) \to 0$. As the sequences $(A(t+t_n, x))_n$, $(q(t+t_n, x))_n$ and $(f(t+t_n, x, s))_n$ are uniformly locally Hölder continuous with respect to $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, uniformly with respect to $s \in [0, \|p\|_\infty]$, one can assume, up to extraction, that they converge to some functions $A_\infty$, $q_\infty$ and $f_\infty$ in $\mathcal{C}^{\frac{\delta'}{2},1+\delta'}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ for all $0 < \delta' < \delta$ (the convergence of $(f_n)_n$ holds in $\mathcal{C}^{\frac{\delta'}{2},\delta',0}_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}_+)$). We can also assume that the sequence $(x_n)_n$ converges to some $x_\infty$.

Set $p_n(t,x) = p(t+t_n, x)$. This function satisfies:

$$\partial_t p_n - \nabla\cdot(A(t+t_n,x)\nabla p_n) + q(t+t_n,x)\cdot\nabla p_n = f(t+t_n, x, p_n)$$

The classical Schauder estimates yield that one can assume that $p_n$ converges to a function $p_\infty$ in $C^{1,2}_{loc}$ such that:

$$\partial_t p_\infty - \nabla\cdot(A_\infty(t,x)\nabla p_\infty) + q_\infty(t,x)\cdot\nabla p_\infty = f_\infty(t,x,p_\infty)$$

and $p_\infty(0, x_\infty) = 0$. As $p_\infty$ is nonnegative, the strong maximum principle yields that for all $t \leq 0$, for all $x, p_\infty(t,x) = 0$. On the other hand, set $\varepsilon = \inf_{t\in\mathbb{R}} p(t, x_0) > 0$. Then for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, one has $p_n(t, x_0) \geq \varepsilon$ and then for all $t \in \mathbb{R}$, $p_\infty(t, x_0) \geq \varepsilon > 0$, which is a contradiction.

Now, we know from Lemma 3.2 that there exist some positive constants $\mu, r$ and $R$ and a function $\psi \in \mathcal{C}^2(\mathbb{R}^N)$ such that for all $x_0 \notin B_{r+R}(0)$, one has:

$$\begin{cases} \psi(x) > 0 \text{ in } B_r(0), \\ \psi(x) = 0 \text{ in } \mathbb{R}^N \backslash B_r(0), \\ -\nabla\cdot(A(t,x)\nabla\psi(x-x_0)) + q(t,x)\cdot\nabla\psi(x-x_0) \\ \qquad\qquad < (f'_u(t,x,0) - \mu)\psi(x-x_0) \text{ in } \mathbb{R}\times B_r(x_0). \end{cases} \tag{3.45}$$

Moreover, there exists some $\varepsilon > 0$ such that for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, for all $s \in [0, \varepsilon]$:

$$f(t,x,s) \geq (f'_u(t,x,0) - \frac{\mu}{2})s.$$

We now fix $y \notin B_{R+r}(0)$ and we can assume, without loss of generality, that

$$0 < \varepsilon < \frac{\inf_{(t,x)\in\mathbb{R}\times B_r(y)} p(t,x)}{\|\psi\|_\infty}.$$

This is possible since $\inf_{(t,x)\in\mathbb{R}\times B_r(y)} p(t,x) > 0$. We set $\underline{p} = \varepsilon\psi$. For all $x_0 \notin B_{r+R}(0)$, this function satisfies:

$$-\nabla\cdot(A(t,x)\nabla\underline{p}(x-x_0)) + q(t,x)\cdot\nabla\underline{p}(x-x_0) \\ < f(t,x,\underline{p}(x-x_0)) - \frac{\mu}{2}\underline{p}(x-x_0) \text{ in } \mathbb{R}\times B_r(x_0). \tag{3.46}$$

We will prove that

$$\inf_{\mathbb{R}\times(\mathbb{R}^N\backslash B_{r+R}(0))} p \geq \underline{p}(0) = \varepsilon\psi(0). \tag{3.47}$$

As $\inf_{\mathbb{R}\times B_{r+R}(0)} p > 0$, this would end the proof of the lemma.

To prove (3.47), take $z \notin B_{r+R}(0)$ and consider a curve $\gamma : [0,1] \to \mathbb{R}^N \backslash B_{r+R}(0)$ such that $\gamma(0) = y$ and $\gamma(1) = z$. We know that $\underline{p}(x-\gamma(0)) \leq p(t,x)$ for all $(t,x) \in \mathbb{R}\times B_r(\gamma(0))$ since $\|\underline{p}\|_\infty = \varepsilon\|\psi\|_\infty \leq \inf_{\mathbb{R}\times B_r(y)} p$. Call

$$\xi^* = \sup\{\xi \in [0,1],\ \forall\, 0 \leq s \leq \xi, \forall\ (t,x) \in \mathbb{R}\times B_r(\gamma(s)),\ \underline{p}(x-\gamma(s)) \leq p(t,x)\}.$$

Suppose by contradiction that $\xi^* < 1$. Then $\underline{p}(x-\gamma(\xi^*)) \leq p(t,x)$ for all $(t,x) \in \mathbb{R}\times B_r(\gamma(\xi^*))$ and there exist some sequences $\xi_n \to \xi^*$ and $(t_n,x_n) \in \mathbb{R}\times B_r(\gamma(\xi_n))$ such that for all $n$:

$$\underline{p}(x_n - \gamma(\xi_n)) > p(t_n,x_n).$$

Set $p_n(t,x) = p(t+t_n,x)$, $A_n(t,x) = A(t+t_n,x)$, $q_n(t,x) = q(t+t_n,x)$, $f_n(t,x,s) = f(t+t_n,x,s)$ for all $(t,x,s) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}_+$. As $(A_n,q_n,f_n)_n$ is bounded in some Hölder space, it is possible to assume, up to extraction, that this sequence converges to some limit $(A_\infty,q_\infty,f_\infty)$ in some Hölder space with a lower rate. Thus, the Schauder parabolic estimates yield that the sequence $(p_n)_n$ can be assumed to converge to a function $p_\infty$ in $\mathcal{C}^{\frac{\delta'}{2},1+\delta'}(\mathbb{R}\times\mathbb{R}^N)$ for all $0 < \delta' < \delta$. Hence, this function is a solution of:

$$\partial_t p_\infty - \nabla\cdot(A_\infty(t,x)\nabla p_\infty) + q_\infty(t,x)\cdot\nabla p_\infty = f_\infty(t,x,p_\infty). \tag{3.48}$$

As $(x_n)_n$ is bounded, one can assume that this sequence converges to some $x_\infty \in \overline{B_r(\gamma(\xi^*))}$. Moreover:

$$\underline{p}(x_\infty - \gamma(\xi^*)) = \lim_{n\to+\infty} \underline{p}(x_n - \gamma(\xi_n)) \geq \lim_{n\to+\infty} p(t_n,x_n) = p_\infty(0,x_\infty).$$

Hence $\underline{p}(x_\infty - \gamma(\xi^*)) = p_\infty(0,x_\infty)$. As $\underline{p} \equiv 0$ on $\partial B_r(0)$ and $\inf_{\mathbb{R}\times B_r(\gamma(\xi^*))} p_\infty \geq \inf_{\mathbb{R}\times B_r(\gamma(\xi^*))} p > 0$, the point $x_\infty$ belongs to $B_r(\gamma(\xi^*))$. Furthermore, the function $(t,x) \mapsto p_\infty(t,x) - \underline{p}(x-\gamma(\xi^*))$ reaches a local minimum at $(0,x_\infty)$. But, from (3.46), the function $\underline{p}(\cdot - \gamma(\xi^*))$ is a strict subsolution of equation (3.48) in $\mathbb{R}\times B_r(\gamma(\xi^*))$. The strong parabolic maximum principle then leads to a contradiction.

