The speed of propagation for KPP type problems. I - Periodic framework

Henri Berestycki [a], François Hamel [b] and Nikolai Nadirashvili [c*]

[a] EHESS, CAMS, 54 Boulevard Raspail, F-75006 Paris, France

[b] Université Aix-Marseille III, LATP, Faculté des Sciences et Techniques, Case cour A

Avenue Escadrille Normandie-Niemen, F-13397 Marseille Cedex 20, France

[c] University of Chicago, Department of Mathematics,

5734 S. University Avenue, Chicago, IL 60637-1546, USA

Abstract. This paper is devoted to some nonlinear propagation phenomena in periodic and more general domains, for reaction-diffusion equations with Kolmogorov-Petrovsky-Piskunov (KPP) type nonlinearities. The case of periodic domains with periodic underlying excitable media is a follow-up of the article [7]. It is proved that the minimal speed of pulsating fronts is given by a variational formula involving linear eigenvalue problems. Some consequences concerning the influence of the geometry of the domain, of the reaction, advection and diffusion coefficients are given. The last section deals with the notion of asymptotic spreading speed. The main properties of the spreading speed are given. Some of them are based on some new Liouville type results for nonlinear elliptic equations in unbounded domains.

Contents

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Introduction

This paper is the first in a series of two in which we address spreading and propagation properties attached with reaction-diffusion type equations in a general framework. We consider reaction-terms of the type associated with Fisher or KPP (for Kolmogorov, Petrovsky and Piskunov) equations. These properties are well understood in the homogeneous framework which we recall eblow. Here and in part II we consider *heterogeneous* problems. Part II will be devoted to propagation properties in very general domains. The present paper deals with the periodic case where both the equation and the domain have periodic structures. The precise setting and assumptions will be given shortly. But before that, let us recall some of the basic features of the homogeneous equations.

Consider the Fisher-KPP equation :

$$u_t - \Delta u = f(u) \quad \text{in } \mathbb{R}^N. \tag{0.1}$$

It has been introduced in the celebrated papers of Fisher (1937) and KPP (1937) originally motivated by models in biology. Here the main assumption is that f is say a C^1 function satisfying

$$f(0) = f(1) = 0, \quad f'(1) < 0, \quad f'(0) > 0, \quad f > 0 \text{ in } (0,1), \quad f < 0 \text{ in } (1, +\infty), \tag{0.2}$$

$$f(s) \leq f'(0)s, \quad \forall s \in [0,1]. \tag{0.3}$$

Archetypes of such nonlinearities are $f(s) = s(1-s)$ or $f(s) = s(1-s^2)$.

Two fundamental features of this equation account for its success in representing propagation (or invasion) and spreading. First, this equaton has a family of planar travelling fronts. These are solutions of the form

$$u(t,x) = U(x \cdot e - ct) \tag{0.4}$$

where e is a fixed vector of unit norm which is the direction of propagation, and $c > 0$ is the speed of the front. Here $U : \mathbb{R} \mapsto \mathbb{R}$ is given by

$$\begin{cases} -U'' - cU' = f(U) & \text{in } \mathbb{R} \\ U(-\infty) = 1, \ U(+\infty) = 0. \end{cases} \tag{0.5}$$

In the original paper of Kolmogorov, Petrovsky and Piskunov, it was proved that, under the above assumptions, there is a threshold value $c^* = 2\sqrt{f'(0)} > 0$ for the speed c. Namely, no fronts exist for $c < c^*$, and, for each $c \geq c^*$, there is a unique front of the type (0.4-0.5). Uniqueness is up to shift in space or time variables.

Another fundamental property of this equation was established mathematically by Aronson and Weinberger (1978). It deals with the asymptotic speed of spreading. Namely, if u_0 is a nonnegative continuous function in $\mathbb{R}^N$ with compact support and $u_0 \not\equiv 0$, then the solution $u(t,x)$ of (0.1) with initial condition u_0 at time $t = 0$ spreads with the speed c^* in all directions for large times : as $t \to +\infty$, $\max_{|x| \le ct} |u(t,x) - 1| \to 0$ for each $c \in [0, c^*)$, and $\max_{|x| \ge ct} u(t,x) \to 0$ for each $c > c^*$.

In this paper, we consider a general heterogeneous periodic framework extending (0.1). The heterogeneous character arises both in the equation and in the underlying domain. The types of equations we consider here are :

$$\left\{ \begin{array}{l} u_t - \nabla \cdot (A(x)\nabla u) + q(x) \cdot \nabla u = f(x,u) \ \text{ in } \Omega \\ \nu \cdot A\nabla u = 0 \ \text{ on } \partial\Omega, \end{array} \right. \tag{0.6}$$

where ν denotes the outward unit normal on $\partial\Omega$. It will be assumed throughout this paper that the matrix $A(x)$, the vector $q(x)$ and the reaction term $f(x,s)$ as well as the geometry Ω are periodic. Precise assumptions will be described shortly. Note that even equation (0.1), if set in a periodic domain (e.g. the space with a periodic array of holes), acquires the features of a non-homogeneous equation. That equation will be considered in general (non-periodic) domains in Part II [10].

Here, in the periodic setting, we address three types of questions.

1) What is the speed of generalized travelling fronts in periodic structures –we recall the definition of such fronts below– ? A formula which we announced in [7] is proved here.

2) Using a formula of Gärtner and Freidlin [39], we relate the asymptotic speed of spreading in a periodic domain to that of the minimal speed of propagation. Contrarily to the homogeneous equation, as we will see on an example, these two speeds may not be the same.

3) We then proceed to derive several important consequences on the minimal speed of propagation and on the asymptotic spreading speed. Effects of stirring, of reaction, and of geometry will be established here rigorously. These formulas indeed allow us to prove properties of the following kind. The presence of holes or of an undulating boundary always hinder the progression or the spreading. On the contrary, any stirring by a flow always increases that speed.