Thus $\xi^* = 1$ and then for all $t \in \mathbb{R}$, one has $p(t,z) \geq \underline{p}(z-\gamma(1)) = \varepsilon\psi(0)$. $\square$

## 3.5 Upper estimates of the spreading radii

In this subsection, we prove the general upper bound for the spreading speeds which is stated in Theorem 1.10. The proof mainly relies on the properties of the generalized principal eigenvalues $k_\lambda(\eta)$ defined in Section 1.

**Proof of Proposition 1.9.** Fix $\lambda \in \mathbb{R}^N$ and set

$$\mathcal{A} = \{k > 0,\ \exists \phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N),\ \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \phi(t,x) > 0,\ P_\lambda\phi \geq k\phi\}.$$

We need to define that this set is not empty and admits an upper bound. First of all, take $k \leq -\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N}\big(\lambda A(t,x)\lambda - \nabla\cdot(A(t,x)\lambda) + q(t,x)\cdot\lambda + \eta(t,x)\big)$ and $\phi = 1$. Then one can easily check that $P_\lambda\phi \geq k\phi$ and thus $\mathcal{A}$ is not empty.

Next, in order to prove that $\mathcal{A}$ is bounded from above, we can assume that $\lambda = 0$ by considering $\widetilde{\eta} = \lambda A \lambda - \nabla \cdot (A\lambda) + q \cdot \lambda + \eta$ and $\widetilde{q} = q + 2\lambda A$. Take $k \in \mathcal{A}$ and consider $\phi$ an associated test function. Assume that

$$k > \frac{\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \big(|q(t,x)|^2 - 4\gamma(t,x)\eta(t,x)\big)}{\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} 4\gamma(t,x)}$$

and try to reach a contradiction, which would provide the upper bound on $\mathcal{A}$.

This would give that $\nu = \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \big(4\gamma(t,x)(\eta(t,x)+k) - |q(t,x)|^2\big) > 0$ and thus we know from Lemma 3.2 that there exists a function $\psi \in \mathcal{C}^2(\mathbb{R}^N)$ and a radius $r > 0$ such that $\psi$ is compactly supported in $B_r(0)$ and

$$-\nabla \cdot (A(t,x)\nabla\psi) + q(t,x) \cdot \nabla\psi - (\eta(t,x)+k)\psi \leq 0 \text{ for all } (t,x) \in \mathbb{R} \times B_r(0).$$

Set $\kappa = \sup_{(t,x)\in\mathbb{R}\times B_r(0)} \frac{\psi(x)}{\phi(t,x)}$. This quantity is a real number since $\varepsilon = \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \phi(t,x) > 0$. Define $z = \kappa\phi - \psi$. This function is nonnegative and $\inf_{\mathbb{R}\times\mathbb{R}^N} z = 0$. Since $z(t,x) \geq \kappa\varepsilon > 0$ as soon as $|x| \geq r$, and since our estimates are uniform with respect to $t \in \mathbb{R}$, we may assume that there exists $(t_0, x_0) \in \mathbb{R} \times B_r(0)$ such that $z(t_0,x_0) = 0$. Otherwise, one only need to consider some translations in time as in the proof of Proposition 3.5 and the contradiction that follows also holds for the limit function. Furthermore, one has

$$\partial_t z - \nabla \cdot (A(t,x)\nabla z) + q(t,x) \cdot \nabla z - (\eta(t,x)+k)z \geq 0 \text{ in } \mathbb{R} \times B_r(0).$$

The strong maximum principle thus shows that $z \equiv 0$ in $(-\infty, t_0] \times B_r(0)$, which is impossible since $z$ is continuous and $z \geq \kappa\varepsilon > 0$ on $\mathbb{R} \times \partial B_r(0)$. $\square$

**Proposition 3.6** *The function $(\lambda, \eta) \in \mathbb{R}^N \times \mathcal{C}^0(\mathbb{R}\times\mathbb{R}^N) \mapsto k_\lambda(\eta)$ is concave and continuous. Moreover, if $\eta_1 \geq \eta_2$, then for all $\lambda \in \mathbb{R}^N$, one has $k_\lambda(\eta_1) \leq k_\lambda(\eta_2)$.*

**Proof.** Set $F(\lambda,\eta) = k_\lambda(\eta)$ and let $\lambda_1, \lambda_2$ be two points in $\mathbb{R}^N$, $\eta_1, \eta_2 \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R}^N)$ and $r \in [0,1]$. We want to show that:

$$F(r(\lambda_1,\eta_1) + (1-r)(\lambda_2,\eta_2)) \geq rF(\lambda_1,\eta_1) + (1-r)F(\lambda_2,\eta_2).$$

Set $\lambda = r\lambda_1 + (1-r)\lambda_2$ and $\eta = r\eta_1 + (1-r)\eta_2$. Set:

$$E_\lambda = \{\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N), \phi e^{\lambda\cdot x} \in W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N) \text{ and } \inf_{\mathbb{R}\times\mathbb{R}^N} \phi e^{\lambda\cdot x} > 0\}$$

One can write the definition of $k_\lambda(\eta)$ as:

$$k_\lambda(\eta) = \sup\{k > 0,\ \exists \phi \in E_\lambda,\ \phi > 0,\ \mathcal{P}\phi \geq k\phi\}. \tag{3.49}$$

Let $\phi_1, \phi_2$ be arbitrarily chosen in $E_{\lambda_1}$ and $E_{\lambda_2}$ respectively. Define $z_1 = \ln(\phi_1)$, $z_2 = \ln(\phi_2)$, $z = rz_1 + (1-r)z_2$ and $\phi = e^z \in E_\lambda$. Therefore, it follows from (3.49) that:

$$k_\lambda(\eta) \geq \inf_{\mathbb{R}\times\mathbb{R}^N} \Big(\frac{\partial_t\phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi}{\phi} - \eta\Big).$$

On the other hand, one can compute that:

$$\frac{\partial_t\phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi}{\phi} = \partial_t z - \nabla\cdot(A\nabla z) - \nabla z A\nabla z + q\cdot\nabla z$$

and:

$$\begin{array}{rcl} \nabla z A\nabla z & = & r\nabla z_1 A\nabla z_1 + (1-r)\nabla z_2 A\nabla z_2 - r(1-r)(\nabla z_1 - \nabla z_2)A(\nabla z_1 - \nabla z_2) \\ & \leq & r\nabla z_1 A\nabla z_1 + (1-r)\nabla z_2 A\nabla z_2. \end{array}$$

Hence,

$$\begin{array}{rcl} \dfrac{\partial_t\phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi}{\phi} - \eta & \geq & r(\partial_t z_1 - \nabla\cdot(A\nabla z_1) - \nabla z_1 A\nabla z_1 + q\nabla z_1 - \eta_1) \\ & & +(1-r)(\partial_t z_2 - \nabla\cdot(A\nabla z_2) - \nabla z_2 A\nabla z_2 + q\nabla z_2 - \eta_1) \\ & = & r\Big(\dfrac{\partial_t\phi_1 - \nabla\cdot(A\nabla\phi_1) + q\cdot\phi_1}{\phi_1} - \eta_1\Big) \\ & & +(1-r)\Big(\dfrac{\partial_t\phi_2 - \nabla.(A\nabla\phi_2) + q\cdot\phi_2}{\phi_2} - \eta_2\Big). \end{array}$$

Then,

$$\begin{array}{rcl} k_\lambda(\eta) & \geq & \inf\limits_{\mathbb{R}\times\mathbb{R}^N}\Big(\dfrac{\partial_t\phi - \nabla.(A\nabla\phi) + q\cdot\nabla\phi}{\phi} - \eta\Big) \\ & \geq & r\inf\limits_{\mathbb{R}\times\mathbb{R}^N}\Big(\dfrac{\partial_t\phi_1 - \nabla\cdot(A\nabla\phi_1) + q\cdot\nabla\phi_1}{\phi_1} - \eta_1\Big) \\ & & +(1-r)\inf\limits_{\mathbb{R}\times\mathbb{R}^N}\Big(\dfrac{\partial_t\phi_2 - \nabla.(A\nabla\phi_2) + q\cdot\nabla\phi_2}{\phi_2} - \eta_2\Big). \end{array}$$

Since $\phi_1$ and $\phi_2$ are arbitrarily chosen in $E_{\lambda_1}$ and $E_{\lambda_2}$, this leads to

$$k_\lambda(\eta) \geq rk_{\lambda_1}(\eta_1) + (1-r)k_{\lambda_2}(\eta_2)$$

Then $f$ is concave and we get the continuity in $\lambda$.

If $\eta_1 \geq \eta_2$, we immediately get from (3.49) that $k_\lambda(\eta_1) \leq k_\lambda(\eta_2)$. Thus, for all $\eta_1, \eta_2$, as $\eta_1 \leq \eta_2 + \|\eta_1 - \eta_2\|_\infty$, one has

$$k_\lambda(\eta_1) \geq k_\lambda(\eta_2 + \|\eta_1 - \eta_2\|_\infty) = k_\lambda(\eta_2) - \|\eta_1 - \eta_2\|_\infty.$$

Similarly, one has

$$k_\lambda(\eta_2) \geq k_\lambda(\eta_1) - \|\eta_1 - \eta_2\|_\infty.$$

This finally shows that for all $\eta_1, \eta_2$,

$$|k_\lambda(\eta_2) - k_\lambda(\eta_1)| \leq \|\eta_1 - \eta_2\|_\infty,$$

which is a sharper result than the classical continuity. □

**Proof of Theorem 1.10.** First of all, we observe that

$$w^{**}(e) = \inf\{w \in \mathbb{R},\ \exists\lambda \in \mathbb{R}^N, k_\lambda(\eta) + w\lambda\cdot e > 0 \text{ and } \lambda\cdot e > 0\}.$$

Next, as $w > w^{**}(e)$, there exist $w' \in [w^{**}(e), w)$ and $\lambda \in \mathbb{R}^N$ such that $k_\lambda(\eta) + w'\lambda \cdot e > 0$ and $\lambda \cdot e > 0$. Set

$$r = \frac{k_\lambda(\eta) + w'\lambda \cdot e}{\lambda \cdot e} > 0.$$

One can find a positive $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R} \times \mathbb{R}^N)$ such that $\mathcal{P}_\lambda \phi \geq (\lambda \cdot e)\ (r/2 - w')\ \phi$ in $\mathbb{R} \times \mathbb{R}^N$. Set

$$\psi(t,x) = \phi(t,x) e^{-\lambda \cdot x + (\lambda \cdot e)(w' - r/2)t}.$$

A straightforward calculation shows that this function is a super solution of equation (1.1). As it is continuous and positive, one may assume, up to multiplication by some positive constant, that $\psi(0,x) \geq u_0(x)$ for all $x \in \mathbb{R}^N$, which implies $\psi \geq u$ in $\mathbb{R}_+ \times \mathbb{R}^N$. Thus

$$u(t, x + wte) \leq \phi(t, x + wte) e^{-\lambda \cdot x} e^{(\lambda \cdot e)(-w + w' - r/2)t} \leq \|\phi\|_\infty e^{-\lambda \cdot x} e^{-(\lambda \cdot e)(r/2)t} \to 0$$

as $t \to +\infty$ locally uniformly with respect to $x \in \mathbb{R}^N$. $\Box$

## 3.6 Example of a sublinear complete spreading

This subsection is dedicated to the proof of Theorem 1.11.

**Proof of Theorem 1.11.** Consider a non-null measurable initial datum such that $0 \leq u_0 \leq 1$ and $u_0$ has a compact support in $\mathbb{R}$. Let $y_- \in \mathbb{R}$ be such that $g(y) \leq 0$ for all $y \leq y_-$. Let $c$ be any positive real number and choose $c'$ such that $0 < c' < c$. We know that $M = \sup_{t \geq 0}(r(t) - c't) < \infty$. Take $t_0 > 0$ such that $M \leq c't_0 + y_-$ and $u_0 = 0$ outside the interval $[-c't_0, c't_0]$. Define

$$v(t,x) = \min\{e^{-c'(|x| - c'(t+t_0))}, 1\}.$$

Notice that $v(0,x) \geq u_0(x)$ for all $x \in \mathbb{R}$. We already know that 1 is a supersolution of

$$\begin{cases} \partial_t u - \partial_{xx} u &= g(r(t) - |x|)u(1-u), \\ u(0,x) &= u_0(x). \end{cases} \tag{3.50}$$

Next, set $\Omega = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R},\ |x| \geq c'(t+t_0)\}$. In order to prove that $v$ is a generalized supersolution of (3.50), it is sufficient to prove that $w : (t,x) \mapsto e^{-c'(|x| - c'(t+t_0))}$ is a supersolution of the parabolic equation over $\Omega$. We compute:

$$\partial_t w - \partial_{xx} w - g(r(t) - |x|)w(1-w) \geq -g(r(t) - |x|)w(1-w) \geq 0$$

since $0 \leq w \leq 1$ and $r(t) - |x| \leq r(t) - c'(t+t_0) \leq M - c't_0 \leq y_-$ in $\Omega$. We conclude that $0 \leq u(t,x) \leq v(t,x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$. In particular, $\sup_{|x| \geq ct} |u(t,x)| \to 0$ as $t \to +\infty$ since $c' < c$.

Let now $\varepsilon$ be any real number in $(0,1)$. For any $R > 0$, there holds

$$\liminf_{t \to +\infty} \inf_{|x| \leq R} f'_u(t, x \pm (1-\varepsilon)r(t), 0) = \liminf_{t \to +\infty} \inf_{|x| \leq R} g(r(t) - |x \pm (1-\varepsilon)r(t)|) = g(+\infty) > 0.$$

Since $r'(t) \to 0$ as $t \to +\infty$, Theorem 1.2 then implies that $\liminf_{t\to+\infty} u(t, \pm(1-\varepsilon)r(t)) > 0$. Define

$$\Omega = \{(t,x) \in [t_0, +\infty) \times \mathbb{R}^N, -(1-\varepsilon)r(t) \le x \le (1-\varepsilon)r(t)\},$$

where $t_0 \ge 0$ is be chosen so that $\kappa_0 = \inf_{\partial\Omega} u > 0$ and $g(s) > 0$ for all $s \ge \varepsilon r(t)$ and $t \ge t_0$. The choice of such a $t_0$ is possible from the previous estimates and from the strong parabolic maximum principle. But the constant function $\kappa_0$ is a subsolution of (3.50) in $\Omega$. Hence, $u \ge \kappa_0$ in $\overline{\Omega}$, which completes the proof. □

# 4 Space-time periodic media

This section is devoted to showing how the ideas developed in this article yield a new an purely PDE approach to the results regarding spreading in periodic media.

We first study the somewhat simpler case of space periodic media where the coefficients in the equation do not depend on $t$. The arguments there are more transparent. Actually, the method presented for this case may be used in the case of space-time periodic media as well. However, in trying to use this method in space-time periodic media, the reader will have to face essentially the same difficulties as with the second method. This is why we prefer to develop still another method for the general space-time periodic media. It will also yield some by-products of independent interest.

## 4.1 The space periodic case

We begin with the proof of Theorem 1.13 in space periodic only media. Actually, we only prove here a slightly weaker version, that is the local propagation of the solution $u$ along the path $t \mapsto wte$ for any direction $e$ and any speed $w$ such that $0 \le w < w^*(e)$. It is not too difficult to prove that $w^*(e)$ is optimal by using the pulsating travelling fronts of [3] and we omit it here.