In the next section we introduce the general setting with precise assumptions and we state the main results achieved in this paper. Their proofs take up the remaining sections.

1 The periodic framework and main results

1.1 Speed of propagation of pulsating travelling fronts in periodic domains

This section deals with pulsating fronts travelling in a given unbounded periodic domain under the effects of diffusion, reaction and possibly advection by a given underlying flow. One of the most important issues in this context is the determination of the speed of propagation of fronts. A variational formula for the minimal speed of propagation is derived.

This notion of propagation of travelling fronts for the homogeneous equation (0.1) can be extended to that of *pulsating* travelling fronts in a more general class of *periodic* domains and for a more general class of reaction-diffusion-advection equations in periodic excitable media.

We now describe the general periodic framework. Let $N \geq 1$ be the space dimension and let d be an integer such that $1 \leq d \leq N$. Call $x = (x_1, \cdots, x_d)$ and $y = (x_{d+1}, \cdots, x_N)$. Let $L_1, \cdots, L_d$ be d positive numbers and let Ω be a C^3 nonempty connected open subset of $\mathbb{R}^N$ such that

$$\left\{ \begin{array}{l} \exists R \geq 0, \quad \forall (x,y) \in \Omega, \quad |y| \leq R, \\ \forall (k_1, \cdots, k_d) \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}, \quad \Omega = \Omega + \displaystyle\sum_{i=1}^{d} k_i e_i, \end{array} \right. \tag{1.1}$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of $\mathbb{R}^N$. Let C be the set defined by

$$C = \{(x,y) \in \Omega, \ x \in (0, L_1) \times \cdots \times (0, L_d)\}.$$

Since $d \geq 1$, Ω is unbounded and C is its periodicity cell. In all what follows, a field w is said to be L-periodic with respect to x in Ω if $w(x_1 + k_1, \cdots, x_d + k_d, y) = w(x_1, \cdots, x_d, y)$ almost everywhere in Ω, for all $k = (k_1, \cdots, k_d) \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}$.

Before going further on, let us point out that this framework includes several types of simpler geometrical configurations. The case of the whole space $\mathbb{R}^N$ corresponds to $d = N$, where $L_1, \cdots, L_N$ are any positive numbers. The case of the whole space $\mathbb{R}^N$ with a periodic array of holes can also be considered. The case $d = 1$ corresponds to domains which have only one unbounded dimension, namely infinite cylinders which may be straight or have oscilatting periodic boundaries, and which may or not have periodic holes. The case $2 \leq d \leq N-1$ corresponds to infinite slabs.

We are interested in propagation phenomena for the following reaction-diffusion-advection equation, with unknown u, set in the periodic domain Ω :

$$\left\{ \begin{array}{rcl} u_t &=& \nabla \cdot (A(x,y)\nabla u) + q(x,y) \cdot \nabla u + f(x,y,u), \quad t \in \mathbb{R}, \ (x,y) \in \Omega, \\ \nu A \nabla u(x,y) &=& 0, \quad t \in \mathbb{R}, \ (x,y) \in \partial\Omega. \end{array} \right. \tag{1.2}$$

Such equations arise especially in simple combustion models for flame propagation [75], [90], [95], as well as in models in biology and for population dynamics [30], [69], [81]. The passive quantity u typically stands for the temperature or a concentration which diffuses and is transported in a periodic excitable medium.

Let us now detail the assumptions on the coefficients of (1.2). First, the diffusion matrix $A(x,y) = (A_{ij}(x,y))_{1 \leq i,j \leq N}$ is a symmetric $C^{2,\delta}(\overline{\Omega})$ (with $\delta > 0$)[1] matrix field satisfying

$$\left\{ \begin{array}{l} A \text{ is } L\text{-periodic with respect to } x, \\ \exists\ 0 < \alpha_1 \leq \alpha_2, \ \forall (x,y) \in \overline{\Omega}, \ \forall \xi \in \mathbb{R}^N, \quad \alpha_1 |\xi|^2 \leq A_{ij}(x,y)\xi_i\xi_j \leq \alpha_2 |\xi|^2 \end{array} \right. \tag{1.3}$$

(we use the usual summation convention with $1 \leq i,j \leq N$). The boundary condition $\nu A \nabla u(x,y)$ stands for $\nu_i(x,y)A_{ij}(x,y)\partial_{x_j} u(t,x,y)$ and ν denotes the unit outward normal

[1] The smoothness assumptions on A, as well as on q and f below, are made to ensure the applicability of some a priori gradients estimates for the solutions of some approximated elliptic equations obtained from (1.2) (see (2.9) in Section 2). These gradient estimates are obtained for smooth (C^3) solutions through a Bernstein-type method, [8]. We however believe that the smoothness assumptions on A, as well as on q and f, could be relaxed, by approximating A, q and f by smoother coefficients.

to Ω. When A is the identity matrix, then this boundary condition reduces to the usual Neumann condition.

The underlying advection $q(x,y) = (q_1(x,y), \cdots, q_N(x,y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) vector field satisfying

$$\left\{\begin{array}{l} q \text{ is } L\text{-periodic with respect to } x, \\ \nabla\cdot q = 0 \ \text{ in } \overline{\Omega}, \\ q\cdot\nu = 0 \ \text{ on } \partial\Omega, \\ \forall 1\le i\le d, \quad \displaystyle\int_C q_i\,dx\,dy = 0. \end{array}\right. \tag{1.4}$$

The divergence-free assumption means that the underlying flow is incompressible. The vector field q is tangent on $\partial\Omega$ and its first d components have been normalized. The flow q may represent some turbulent fluctuations with respect to a mean field.