**Proof of Theorem 1.13 in space periodic media.** First, for all $\lambda \in \mathbb{R}^N$, we define $k_\lambda$ to be the space periodic principal eigenvalue associated with the operator $L_\lambda$ defined for all $\phi \in \mathcal{C}^2(\mathbb{R}^N)$ by

$$L_\lambda \phi = -\nabla \cdot (A(x)\nabla\phi) + \big(q(x) - 2\lambda A(x)\big) \cdot \nabla\phi - \big(\lambda A(x)\lambda + \nabla \cdot (A(x)\lambda) - q(x) \cdot \lambda f'_u(x,0)\big)\phi.$$

Next, take any $e \in \mathbb{S}^{N-1}$ and $w \in [0, w^*(e))$. Define $T = \frac{\sum_{i=1}^N e_i L_i}{w}$ and for all $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$:

$$\mathcal{L}_w \phi = \partial_t \phi - \nabla \cdot (A(x+wte)\nabla\phi) + (q(x+wte) - we) \cdot \nabla\phi - f'_u(x+wte,0)\phi.$$

The operator $\mathcal{L}_w$ is a parabolic operator with space-time periodic coefficients of periods $(T, L_1, ..., L_N)$ respectively.

Consider the modified operators $L_{w,\lambda}$ for all $\lambda \in \mathbb{R}^N$, where for all $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$:

$$\begin{aligned}
&L_{w,\lambda}\phi = e^{-\lambda \cdot x}\mathcal{L}_w(e^{\lambda \cdot x}\phi) \\
&= \partial_t\phi - \nabla \cdot (A(x+wte)\nabla\phi) + \big(q(x+wte) - we - 2\lambda A(x+wte)\big) \cdot \nabla\phi \\
&-\big(\lambda A(x+wte)\lambda + \nabla \cdot (A(x+wte)\lambda) - q(x+wte) \cdot \lambda + we \cdot \lambda + f'_u(x+wte,0)\big)\phi.
\end{aligned}$$

Set $k_\lambda^w$ the space-time periodic principal eigenvalue associated with the operator $L_{w,\lambda}$. Doing the change of variables $y = x + wte$, one can easily remark that for all $\lambda \in \mathbb{R}^N$:

$$k_\lambda^w = k_\lambda + w\lambda \cdot e.$$

For all $R > 0$, we define the principal eigenvalue associated with $\mathcal{L}_w$ and with time periodic boundary conditions and Dirichlet boundary conditions in space:

$$\begin{cases} \mathcal{L}_w \phi &= \lambda_1^w(B_R)\phi, \\ \phi &> 0 \text{ in } \mathbb{R} \times B_R, \\ \phi &\text{is} \quad \text{T-periodic}, \\ \phi &= 0 \text{ in } \mathbb{R} \times \partial B_R. \end{cases} \tag{4.51}$$

It has been proved in [23] that

$$\lambda_1^w(B_R) \to \max_{\lambda \in \mathbb{R}^N} k_\lambda^w = \max_{\lambda \in \mathbb{R}^N} (k_\lambda + w\lambda \cdot e).$$

But as

$$0 \leq w < w^*(e) = \min_{\lambda \in \mathbb{R}^N, \lambda \cdot e > 0} \frac{-k_\lambda}{\lambda \cdot e},$$

one has $\lambda_1^w(B_R) < 0$ when $R$ is large enough. Fix such a $R$ and some principal eigenfunction $\phi_R^w$ associated with $\lambda_1^w(B_R) < 0$, that we extend to the whole space $\mathbb{R}^N$ by setting $\phi_R^w(t,x) = 0$ if $|x| \geq R$.

Consider some bounded measurable nonnegative initial datum $u_0 \not\equiv 0$ and $u$ the solution of the Cauchy problem associated with $u_0$. Up to some shift in time, we can assume that $u(0,\cdot)$ is continuous and positive. Thus there exists some small $\kappa$ such that $u(0,x) \geq \kappa \phi_R^w(0,x)$ for all $x \in \mathbb{R}^N$. As $f$ is of class $\mathcal{C}^1$ in $s = 0$ and $\lambda_1^w(B_R) < 0$, we can assume that $\kappa$ is small enough such that for all $(t,x) \in \mathbb{R}_+ \times B_R$:

$$\partial_t \kappa \phi_R^w - \nabla \cdot (A(x+wte)\nabla \kappa \phi_R^w) + (q(x+wte) - we) \cdot \nabla \kappa \phi_R^w \leq f(x+wte, \kappa \phi_R^w).$$

Set $\psi_R^w(t,x) = \phi_R^w(t, x - wte)$ if $|x - wte| \leq R$ and $0$ otherwise, then $\kappa \psi_R^w$ is a subsolution of the Cauchy problem (1.2) and thus $u(t,x) \geq \kappa \psi_R^w(t,x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Therefore, $u(t, x + wte) \geq \kappa \phi_R^w(t,x)$ and thus

$$\liminf_{t \to +\infty} u(t, wte) \geq \kappa \min_{t \in \mathbb{R}} \phi_R^w(t, 0) > 0,$$

which yields the propagation property. The proof of Theorem 1.13 in space periodic media is thus complete. □

## 4.2 Approximation of the lower spreading speed

We now turn to the case where the coefficients of the equation have a periodic time dependence in addition to the space periodic dependence. First, we prove that the spreading speed $w^*(e,\mu)$ defined as

$$w^*(e,\mu) = \min_{\lambda \in \mathbb{R}^N, \lambda \cdot e > 0} \frac{-k_\lambda(\mu)}{\lambda \cdot e},$$

is the limit of the sequence of the spreading speeds associated with increasing cylinders of direction $e$. These approximating spreading speeds are not always defined and one requires the direction $e$ to meet the periodicity network so that the coefficients are periodic in the direction $e$. This condition however is not restrictive since the set of all the directions $e$ that meet the periodicity network is dense in $\mathbb{S}^{N-1}$.

**Definition 7** *The periodicity network is the set $L_1\mathbb{Z} \oplus ... \oplus L_N\mathbb{Z}$. We define:*

$$\Sigma = \{e \in \mathbb{S}^{N-1}, \mathbb{R}e \cap (L_1\mathbb{Z} \oplus ... \oplus L_N\mathbb{Z}) \text{ is not empty}\}.$$

**Proposition 4.1** *The set $\Sigma$ is dense in $\mathbb{S}^{N-1}$.*

This is a standard property. For the sake of completeness, we recall it here.

**Proof.** Take $\xi \in \mathbb{S}^{N-1}$. For all $i \geq 2$, there exist two sequences $(p_i^{(n)}, q_i^{(n)})_{n\in\mathbb{N}}$, where $p_i^{(n)} \in \mathbb{Z}, q_i^{(n)} \in \mathbb{N}^*$, such that:

$$\frac{p_i^{(n)}}{q_i^{(n)}} \to \frac{\xi_i L_1}{L_i \xi_1}.$$

Set $k_1^{(n)} = \Pi_{i=2}^N q_i^{(n)}$ and for all $i \geq 2$, $k_i^{(n)} = k_1^{(n)} \frac{p_i^{(n)}}{q_i^{(n)}}$ and $\xi_i'^{(n)} = k_i^{(n)} L_i \xi_1$.

Using the preceding construction, it is readily seen that the vector $\xi^{(n)} = \frac{\xi'^{(n)}}{\|\xi'^{(n)}\|}$ belongs to $\Sigma$ for all $n$ and that $\xi_i'^{(n)} \to \xi_i$ as $n \to +\infty$ for all $i$. As $\|\xi\| = 1$, one has $\|\xi'^{(n)}\| \to 1$ as $n \to +\infty$ and thus $\xi^{(n)} \to \xi$ as $n \to +\infty$. $\square$

We now set in all the sequel of this subsection:

$$\widetilde{A}(t, x_1, ..., x_N) = A(t, L_1x_1, ..., L_Nx_N),$$

$$\widetilde{q}(t, x_1, ..., x_N) = q(t, L_1x_1, ..., L_Nx_N),$$

$$\widetilde{\mu}(t, x_1, ..., x_N) = \mu(t, L_1x_1, ..., L_Nx_N).$$

Using this change of variables, we can assume without loss of generality that

$$L_1 = ... = L_N = 1.$$

In this case, observe that one can choose a convenient basis as in the next lemma.

**Lemma 4.2** *If $e \in \Sigma$, one can find an orthonormal basis $(e^1, ..., e^N)$ of $\mathbb{R}^N$ such that $e^1 = e$ and $e^k \in \Sigma$ for all $k \in [1, N]$.*

**Proof.** We prove our proposition by induction. If $N = 2$, assume that $re \in \mathbb{Q}^2$ and set $e^2 = (-e_2, e_1)$. Then $re^2 \in \mathbb{Q}^2$ and $(e, e^2)$ is an orthonormal basis.

Assume that the property is true at the rank $N$ and take a unit vector $e$ associated with some $r \in \mathbb{R}$ such that $re \in \mathbb{Q}^{N+1}$. Set $(\varepsilon_1, ..., \varepsilon_{N+1})$ the canonical basis of $\mathbb{R}^{N+1}$. Applying the proposition in the space $Span(\varepsilon_1, e)$, one knows that there exists some $e_2 \in \Sigma$ such that

$(e, e^2)$ is an orthonormal basis of $Span(\varepsilon_1, e)$. Set $V = e^\perp$, applying the proposition at rank $N$ in the space $V$ to the unit vector $e^2 \in \Sigma$, one can find an orthonormal basis that satisfy the good conditions. $\square$

For all $e \in \Sigma$, set

$$C_R(e) = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N, \text{ such that } \|x - (x \cdot e)e\| < R\}.$$

Take $r \in \mathbb{R}$ such that $re \cap (L_1\mathbb{Z} \oplus ... \oplus L_N\mathbb{Z})$ is not empty. We define the following eigenelements for all $\lambda > 0$:

$$\left\{\begin{array}{rcl} e^{-\lambda x\cdot e}\mathcal{L}(e^{\lambda x\cdot e}\phi^R_{\lambda e}) & = & k^R_{\lambda e}\phi^R_{\lambda e}, \\ \phi^R_{\lambda e}(t+T, x) & = & \phi^R_{\lambda e}(t, x) \text{ for all } (t, x) \in \mathbb{R} \times C_R(e), \\ \phi^R_{\lambda e}(t, x+re) & = & \phi^R_{\lambda e}(t, x) \text{ for all } (t, x) \in \mathbb{R} \times C_R(e), \\ \phi^R_{\lambda e} > 0 & \text{on} & C_R(e), \\ \phi^R_{\lambda e} \equiv 0 & \text{on} & \partial C_R(e). \end{array}\right. \tag{4.52}$$

The medium is $r$-periodic in the direction $e$, periodic in time and we impose Dirichlet boundary conditions on the boundary of $C_R(e)$. The following proposition is a generalization of theorem 2.7 of [23]. It can be proved with the same method as in [23] and we do not repeat the proof here.

**Proposition 4.3** *These eigenelements are well-defined for all $e \in \Sigma$ and unique up to multiplication of $\phi^R_{\lambda e}$ by a positive constant. Moreover, the function $\lambda \mapsto k^R_{\lambda e}$ is concave and continuous.*

Set $w^*_R(e) = \min_{\lambda>0} \frac{-k^R_{\lambda e}}{\lambda}$. Using the concavity of the function $\lambda \mapsto k_\lambda$, one easily gets the following characterization:

$$w^*_R(e) = \min\{w \in \mathbb{R}, \exists \lambda \in \mathbb{R}, k^R_{\lambda e} + \lambda w = 0\}.$$

It may be shown that this quantity is the spreading speed associated with equation (1.1) with Dirichlet boundary conditions on the boundary of the cylinder $C_R(e)$. This is derived by the same methods as in this paper. But now, this property is not the main goal of the present paper and we leave the details out. Travelling fronts and spreading properties in cylinder have been widely investigated (see [3, 19] for example), but in general, only Neumann boundary conditions, are used. Using this interpretation, it is natural to try to identify the limit of the function $R \mapsto w^*_R(e)$. We now prove that this limit is $w^*(e)$.

**Proposition 4.4** *There holds $k^R_{\lambda e} \searrow \max_{\beta\cdot e=0} k_{\lambda e+\beta}$ as $R \to +\infty$.*

**Corollary 4.5** *The following convergence holds as $R \to +\infty$:*

$$w^*_R(e) \nearrow w^*(e)$$

**Proof.** Using Proposition 4.4, one gets

$$w_R^*(e) \nearrow \min_{\lambda>0}\min_{\beta\cdot e=0}\frac{-k_{\lambda e+\beta}}{\lambda} = \min_{\gamma\cdot e>0}\frac{-k_\gamma}{\gamma\cdot e} = w^*(e).$$

□

The proof of Proposition 4.4 is based on two lemmas.

**Lemma 4.6** *The map $R \mapsto k_{\lambda e}^R$ is decreasing for all $\lambda > 0, e \in \Sigma$.*

**Proof.** Take $R_1 < R_2$ and assume that $k_{\lambda e}^{R_1} \leq k_{\lambda e}^{R_2}$. Take $\phi_\lambda^{R_1}, \phi_\lambda^{R_2}$ two eigenfunction associated with $k_{\lambda e}^{R_1}$ and $k_{\lambda e}^{R_2}$. One has:

$$\partial_t\phi_\lambda^{R_1} - \nabla\cdot(A\nabla\phi_\lambda^{R_1}) + q\cdot\phi_\lambda^{R_1} - \mu\phi_\lambda^{R_1} - k_\lambda^{R_2}\phi_\lambda^{R_1} = (k_\lambda^{R_1} - k_\lambda^{R_2})\phi_\lambda^{R_1} \leq 0 \text{ in } C_{R_2}(e).$$

Thus $\phi_\lambda^{R_1}$ is a subsolution of the equation satisfied by $\phi_\lambda^{R_2}$ on $C_{R_2}(e)$. Next, set:

$$\kappa^* = \sup\{\kappa > 0, \kappa\phi_\lambda^{R_1} < \phi_\lambda^{R_2} \text{ in } \overline{C_{R_2}(e)}\}.$$

As $\phi_\lambda^{R_2}$ is bounded and $\phi_\lambda^{R_1}$ has a positive infimum over $\overline{C_{R_2}(e)}$ since it is an periodic function, $\kappa^*$ is finite and positive. Set $z = \phi_\lambda^{R_2} - \kappa^*\phi_\lambda^{R_1}$. There exists a sequence $(t_n, x_n) \in \overline{C_{R_2}(e)}$ such that $z(t_n, x_n) \to 0$. Set $z_n(t,x) = z(t+t_n, x+x_n)$, this function satisfies:

$$\partial_t z_n - \nabla\cdot(A(t+t_n, x+x_n)\nabla z_n) + q(t+t_n, x+x_n)\cdot z_n - \mu(t+t_n, x+x_n)z_n - k_\lambda^{R_2}z_n \leq 0 \text{ in } C_{R_2}(e).$$

The periodicity yields that, up to extraction, we can assume that the sequence $(A(.+t_n, .+x_n), q(.+t_n, .+x_n), \mu(.+t_n, .+x_n))$ converges uniformly in $\overline{C_R(e)}$ to a function $(A_\infty, q_\infty, \mu_\infty)$. From the classical Schauder estimates we infer that, up to extraction of a subsequence, the sequence $(z_n)$ uniformly converges to a function $z_\infty$ that satisfies:

$$\partial_t z_\infty - \nabla\cdot(A_\infty\nabla z_\infty) + q_\infty\cdot z_\infty - \mu_\infty z_\infty - k_\lambda^{R_2}z_\infty \leq 0 \text{ in } C_{R_2}(e).$$

As $z_\infty \geq 0$ and $z_\infty(0,0) = 0$, the strong parabolic maximum principle yields that for all $t \leq 0, x \in \mathbb{R}^N$, one has $z_\infty(t,x) \equiv 0$. The periodicity thus yields that $z_\infty \equiv 0$. In the other hand, one has $z(t,x) = z_n(t-t_n, x-x_n)$. The uniform convergence thus yields that $z \equiv 0$, which is a contradiction since $\phi_\lambda^{R_1} > 0$ in $\overline{C_{R_2}(e)}$. □