Lastly, let $f(x,y,u)$ be a nonnegative[2] function defined in $\overline{\Omega}\times[0,1]$, such that

$$\left\{\begin{array}{l} f\ge 0,\ f \text{ is } L\text{-periodic with respect to } x \text{ and of class } C^{1,\delta}(\overline{\Omega}\times[0,1]), \\ \forall (x,y)\in\overline{\Omega}, \quad f(x,y,0)=f(x,y,1)=0, \\ \exists \rho\in(0,1), \quad \forall(x,y)\in\overline{\Omega}, \quad \forall\, 1-\rho\le s\le s'\le 1, \quad f(x,y,s)\ge f(x,y,s'), \\ \forall s\in(0,1), \quad \exists(x,y)\in\overline{\Omega}, \quad f(x,y,s)>0, \\ \forall(x,y)\in\overline{\Omega}, \quad f'_u(x,y,0) := \lim_{u\to 0^+} f(x,y,u)/u > 0. \end{array}\right. \tag{1.5}$$

The simplest case of such a monostable function $f(x,y,u)$ satisfying (1.5) is when $f(x,y,u) = g(u)$ and the $C^{1,\delta}$ function g satisfies : $g(0)=g(1)=0$, $g>0$ on $(0,1)$, $g'(0)>0$ and $g'(1)<0$. Such nonlinearities arise in combustion and biological models (*see* Fisher [30], Kolmogorov, Petrovsky, Piskunov [57], Aronson, Weinberger [1]). Another example of such a function f is $f(x,y,u) = h(x,y)\tilde{f}(u)$ where $\tilde{f}$ is as before and h is L-periodic with respect to x, Lipschitz-continuous and positive in $\overline{\Omega}$.

This section is concerned with special solutions, which are called pulsating travelling fronts (or periodic travelling fronts, *see* [82]), and which are classical time-global solutions u of (1.2) satisfying $0\le u\le 1$ and

$$\left\{\begin{array}{l} \forall k\in\displaystyle\prod_{i=1}^d L_i\mathbb{Z}, \quad \forall(t,x,y)\in\mathbb{R}\times\overline{\Omega}, \quad u\left(t-\frac{k\cdot e}{c},x,y\right) = u(t,x+k,y), \\ \qquad\qquad u(t,x,y) \underset{x\cdot e\to+\infty}{\longrightarrow} 0, \quad u(t,x,y) \underset{x\cdot e\to-\infty}{\longrightarrow} 1, \end{array}\right. \tag{1.6}$$

where the above limits hold locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e. Here, $e = (e^1, \cdots, e^d)$ is a given unit vector in $\mathbb{R}^d$. Such a solution satisfying (1.6) is then called a *pulsating travelling front* propagating in direction e. We say that c is the *effective* unknown speed $c\neq 0$. Let us mention here that, without the uniformity of the limits in (1.6), many other fronts may exist, whose level sets may for instance have conical shapes (*see e.g.* [19], [42], [43]).

[2] In [7], this assumption of f being nonnegative was explicit in formula (1.7) for a function $f = f(u)$ depending only on u. However, although this assumption was obviously also used for the general periodic nonlinearity $f(x,y,u)$ described in [7], it was not mentioned there explicitely. An extension for divergence-type equations with a function f which may change sign is proved in [11].

Under the above assumptions, the first two authors proved in [7] that there exists $c^*(e) > 0$ such that pulsating travelling fronts u in the direction e with the speed c exist if and only if $c \geq c^*(e)$; furthermore, all such pulsating fronts are increasing in time t (other results with more general nonlinearities f were proved in [7], *see* below). The following Theorem gives a variational characterization of this minimal speed $c^*(e)$ under an additional assumption on the nonlinearity f.

Assume that Ω, A and q satisfy (1.1), (1.3) and (1.4), and that f satisfies (1.5) and

$$\forall (x,y,s) \in \overline{\Omega} \times (0,1), \quad 0 < f(x,y,s) \leq f'_u(x,y,0)s. \tag{1.7}$$

Call $\zeta(x,y) := f'_u(x,y,0)$ and denote $\tilde{e}$ the vector defined by $\tilde{e} = (e^1, \cdots, e^d, 0, \cdots, 0) \in \mathbb{R}^N$.

Theorem 1.1 *Under the above assumptions, let $c^*(e)$ be the minimal speed of pulsating travelling fronts propagating in the direction e and solving (1.2) and (1.6). Then*

$$c^*(e) = \min_{\lambda > 0} \frac{k(\lambda)}{\lambda} \tag{1.8}$$

where $k(\lambda)$ is the principal eigenvalue of the operator

$$L_\lambda \psi := \nabla \cdot (A \nabla \psi) - 2\lambda \tilde{e} A \nabla \psi + q \cdot \nabla \psi + [-\lambda \nabla \cdot (A\tilde{e}) - \lambda q \cdot \tilde{e} + \lambda^2 \tilde{e} A \tilde{e} + \zeta]\psi \tag{1.9}$$

acting on the set $E = \{\psi \in C^2(\overline{\Omega}),\ \psi$ is L-periodic with respect to x and $\nu A \nabla \psi = \lambda(\nu A \tilde{e})\psi$ on $\partial\Omega\}$.

Before studying the consequences of Theorem 1.1, let us briefly explain the formula for the minimal speed c^* and mention some earlier results about front propagation, starting from the simplest case of planar fronts in homogeneous media.

Assumption (1.7) is often called the Fisher-KPP assumption (*see* Fisher [32] and Kolmogorov, Petrovsky and Piskunov [57]). It is especially satisfied for the canonical example $f(u) = u(1-u)$, or more generally when $f = f(u)$ is a C^2 concave function on $[0,1]$, positive on $(0,1)$. Thus, under the KPP assumption (1.7), the minimal speed $c^*(e)$ can be explicitely given in terms of e, the domain Ω, the coefficients q and A and of $f'_u(\cdot,\cdot,0)$. We point out that the dependance of $c^*(e)$ on the function f is only through the derivative of f with respect to u at $u = 0$. When $\Omega = \mathbb{R}^N$, $A = I$, $q = 0$ and $f = f(u)$ (with $f(u) \leq f'(0)u$ in $[0,1]$), formula (1.8) then reduces to the well-known KPP formula $c^*(e) = 2\sqrt{f'(0)}$ for the minimal speed of planar fronts for the reaction-diffusion equation $u_t = \Delta u + f(u)$ in $\mathbb{R}^N$.