**Lemma 4.7** *For all $\lambda > 0, e \in \Sigma$ and $\beta \in \mathbb{R}^N$ such that $\beta\cdot e = 0$, the following inequality holds:*

$$k_{\lambda e}^R > k_{\lambda e+\beta}.$$

**Proof.** Assume that $k_{\lambda e}^R \leq k_{\lambda e+\beta}$ and consider $\phi_{\lambda e}^R$ and $\psi_{\lambda e+\beta}$ two eigenfunctions associated with $k_{\lambda e}^R$ and $k_{\lambda e+\beta}$. It is easy to see that $(t,x) \mapsto e^{-\beta\cdot x}\phi_{\lambda e}^R(t,x)$ is a subsolution of the equation satisfied by $\psi_{\lambda e+\beta}$ in $C_R(e)$. Set:

$$\kappa^* = \sup\{\kappa > 0, \kappa e^{-\beta\cdot x}\phi_\lambda^R < \psi_{\lambda e+\beta} \text{ in } \overline{C_R(e)}\}.$$

This quantity is finite and positive. We define $z = \psi_{\lambda e+\beta} - \kappa^* e^{-\beta\cdot x}\phi^R_\lambda$. Take $(t_n, x_n) \in \overline{C_R(e)}$ and consider the sequence $z_n(t,x) = z(t+t_n, x+x_n)$. As in the proof of the preceding lemma, it is possible to extract a subsequence that uniformly converges to a function $z_\infty$. The strong maximum principle and the periodicity yield $z_\infty \equiv 0$ and thus $z \equiv 0$, which is a contradiction. □

**Proof of Proposition 4.4.** First of all, thanks to Lemma 4.2, one can find an orthonormal basis $(e^1, ..., e^N)$ such that $e = e^1$ and $e^k \in \Sigma$ for all $k$. Therefore, the coefficients $A, q$ and $\mu$ are all space-periodic in the directions $e^1, ..., e^N$. Thus, up to some rotation, we can assume that $e = \varepsilon^1$ in the sequel, where $\varepsilon^1$ stands for the first vector of the canonical basis. In other words, $re = L_1$.

As $R \mapsto k^R_{\lambda e}$ is a decreasing bounded function, it admits a limit $k^\infty_{\lambda e}$ as $R \to +\infty$. The Schauder classical estimates enable us to extract a sequence $R_n \to +\infty$ such that the eigenfunctions sequence of $(\phi^{R_n}_{\lambda e})$, normalized by the condition $\phi^{R_n}_{\lambda e}(0,0) = 1$, converges to a nonnegative function $\phi^\infty_{\lambda e}$ that satisfies:

$$\left\{\begin{array}{rcl} e^{-\lambda x\cdot e}\mathcal{L}(e^{\lambda x\cdot e}\phi^\infty_{\lambda e}) & = & k^\infty_{\lambda e}\phi^\infty_{\lambda e}, \\ \phi^\infty_{\lambda e}(t+T,x) & = & \phi^\infty_{\lambda e}(t,x) \text{ for all } (t,x) \in \mathbb{R}\times\mathbb{R}^N, \\ \phi^\infty_{\lambda e}(t,x+L_1) & = & \phi^\infty_{\lambda e}(t,x) \text{ for all } (t,x) \in \mathbb{R}\times\mathbb{R}^N, \\ \phi^\infty_{\lambda e}(0,0) & = & 1. \end{array}\right. \tag{4.53}$$

The strong maximum principle yields that this function is positive.

Next, we set $\varphi(t,x) = \phi^\infty_{\lambda e}(t,x)e^{\lambda e\cdot x}$ and $\psi(t,x) = \frac{\varphi(t,x+L_2)}{\varphi(t,x)}$, then $\psi$ satisfies:

$$\partial_t\psi - \nabla\cdot(A(t,x)\nabla\psi) + q(t,x)\cdot\nabla\psi - 2\frac{\nabla\varphi}{\varphi}A(t,x)\nabla\psi = 0.$$

As the coefficients $A$, $q$ and $f(\cdot,\cdot,s)$ are of class $\mathcal{C}^{\delta/2,\delta}(\mathbb{R}\times\mathbb{R}^N)$ for all $s \geq 0$ uniformly over $\mathbb{R}\times\mathbb{R}^N$, the Krylov-Safonov-Harnack inequality yields that $\psi$ is uniformly bounded over $\mathbb{R}\times\mathbb{R}^N$. Set $m = \sup_{\mathbb{R}\times\mathbb{R}^N}\psi > 0$ and $(x_n, t_n) \in [0,T]\times\mathbb{R}^N$ such that: $\psi(x_n,t_n) \to m$ as $n \to \infty$.

There exists $y_n \in \overline{C}$ so that for all n, $x_n - y_n \in L_1\mathbb{Z}\times ...\times L_N\mathbb{Z}$. We may assume that $y_n \to y_\infty \in \overline{C}$ and $t_n \to t_\infty \in [0,T]$.

Set $\psi_n(t,x) = \psi(t+t_n, x+x_n)$ and $\varphi_n(t,x) = \frac{\varphi(t+t_n,x+x_n)}{\varphi(t_n,x_n)}$. The function $\varphi_n$ satisfies:

$$\partial_t\varphi_n - \nabla\cdot(A(t+t_n,x+y_n)\nabla\varphi_n) + q(t+t_n,x+y_n)\cdot\nabla - \mu(t+t_n,x+y_n)\varphi_n = k^\infty_{\lambda e}\varphi_n.$$

Using the classical parabolic estimates, we may suppose, up to extraction, that $\varphi_n \to \varphi_\infty$ in $C^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^N)$. The function $\varphi_\infty$ satisfies:

$$\left\{\begin{array}{l} \partial_t\varphi_\infty - \nabla\cdot(A(t+t_\infty,x+y_\infty)\nabla\varphi_\infty) + q(t+t_\infty,x+y_\infty)\cdot\nabla\varphi_\infty, \\ -\mu(t+t_\infty,x+y_\infty)\varphi_\infty = k^\infty_{\lambda e}\varphi_\infty, \\ \varphi_\infty \text{ periodic in t}, \\ \varphi_\infty > 0,\ \varphi_\infty(0,0) = 1. \end{array}\right.$$

On the other hand, $\psi_n$ is the solution of:

$$\partial_t\psi_n - \nabla\cdot(A(t+t_n,x+y_n)\nabla\psi_n) + q(t+t_n,x+y_n)\cdot\nabla\psi_n - 2\frac{\nabla\varphi_n}{\varphi_n}A(x+y_n,t+t_n)\nabla\psi_n = 0.$$

So, we may assume, up to extraction, that $\psi_n \to \psi_\infty$, where $\psi_\infty$ satisfies:

$$\partial_t\psi_\infty - \nabla\cdot(A(t+t_\infty, x+y_\infty)\nabla\psi_\infty) + q(t+t_\infty, x+y_\infty)\cdot\nabla\psi_\infty - 2\frac{\nabla\varphi_\infty}{\varphi_\infty}A(t+t_\infty, x+y_\infty)\nabla\psi_\infty = 0.$$

Furthermore, $\psi_\infty \leq m$ and, as $\psi_n(0,0) = \psi(t_n, x_n) \to m$, $\psi_\infty(0,0) = m$. Using the strong parabolic maximum principle and the time periodicity, we get $\psi_\infty \equiv m$.