A planar front is a solution of the type $\phi(x \cdot e - ct)$, where the planar profile ϕ solves $\phi'' + c\phi' + f(\phi) = 0$ in $\mathbb{R}$ with the limiting conditions $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$. Such a solution propagates with constant speed c in the direction e and its shape is invariant in the frame moving with speed c in the direction e. Many papers were devoted to such planar fronts, as well as for other classes of nonlinear functions $f(u)$ (*see e.g.* [1], [15], [30], [31], [52]). For a detailed study of planar fronts for systems of reaction-diffusion equations, we refer to the book of Volpert, Volpert and Volpert [87] and to the references therein.

Equations with periodic nonlinearities $f(x,u)$ in space dimension 1, without advection, were first considered by Shigesada, Kawasaki and Teramoto [82], and by Hudson and Zinner

[50]. The notion of travelling fronts propagating with constant speed c no longer holds in general and has to be replaced with the more general one of pulsating travelling fronts, as defined in (1.6) (*see* [82]). The profile of such a front is not invariant anymore, but, in one space dimension, the profile is periodic in time in the frame moving with speed c along the direction of propagation. In [50], a formula similar to the right-hand side of (1.14) in dimension 1 was given and it was proved that for any speed not smaller than the right-hand side of (1.14), then pulsating travelling fronts exist. The case of a periodic nonlinearity $f(x,u)$ changing sign with respect to x, based on a patch invasion model in ecology was considered in [81] and [82], and recently revisited from a rigorous mathematical and more general point of view in [11] and [12] in dimensions 1 and higher, and for more general reaction terms. Lastly, the case of periodic diffusion with bistable type nonlinearity (*see* (1.11) below) was investigated by Nakamura [71] in dimension 1.

The case of shear flows $q=(\alpha(y),0,\cdots,0)$ in straight infinite cylinders $\Omega=\mathbb{R}\times\omega$ was dealt with by Berestycki, Larrouturou, Lions [13], and Berestycki and Nirenberg [17]. Under the assumption that all coefficients of equation (1.2) do not depend on the x_1 variable, the period L_1 can be any arbitrary positive number and pulsating travelling fronts reduce in this case to travelling fronts $\phi(x_1-ct,y)$ which move with constant instantaneous speed c and keep a constant shape. Formula (1.13) was derived in this framework in [17] for the minimal speed of travelling fronts with a nonlinearity $f=f(u)$ satisfying (1.5) and (1.7). Other nonlinearities $f(u)$ were treated in [17] : for a combustion-type nonlinearity f such that

$$\begin{array}{c}\exists\theta\in(0,1)\ \text{(called ignition temperature)},\\ f=0\text{ on }[0,\theta]\cup\{1\},\ \ f>0\text{ on }(\theta,1),\ \ f'(1)<0\end{array}\tag{1.10}$$

(*see* [52]), there exists a unique speed c and a unique (up to shift in time, or equivalently in x_1) travelling front $\phi(x_1-ct,y)$; for a bistable nonlinearity f such that

$$\begin{array}{c}\exists\theta\in(0,1),\quad f(0)=f(\theta)=f(1)=0,\\ f<0\text{ on }(0,\theta),\ \ f>0\text{ on }(\theta,1),\ \ f'(0)<0,\ \ f'(1)<0,\end{array}\tag{1.11}$$

there still exists a unique speed c and a unique (up to shift) travelling front, under the additional assumption that the section ω of the cylinder is convex. Min-max type variational formulas –involving the values of $f(u)$ for all $u\in(0,1)$– for the unique or minimal speeds of propagation of these travelling fronts were obtained by Hamel [41] and Heinze, Papanicolaou and Stevens [49], generalizing some results for equations [40] or systems [54], [87] in dimension 1 (*see* also Benguria, Depassier [5] for integral formulations in dimension 1, and Coutinho, Fernandez [26], Harris, Hudson and Zinner [45] for similar problems with discrete diffusion). Several lower and upper bounds for the speeds of travelling fronts in infinite cylinders, as well as some asymptotics for large advection and for other regimes, were derived by Audoly, Berestycki and Pomeau [3], Berestycki [6], Constantin, Kiselev and Ryzhik [25], [56] and Heinze [48] for combustion-type and/or general positive nonlinearities $f(u)$. Rotating flows were also considered in [3] and [56], and percolating-type flows were dealt with in [56], where estimates for the more general notion of bulk burning rate (*see* [24]) are given. Dirichlet type boundary conditions on $\partial\Omega$, instead of Neumann conditions, were dealt with by Gardner [38] and Vega [86] in infinite cylinders. Let us also mention here that a formula similar to (1.8) for a nonlinear source term $f(u)$ of the KPP type (1.7) has recently been obtained by Schwetlick for a similar *hyperbolic* transport equation [80].

Whereas usual travelling fronts of the type $\phi(x_1-ct,y)$ exist in straight infinite cylinders in the case of shear flows –assuming that all coefficients in (1.2) are invariant with respect to the variable x_1–, this is not the case anymore in infinite cylinders $\Omega=\{(x_1,y),\ y\in\omega(x_1)\}$ with oscillating boundaries (ω being periodic in x_1), even, say, for the equation $u_t=\Delta u+f(u)$ without advection. Such a geometrical configuration was first considered for a bistable nonlinearity f by Matano [66], and the case of ondulating cylinders whose boundaries have small spatial periods with small amplitudes was recently dealt with by Lou and Matano [61].

The case of the whole space $\mathbb{R}^N$ with periodic diffusion and advection was first considered by Xin [91], [93] for a combustion-type nonlinearity f satisfying (1.10), for which the speed of propagation of the pulsating fronts was proved to be unique in any given direction. Note that usual travelling fronts propagating with constant speed and constant shape do not exist anymore for general advection or diffusion and one has to extend these notions. The homogenization limit in $\mathbb{R}^N$ with coefficients having small periods was investigated by Caffarelli, Lee and Mellet [21], Freidlin [35], Heinze [46], Majda and Souganidis [63], and Xin [94]. Heinze also considered the case of the whole space with small periodic holes [47]. Freidlin [35] and Xin [94] also studied questions related to front propagation in random media.

The more general framework of periodic domains and periodic excitable media was considered by the first two authors of this paper in [7]. It was especially proved that for a nonnegative combustion-type nonlinearity $f(x,y,u)$ satisfying the following assumptions, more general than (1.10) :

$$\left\{\begin{array}{l} f \text{ is } L\text{-periodic with respect to } x,\\ f \text{ is globally Lipschitz-continuous and } \exists\delta>0,\ f \text{ is } C^{1,\delta} \text{ with respect to } u,\\ \exists\theta\in(0,1),\quad \forall(x,y)\in\overline{\Omega},\quad \forall s\in[0,\theta]\cup\{1\},\quad f(x,y,s)=0,\\ \exists\rho\in(0,1-\theta),\ \forall(x,y)\in\overline{\Omega},\ \forall 1-\rho\le s\le s'\le 1,\quad f(x,y,s)\ge f(x,y,s'),\\ \forall s\in(\theta,1),\quad \exists(x,y)\in\overline{\Omega},\quad f(x,y,s)>0,\end{array}\right. \tag{1.12}$$

and given a direction e of $\mathbb{R}^d$, there exists a unique effective speed of propagation $c(e)$ and a unique (up to shift in time) pulsating travelling front u satisfying (1.2) and (1.6). As already emphasized, paper [7] also gives the proof of the existence of a minimal speed $c^*(e)$ of propagation of pulsating fronts for a function f satisfying (1.5). Furthermore, under the notations of Theorem 1.1, the inequality

$$c^*(e)\ge\min_{\lambda>0}\frac{k(\lambda)}{\lambda}$$

holds as soon as f satisfies (1.5) (*see* Remark 1.16 in [7]). However, the question of the uniqueness, up to shift, of the fronts for any given effective speed $c\ge c^*(e)$ is still an open problem.

Remark 1.2 (Equivalent formulas) It can be easily checked in the general framework described above that formula (1.8) can be rewritten in the following equivalent formulations :

$$c^*(e)=\min\{c,\ \exists\lambda>0,\ k(\lambda)=\lambda c\} \tag{1.13}$$

and

$$c^*(e) = \min_{\lambda>0} \min_{\psi\in F} \max_{(x,y)\in\overline{\Omega}} \frac{L_\lambda\psi(x,y)}{\lambda\psi(x,y)} \tag{1.14}$$

where $F = \{\psi \in E, \psi \in C^2(\overline{\Omega}),\ \psi > 0 \text{ in } \overline{\Omega}\}$. Formula (1.14) is obtained from (1.8) and from some characterizations of principal eigenvalues of elliptic operators ([18], [74]). We also refer to [7] for a detailed study of the above eigenvalue problems with periodic and Neumann type boundary conditions. Such operators L_λ also arise in Bloch eigenvalue problems in homogenization theory (*see* [22], [23], [58]).

The proof of formula (1.8), which was announced in [7], is based on the methods developped in [7] and [17] (sub- and supersolutions, regularizing approximations in bounded domains). The authors also mention that a formula equivalent to (1.8) was recently obtained independently with different tools by Weinberger [89] for similar problems.

1.2 Influence of the geometry of the domain and of the underlying medium

As we have just seen, several equivalent variational formulas for the minimal speed of propagation of pulsating travelling fronts in general periodic excitable media were given. We now analyze the influence of the geometry of the domain and of the coefficients of the medium (reaction, diffusion and advection coefficients) on the minimal speed of propagation. Since the influence of these data may be opposite, we shall investigate each of them separately.

Let us first study the influence of the geometry of the domain. Under the assumptions of the previous subsection, it easily follows from formula (1.13) that even for a homogeneous equation, due to the geometry, the minimal speed $c^*(e)$ depends continuously on e in the unit sphere S^{d-1} of $\mathbb{R}^d$. Note that the speed $c^*(e)$ does depend on the direction e in general because of the geometry of the domain and because of the spatial heterogeneity of the coefficients of equation (1.2). This situation is in contrast with the homogeneous equation

$$u_t = \Delta u + f(u) \tag{1.15}$$

in the whole space $\mathbb{R}^N$, for which pulsating travelling fronts are actually planar travelling fronts and the minimal speed has the same value, $c^*(e) = 2\sqrt{f'(0)}$ in all directions e of $\mathbb{R}^N$.

Let us now consider the above homogeneous equation $u_t = \Delta u + f(u)$, but now set in a periodic domain $\Omega \subset \mathbb{R}^N$ satisfying (1.1). Assume that the function f satisfies (1.5) and (1.7). If $\Omega = \mathbb{R}^N$, then $c^*(e) = 2\sqrt{f'(0)}$ for all $e \in \mathbb{R}^N$ with $|e| = 1$. The following statement shows that this value $2\sqrt{f'(0)}$ is always an upper bound whatever Ω is –provided it satisfies (1.1)–, and is optimal in some sense :

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^N$ satisfy (1.1) and let $f = f(u)$ satisfy (1.5) and (1.7). Let $e = (e^1, \cdots, e^d) \in \mathbb{R}^d$ be such that $|e| = 1$. Let $c^*(e)$ be the minimal speed of pulsating travelling fronts satisfying (1.15) and (1.6) together with the Neumann boundary conditions $\partial_\nu u = 0$ on $\partial\Omega$. Then,*

$$0 < c^*(e) \le 2\sqrt{f'(0)},$$

and $c^*(e) = 2\sqrt{f'(0)}$ if and only if the domain Ω is invariant in the direction $\tilde{e}$, namely $\Omega + \tau\tilde{e} = \Omega$ for all $\tau \in \mathbb{R}$, where $\tilde{e} = (e^1, \cdots, e^d, 0, \cdots, 0) \in \mathbb{R}^N$.