Since $m > 0$, we can define $\beta_2 = \frac{1}{|L_2|}\ln(m)$. Then the function $\varphi_\infty exp(-\beta_2 x_2)$ is $L_2$-periodic. Going on the construction, one can find a $\beta_i$ for all $i \geq 2$ and then get a function $\theta$ verifying:

$$\begin{cases} \partial_t\theta - \nabla\cdot(A(t+t_\infty, x+y_\infty)\nabla\theta) + q(t+t_\infty, x+y_\infty)\cdot\nabla\theta - \mu(t+t_\infty, x+y_\infty)\theta &= k^\infty_{\lambda e}\theta, \\ \theta(t,x)exp(-(\lambda e+\beta)\cdot x) \text{ is periodic in } t, x_1, ..., x_N,\ \theta > 0,\ \theta(0,0) &= 1. \end{cases}$$

Therefore, since the periodic principal eigenvalue $k_\lambda$ is invariant under a translation in $(t,x)$ of the coefficients, there exists a positive constant $C$ such that the function $\theta$ is equal to $C\phi_{\lambda+\beta}$ and $k^\infty_{\lambda e} = k_{\lambda+\beta}$, where $\beta\cdot e = \beta_1 = 0$. On the other hand, Lemma 4.7 yields that $k^\infty_{\lambda e} \geq \max_{\beta\cdot e=0} k_{\lambda e+\beta}$. As the equality holds for at least one $\beta$ such that $\beta\cdot e = 0$, we finally have $k^\infty_{\lambda e} = \max_{\beta\cdot e=0} k_{\lambda e+\beta}$. $\square$

## 4.3 Proof of Theorem 1.13

We now prove Theorem 1.13. First, up to a shift in time, one can assume that $u_0$ is positive and continuous. We begin with the following lemma, which is a generalization of a theorem that had been proved by Mallordy and Roquejoffre [19]:

**Lemma 4.8** *For all $R > 0$ and $e \in \Sigma$, there exists $\delta > 0$ which does not depend on $e$ such that for all $w \in [w^*_R(e) - \delta, w^*_R(e))$, there exists a complex $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and a solution $\phi_\lambda \in \mathcal{C}^{1,2}(\mathbb{R}^N, \mathbb{C})$ of:*

$$\begin{cases} e^{-\lambda x\cdot e}\mathcal{L}(e^{\lambda x\cdot e}\phi_\lambda) &= -\lambda w\psi, \\ \phi_\lambda(t+T, x) &= \phi_\lambda(t,x) \text{ for all } (t,x) \in \mathbb{R}\times C_R(e), \\ \phi_\lambda(t, x+re) &= \phi_\lambda(t,x) \text{ for all } (t,x) \in \mathbb{R}\times C_R(e), \\ Re(\phi_\lambda) > 0 & \text{on} \quad C_R(e), \\ Re(\phi_\lambda) \equiv 0 & \text{on} \quad \partial C_R(e). \end{cases} \tag{4.54}$$

In order to understand this lemma, it is useful to think about the homogeneous one-dimensional case. In this case, the linearized equation admits positive exponential solutions, that is solutions of the form $(t,x) \mapsto e^{\lambda\cdot x+wt}\phi_\lambda(t,x), \lambda \in \mathbb{R}$, if and only if $w \geq 2\sqrt{f'(0)}$. Otherwise, there exist exponential solutions, but with $\lambda \in \mathbb{C}$ and these solutions cannot be uniformly positive. The preceding lemma selects this kind of solutions.

**Proof of Lemma 4.8.** Let us first prove this lemma in a neighborhood of all $e \in \Sigma$. Fix $e$ and set $\lambda^* = \lambda_{w^*_R(e)}$. The family of operators $L_{\lambda e}$ depends analytically on $\lambda$, in the sense of Kato [17]. From the Kato-Rellich theorem, there exists a neighborhood $V$ of $\lambda^*$ in $\mathbb{C}$, such that there exists a simple eigenvalue $\widetilde{k}^R_{\lambda e}$ continuing $k^R_\lambda$ on all $V$ analytically and a family of eigenfunctions $\phi_\lambda$ analytic in $\lambda$, where $\phi_{\lambda^*}$ is the positive principal eigenfunction associated with $w_R^*(e)$.

For all $\xi \in \Sigma$, $w \in \mathbb{R}$, set $F_{w,\xi}(\lambda) = \widetilde{k}_{\lambda\xi}^R + \lambda w$. This function is analytic in $\lambda$ and converges locally uniformly to $F_{w_R^*(e)}$ as $\xi \to e$ and $w \to w_R^*(e)$. As $F_{w_R^*(e),e}(\lambda^*) = 0$, the Rouché theorem yields the existence of some neighborhood $V_e$ of $(w_R^*(e), e)$ such that for all $(w, \xi) \in V_e$, there exists some $\lambda_{w,\xi} \in \mathbb{C}$ such that $F_{w,\xi}(\lambda_{w,\xi}) = 0$ and $\lambda_{w,\xi} \to \lambda^*$ as $\xi \to e$ and $w \to w_R^*(e)$.

Using the Schauder estimates, one can prove that $\phi_{\lambda_{w,\xi}} \to \phi_{\lambda^*}$ uniformly in $t$ and $x$. Thus $Re(\phi_{\lambda_{w,\xi}}) \to \phi_{\lambda^*} > 0$ and taking $V_e$ small enough, we can assume that $Re(\phi_{\lambda_{w,\xi}}) > 0$ for all $(\xi, w) \in V_e$. Lastly, if $-w_R^*(-\xi) < w < w_R^*(\xi)$, it is impossible to have $\lambda_{w,\xi} \in \mathbb{R}$. Otherwise, this would contradict the definition of $w_R^*(\xi)$.

Next, as $e \in \Sigma \mapsto w_R^*(e)$ is continuous, $\mathbb{S}^{N-1}$ is compact and $\Sigma$ is dense in $\mathbb{S}^{N-1}$, we can extract a finite family $(V_{e_k})_{1\geq k\geq m}$ such that $\{(w_R^*(e), e), e \in \Sigma\} \subset \cup_{1\geq k\geq m} V_{e_k}$. Thus, there exists some $\delta > 0$ such that for all $e \in \Sigma$, for all $w \in [w_R^*(e) - \delta, w_R^*(e))$, there exists $\lambda_{w,e} \in \mathbb{C}\backslash\mathbb{R}$ such that $F_{w,e}(\lambda_{w,e}) = 0$ and the proposition is proved. $\square$

We are now able to construct a subsolution with compact support as in Proposition 2.1 for all $e \in \Sigma$. Take $w \in (w_R^*(e) - \delta, w_R^*(e))$, Lemma 4.8 yields some $\lambda$ and $\phi_\lambda$ associated with $w$. Set:

$$v_0(t, x) = Re(\phi_\lambda(t, x)e^{\lambda(x\cdot e+wt)}).$$

One has:

$$v_0(t, x) = e^{\lambda_r(x\cdot e+wt)}[\phi_{\lambda,r}cos(\lambda_i(x \cdot e + wt)) + \phi_{\lambda,i}sin(\lambda_r(x \cdot e + wt))], \tag{4.55}$$

where $\phi_{\lambda,i}, \phi_{\lambda,r}, \lambda_i, \lambda_r$ denote the imaginary and real parts of $\lambda$ and $\phi$. For all $n \in \mathbb{Z}$, if $(e \cdot x + ct) = 2n\pi/\lambda_i$, then $w_0(t, x) > 0$. Similarly, for all $n \in \mathbb{Z}$, if $(e \cdot x + ct) = (2n+1)\pi/\lambda_i$, then $v_0(t, x) < 0$. Thus, it follows from (4.55) that there exist an interval $(b_1, b_2) \subset \mathbb{R}$ and an unbouded domain $D \subset C_R(e)$ such that:

$$\begin{cases} D \subset \big\{(t, x) \in C_R(e), x \cdot e + wt \in [b_1, b_2]\big\}, \\ 0 < v_0(t, x) < \varepsilon, \text{for all } (t, x) \in D, \\ v_0(t, x) = 0, \text{ for } (t, x) \in \partial D. \end{cases} \tag{4.56}$$

where $\varepsilon = \inf_{x\in B_R} u_0(x) > 0$.