In other words, Theorem 1.3 implies that the presence of holes (perforations) in the domain always hinder the propagation with respect to the case of the whole space. Similarly, the fronts propagate strictly slower in an infinite cylinder with oscillating boundaries than in a straight infinite cylinder. The homogenization limit of small holes with a combustion type nonlinearity was dealt with by Heinze [47] (*see* also [72] for homogenization of linear diffusion equations with small holes).

After having proved that holes make the propagation of pulsating fronts slower than in the case of the whole space $\mathbb{R}^N$, it is now natural to wonder whether the minimal speed $c^*(e)$ is all the smaller the bigger the holes. Actually, the answer is no in general :

Theorem 1.4 *Let $N \geq 2$ and e be any unit direction in $\mathbb{R}^N$. Let $f = f(u)$ satisfy (1.5) and (1.7). Then there exist some positive numbers $L_1, \cdots, L_N$, a family of domains $(\Omega_\alpha)_{0\leq\alpha<1}$ satisfying (1.1) with $d = N$ and*

$$\Omega_0 = \mathbb{R}^N, \quad \Omega_\alpha \supset \Omega_{\alpha'} \text{ for all } 0 \leq \alpha \leq \alpha' < 1, \qquad \bigcap_{0\leq\alpha<\alpha'} \Omega_\alpha = \Omega_{\alpha'} \text{ for all } 0 < \alpha' < 1, \quad (1.16)$$

such that, if c_α denotes the minimal speed $c_\alpha = c^*(e, \Omega_\alpha)$ of the pulsating fronts satisfying (1.15) and (1.6) in Ω_α with Neumann boundary conditions on $\partial\Omega_\alpha$, then the function $\alpha \mapsto c_\alpha$ is continuous on $[0, 1)$, $c_0 = 2\sqrt{f'(0)}$, $c_\alpha < 2\sqrt{f'(0)}$ for all $\alpha \in (0, 1)$ and $c_\alpha \to 2\sqrt{f'(0)}$ as $\alpha \to 1^-$.

Theorem 1.4 says that the minimal speed of propagation for the homogeneous equation (1.15) may not be monotone with respect to the size of the holes. Furthermore, under the notations of Theorem 1.4, one can say that there exists at least one value of α_0 in $(0, 1)$ for which the minimal speed of pulsating fronts is minimal in Ω_{α_0} among all the domains Ω_α for $0 < \alpha < 1$.

Remark 1.5 Theorem 1.3 no longer holds for equations with periodic *heterogeneous* coefficients even if the equation is invariant in direction $\tilde{e}$. For instance, let $\Omega' \subset \mathbb{R}^{N-1}$ be a periodic domain satisfying (1.1) with $d \leq N - 1$, and such that $\Omega' \neq \mathbb{R}^{N-1}$. Let $\Omega = \Omega' \times \mathbb{R} = \{x = (x', x_N),\ x' \in \Omega',\ x_N \in \mathbb{R}\}$. Let $f(x, u)$ be a function satisfying (1.5) and (1.7), and assume that $f(x, u)$ is written as $f(x, u) = h(x')\tilde{f}(u)$, where $\tilde{f}$ satisfies (1.7), $0 < h(x') \leq 1$ in $\mathbb{R}^{N-1}$, $h(x') = 1$ in Ω' and $h \not\equiv 1$ in $\mathbb{R}^{N-1}$. Let $e = e_N$ be the unit vector in the x_N-direction. Then the minimal speed of propagation of pulsating fronts solving

$$u_t = \Delta u + f(x, u)$$

and (1.6) in Ω, together with $\partial_\nu u = 0$ on $\partial\Omega$, is equal to $2\sqrt{\tilde{f}'(0)}$. But the minimal speed for the same equation set in the whole space $\mathbb{R}^N$ is *strictly* less than $2\sqrt{\tilde{f}'(0)}$ (*see* the proof of Theorem 1.6 below for more details).

Let us now investigate the influence of the reaction coefficients on the minimal speed of propagation.

Theorem 1.6 *Under the assumptions (1.1), (1.3) and (1.4), let $f = f(x, y, u)$, resp. $g = g(x, y, u)$, be a nonnegative nonlinearity satisfying (1.5) and (1.7). Let e be a unit direction of $\mathbb{R}^d$ and let $c^*(e, f)$, resp. $c^*(e, g)$, be the minimal speed of propagation of pulsating fronts solving (1.2) and (1.6) with nonlinearity f, resp. g.*

a) If $f'_u(x, y, 0) \le g'_u(x, y, 0)$ for all $(x, y) \in \overline{\Omega}$, then $c^*(e, f) \le c^*(e, g)$, and if $f'_u(x, y, 0) \le, \not\equiv g'_u(x, y, 0)$, then $c^*(e, f) < c^*(e, g)$.

b) If $c^*(e, Bf)$ denotes the minimal speed for the nonlinearity Bf, with $B > 0$, then $c^*(e, Bf)$ is increasing in B and

$$\limsup_{B\to+\infty} \frac{c^*(e, Bf)}{\sqrt{B}} < +\infty.$$

Furthermore, if $\Omega = \mathbb{R}^N$ or if $\nu A\tilde{e} \equiv 0$ on $\partial\Omega$, then $\liminf_{B\to+\infty} c^*(e, Bf)/\sqrt{B} > 0$.