Set $v$ the function:

$$v(t, x) = \begin{cases} v_0(t, x) \text{ if } (t, x) \in D, \\ 0 \text{ otherwise.} \end{cases} \tag{4.57}$$

This function has a compact support and it is a subsolution of equation (1.1).

One has $v(0, x) \geq u_0(x)$ for all $x \in \mathbb{R}^N$, the maximum principle leads to $u \leq v$. We remark that

$$v(t, x - wte) = Re(e^{\lambda x\cdot e}\phi_\lambda(t, x - wte)).$$

As $\phi_\lambda$ is space-time periodic, one has

$$\inf_{e\in\mathbb{S}^{N-1}} \inf_{t\in\mathbb{R}_+} v(t, x - wte) > 0,$$

for all $x \in \mathbb{R}^N$ and $w \in (w_R^*(e) - \delta, w_R^*(e))$.

We recall that $\Sigma$ is dense in $\mathbb{S}^{N-1}$, thus the continuous function $e \mapsto w_R^*(e)$ admits a continuous extension $e \mapsto \widetilde{w}_R^*(e)$ to the compact set $\mathbb{S}^{N-1}$. For all positive $R$ and $\delta'$, set:

$$\Omega = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \forall e \in \mathbb{S}^{N-1}, -e \cdot x - (\widetilde{w}_R^*(e) - \delta')t < R\}.$$

Then taking $\delta'$ small enough and $R$ large enough, one may assume that:

$$\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, x \in tK\} \subset \Omega.$$

We know from the previous step that there exists some $\varepsilon > 0$ such that

$$\forall (t,x) \in \mathbb{R}_+ \times B_R, \forall e \in \Sigma, u(t, x - (w_R^*(e) - \delta')te) \geq \varepsilon.$$

As $\Sigma$ is dense in $\mathbb{S}^{N-1}$, this inequality can be generalized:

$$\forall (t,x) \in \mathbb{R}_+ \times B_R, e \in \mathbb{S}^{N-1}, u(t, x - (\widetilde{w}_R^*(e) - \delta')te) \geq \varepsilon.$$

As $\inf_{\overline{B_R}} u_0 > 0$, one may assume that $\varepsilon$ is small enough so that $\inf_{\overline{B_R}} u_0 \geq \varepsilon$. Hence for all $(t,x) \in \partial\Omega, u(t,x) \geq \varepsilon$.

As $f$ is of class $\mathcal{C}^1$ in the neighborhood of 0 and $k_0(\mu) < 0$, there exists some $\kappa_0 > 0$ such that

$$\forall 0 < \kappa < \kappa_0,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ f(t,x,\kappa) \leq (\mu(t,x) - k_0(\mu))\kappa.$$

Set $z = u - \phi_0$, where $\phi_0$ is some eigenfunction associated with $k_0(\mu)$ such that $\|\phi_0\|_\infty < \min\{\varepsilon, \kappa_0\}$. One easily remarks that $\phi_0$ is a subsolution of equation (1.1). In order to apply the modified maximum principle proved in Lemma 2.2, define $b(t,x) = \frac{f(t,x,u) - f(t,x,\phi_0)}{u - \phi_0}$. As $f$ is Lipschitz-continuous in $u$ uniformly in $(t,x)$, the function $b$ is bounded. The function $z$ satisfies the equation:

$$\partial_t z - \nabla \cdot (A \nabla z) + q \cdot z + bz \geq 0.$$

Thus, the hypothesis of Lemma 2.2 are satisfied and one has $z \geq 0$, that is $u \geq \phi_0$ in $\Omega$. This shows that

$$\liminf_{t \to +\infty} \inf_{x \in tK} u(t,x) \geq \min_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} \phi_0(t,x) > 0.$$

and thus the proof is complete. □

## 4.4 Proof of additional results

**Proof of Theorem 1.14.** In this case, the spreading speeds $w^*(\mu)$ and $w^{**}(\mu)$ are still well-defined, the main difference is that these two quantities are both negative or positive if $\lambda_1 \geq 0$. Anyway, the preceding proof still works, because the set

$$\Omega = \{(t,x) \in \mathbb{R} \times \mathbb{R}^N, x \in (-R - w_R^* t, R + w_R^{**} t)\}$$

remains bounded and thus it is possible to apply our modified weak maximum principle. If the dimension $N$ is higher than 2, then this set is not bounded anymore and the maximum principle does not necessarily hold. □

**Proof of Proposition 1.12.** We need to adapt the proof of Proposition 2.13 of [23], where we defined the generalized principal eigenvalue with the help of time periodic subsolutions of the linearized equation instead of general supersolutions.

Set $l_\lambda$ the space-time periodic principal eigenvalue and $k_\lambda$ the generalized principal eigenvalue. Taking $\varphi$ a periodic principal eigenfunction associated with $l_\lambda$ as a test-function, one gets $k_\lambda \geq l_\lambda$. Next, take $k > l_\lambda$ and assume that there exists a function $\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)$ such that $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \phi > 0$ and $P_\lambda \phi \geq k\phi$. We now search for a contradiction in order to prove that such a $k$ does not exist and that $l_\lambda \geq k_\lambda$.

Set $\gamma = \inf_{(0,T)\times C} \frac{\phi}{\varphi}$, then $0 < \gamma < \infty$ and one can define $z = \phi - \gamma\varphi$. This function is nonnegative and $\inf z = 0$. Set $\varepsilon = (k - l_\lambda)\min\varphi > 0$. One has $(P_\lambda - k)(z) \geq \gamma\varepsilon > 0$.

Consider a nonnegative function $\theta \in \mathcal{C}^2(\mathbb{R}^N)$ that satisfies:

$$\theta(0) = 0, \lim_{|x|\to+\infty} \theta(x) = 1, \|\theta\|_{\mathcal{C}^2} < \infty.$$

There exists $\kappa > 0$ sufficiently large such that:

$$\forall y \in \mathbb{R}^N, (P_\lambda - k)(\tau_y\theta) > -\kappa\gamma\varepsilon/2,$$

where we denote $\tau_y\theta = \theta(. - y)$.

Since $\inf z = 0$, one can find some $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^N$ such that:

$$z(t_0, x_0) < \min\{\frac{1}{\kappa}, \frac{\gamma\varepsilon}{2\|\zeta - k\|_\infty}\}$$

where

$$\zeta = \eta + \lambda^2 eAe + \lambda\nabla\cdot(Ae) - \lambda q\cdot e$$

and $\|\zeta - k\|_\infty^{-1} = +\infty$ if $\zeta - k \equiv 0$. Since $\lim_{|x|\to+\infty}\theta(x) = 1$, there exists a positive constant $R$ such that $\tau_{x_0}\theta(x)/\kappa > z(t_0, x_0)$ if $|x - x_0| \geq R$. Consequently, setting $\widetilde{z} = z + \tau_{x_0}\theta(x)/\kappa$, one finds for all $|x - x_0| \geq R$, that:

$$\widetilde{z}(t,x) \geq \tau_{x_0}\theta(x)/\kappa > z(t_0,x_0) = \widetilde{z}(t_0,x_0).$$

Hence, if $\alpha = \inf_{\mathbb{R}\times\mathbb{R}^N} \widetilde{z}$, then

$$\alpha \leq \widetilde{z}(t_0,x_0) = z(t_0,x_0) < \frac{\gamma\varepsilon}{2\|\zeta - k\|_\infty}$$

and

$$\begin{array}{rcl}(P_\lambda - k)(\widetilde{z} - \alpha) & = & (P_\lambda - k)(z) + \frac{1}{\kappa}(P_\lambda - k)(\tau_{x_0}\theta(x)) - \zeta(t,x)\alpha + k\alpha \\ & > & \gamma\varepsilon - \frac{\gamma\varepsilon}{2} - \|\zeta - k\|_\infty\alpha > 0\end{array}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$. Thus, Lemma 3.4 yields $\inf_{\mathbb{R}\times\mathbb{R}^N}(\widetilde{z} - \alpha) > 0$, which is impossible. $\square$

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