Part a) of Theorem 1.6 follows immediately from Theorem 1.1 (note that similar monotonicity results also hold for equations with nonlinearities changing sign, *see* [11], [12]). Notice that the inequality $c^*(e, f) \le c^*(e, g)$ holds as soon as f and g satisfy (1.5) and $f \le g$, even if f or g do not satisfy (1.7) (this inequality follows from the construction of the minimal speed by approximation of speeds of fronts with combustion-type nonlinearities satisfying (1.12), *see* [7] and Remark 1.7 below). However, the strict inequality $c^*(e, f) < c(e, g)$ does not hold in general if $f \le g$ and $f \not\equiv g$, even if f and g satisfy (1.5) and (1.7) : indeed, under these assumptions, the dependence on f of the minimal speed $c^*(e, f)$ is only through its derivative $f'_u(x, y, 0)$ at $u = 0^+$.

The condition $\nu A\tilde{e} \equiv 0$ on $\partial\Omega$ especially holds if $A\tilde{e}$ is constant and if Ω is invariant in this direction $A\tilde{e}$ (for instance, $A = I$ and Ω is a straight infinite cylinder in direction $\tilde{e}$). Notice that parts a) and b) of Theorem 1.6 obviously hold for the KPP formula $c^* = 2\sqrt{Bf'(0)}$ in the case of the homogeneous equation (1.15) in $\mathbb{R}^N$ with nonlinearity Bf. However, the precise asymptotic behavior of $c^*(e, Bf)/\sqrt{B}$ as $B \to +\infty$ is not known in general.

Lastly, part b) also holds good if the nonlinearity Bf is replaced by a nonlinearity of the type $Bf + f_0$, with given f and f_0 satisfying (1.5) and (1.7).

Remark 1.7 Similar comparison properties as in Theorem 1.6 also hold for the *unique* speeds $c(e, f)$ and $c(e, g)$ of the pulsating fronts solving (1.2) and (1.6) in the case where the nonnegative nonlinearities $f = f(x, y, u)$ and $g = g(x, y, u)$ satisfy (1.12) and are ordered. Namely, if $f \le g$ in $\overline{\Omega} \times [0, 1]$, then $c(e, f) \le c(e, g)$. Furthermore, in this framework, one has $c(e, f) < c(e, g)$ if $f \le g$ and $f \not\equiv g$. These facts follow easily from the proofs in [7]. However, the behaviour of $c(e, Bf)$ for large B is not known in this case.

The influence of advection on the speed of propagation is more difficult to analyze, because of possible interaction between the stream lines and the geometry of the domain, especially the holes. However, at least in the case where the domain is invariant in the direction $\tilde{e}$, with isotropic diffusion, one can compare the speeds of propagation in direction e when there is, or not, a drift term in the equation.

Theorem 1.8 *Let $\Omega \subset \mathbb{R}^N$ be a domain satisfying (1.1) and $\Omega + \tau\tilde{e} = \Omega$ for all $\tau \in \mathbb{R}$, where e is a unit vector of $\mathbb{R}^d$. Assume that $A = I$ and that $f = f(u)$ satisfies (1.5) and*

(1.7), and that q satisfies (1.4). Let $c^*_q(e)$ be the minimal speed of the pulsating fronts solving (1.2) and (1.6), with advection coefficient q. Then

$$c^*_q(e) \geq c^*_0(e) = 2\sqrt{f'(0)}$$

and equality holds if and only if $q \cdot \tilde{e} \equiv 0$ in $\overline{\Omega}$.

Under the above assumptions, Theorem 1.8 means that the advection, or stirring, makes the propagation faster, whatever the flow is a shear flow or not. Roughly speaking, the presence of turbulence in the medium increases the speed of propagation of the pulsating fronts. Furthermore, the influence of advection on the speed of propagation is minimal if and only if the advection is orthogonal to the direction of propagation.

The influence of large periodic advection, namely where q is replaced with Bq with large B, is analyzed by the authors in [9]. The behaviour of $c^*_{Bq}(e)$ is always at most linear in B for large B, in a general domain Ω which satisfies (1.1) but may not be invariant in the direction $\tilde{e}$. A necessary and sufficient condition for $c^*_{Bq}(e)$ to be at least linear in B is given in [9], involving the first integrals of the velocity field q.

Remark 1.9 It is not clear in general whether, under the assumptions of Theorem 1.8, $c^*_{Bq}(e)$ is nondecreasing with respect to $B > 0$ or not. However, in the case of a shear flow $q = \alpha(x_2, \cdots, x_N)e_1$ in a straight cylinder $\Omega = \mathbb{R} \times \omega$ in the direction e_1, with, say, ω bounded in $\mathbb{R}^{N-1}$, $\alpha \not\equiv 0$ of class C^1 and with zero average, the first author proved in [6] that $c^*_{Bq}(e_1)$ is increasing with $B > 0$, $c^*_{Bq}(e_1)/B$ is decreasing with $B > 0$ and $c^*_{Bq}(e_1)/B \to \rho > 0$ as $B \to +\infty$.

As far as the influence of the diffusion coefficients is concerned, one can compare the minimal speed of propagation in the case of heterogeneous diffusion with that of a homogeneous diffusion in a given direction e. The following theorem also gives a monotonicity result of the speed of propagation with respect to the intensity of diffusion :

Theorem 1.10 *Under the assumptions (1.1), (1.3), (1.5) and (1.7), let $q = 0$. Let e be a unit direction of $\mathbb{R}^d$. Then,*

1)

$$c^*(e) \leq 2\sqrt{M_0 M}, \tag{1.17}$$

where $M_0 = \max_{(x,y)\in\overline{\Omega}} \zeta(x,y)$ and $M = \max_{(x,y)\in\overline{\Omega}} \tilde{e}A(x,y)\tilde{e}$. Furthermore, the equality holds in (1.17) if and only if ζ and $\tilde{e}A\tilde{e}$ are constant, $\nabla \cdot (A\tilde{e}) \equiv 0$ in $\overline{\Omega}$ and $\nu A\tilde{e} = 0$ on $\partial\Omega$ (if $\partial\Omega \neq \emptyset$).

2) Assume furthermore that $f = f(u)$ depends on u alone. Let $c^*_\gamma(e)$ denote the minimal speed of pulsating fronts in the direction e, with diffusion matrix γA, where $\gamma > 0$. Then $c^*_\alpha(e) \leq c^*_\beta(e)$ if $0 < \alpha \leq \beta$.

As a special case of (1.17) we see that $c^*_\alpha(e) \leq C\sqrt{\alpha}$ for all $\alpha > 0$, where C does not depend on $\alpha > 0$. Furthermore, part 2) implies that a larger diffusion speeds up the propagation.

Remark 1.11 The assumption $q = 0$ was made for the sake of simplicity in the derivation of the upper bound (1.17). However, more general bounds can be obtained when $q \neq 0$. Namely, under the assumptions (1.1), (1.3), (1.4), (1.5) and (1.7), one gets as in the proof of Theorem 1.10 :

$$c^*(e) \leq 2\sqrt{M_0 M} + \max_{(x,y)\in\overline{\Omega}} (-q(x,y)\cdot\tilde{e}),$$

where M_0 and M are the same as in Theorem 1.10.

Lower bounds can be obtained as well, but are more restrictive. Namely, under the assumptions (1.1), (1.3), (1.4), (1.5) and (1.7), assume furthermore that $\Omega = \mathbb{R}^N$ or $\nu A\tilde{e} = 0$ on $\partial\Omega$ if $\partial\Omega \neq \emptyset$. Then

$$c^*(e) \geq \min\left(\frac{m_0 m}{b}, -b + 2\sqrt{m_0 m}\right), \tag{1.18}$$

under the convention that $m_0 m/b = +\infty$ if $b = 0$, where $m_0 = \min_{(x,y)\in\overline{\Omega}} \zeta(x,y)$, $m = \min_{(x,y)\in\overline{\Omega}} \tilde{e}A(x,y)\tilde{e}$ and $b = \|\nabla\cdot(A\tilde{e})\|_{L^\infty(\Omega)} + \|q\cdot\tilde{e}\|_{L^\infty(\Omega)}$. Formula (1.18) is proved in Section 3.2. It especially implies that, if $q\cdot\tilde{\varepsilon} \equiv 0$, then $\liminf_{\varepsilon\to 0^+} c^*_\varepsilon(e)/\sqrt{\varepsilon} > 0$, where $c^*_\varepsilon(e)$ denotes the minimal speed of pulsating fronts in the direction e with diffusion matrix εA.

1.3 Spreading speed in periodic domains

The question of the stability of travelling fronts and the asymptotic convergence to travelling fronts for the solutions of Cauchy problems of the type (1.2) with "front-like" initial conditions has been thoroughly studied since the pioneering paper by Kolmogorov, Petrovsky and Piskunov [57] in the one-dimensional case (*see e.g.* [1], [20], [28], [31], [37], [52], [59], [67], [78], [79], [83], [85] for other stability results in the homogeneous 1d case, [2] for the homogeneous multidimensional case, or [14], [44], [65], [76], [77] for the case of infinite cylinders with shear flows). However, few results (*see* [60], [70], [92]) have so far been obtained about the stability of pulsating travelling fronts in periodic media.

Another important notion is that of asymptotic speed of propagation, or spreading (see below for precise meaning), for solutions of Cauchy problem like (1.2) with nonnegative continuous compactly supported initial condition $u_0 \not\equiv 0$. This problem for the homogeneous equation (1.15) in $\mathbb{R}^N$ was solved by Aronson and Weinberger [2]. They proved that, under the above assumptions on u_0 and if f satisfies (1.5) and $\liminf_{u\to 0^+} f(u)/u^{1+2/N} > 0$,[3] then

$$\min_{|z|\leq ct} u(t,z) \to 1 \text{ if } 0 \leq c < c^* \text{ and } \max_{|z|\leq ct} u(t,z) \to 0 \text{ if } c > c^*, \text{ as } t \to +\infty,$$

where c^* is the minimal speed of planar fronts. The speed c^* can then also be viewed as a spreading speed (*see* [1], [2], [31], [51], [53], [78] for similar results with other nonlinearities $f(u)$ in dimensions 1 or higher). These spreading properties were generalized by Mallordy and Roquejoffre [65], [77] for equations with shear flows in straight infinite cylinders.

The case of a reaction-diffusion equation (1.2) without advection in the whole space $\mathbb{R}^N$ with periodic coefficients was considered in the important work of Gärtner and Freidlin [39] and later by Freidlin [34] in the case with advection q (the proofs in [39] and [34] used

[3] The latter is fulfilled if f satisfies (1.7) as well.

probabilistic tools). Namely, under the assumptions (1.3), (1.4), (1.5)[4] and (1.7), if u_0 is nonnegative, continuous and compactly supported, then the solution $u(t,z)$ of (1.2) in $\mathbb{R}^N$ with initial condition u_0 is such that, for any unit vector e of $\mathbb{R}^N$,

$$u(t,z+c\,t\,e)\to 1 \text{ if } 0\le c<w^*(e) \ \text{ and } \ u(t,z+cte)\to 0 \text{ if } c>w^*(e), \ \text{ as } t\to+\infty, \quad (1.19)$$

locally in $x\in\mathbb{R}^N$. Furthermore, Gärtner and Freidlin derive a formula which we call the Gärtner-Freidlin formula :

$$w^*(e)=\min_{\vec{\lambda}\cdot e>0}\frac{\tilde{k}(\vec{\lambda})}{\vec{\lambda}\cdot e} \quad (1.20)$$

and $\vec{\lambda}\in\mathbb{R}^N$ and $\tilde{k}(\vec{\lambda})$ is the first eigenvalue of the operator

$$L_{\vec{\lambda}}:=\nabla\cdot(A\nabla)-2\vec{\lambda}A\nabla+q\cdot\nabla+[-\nabla\cdot(A\vec{\lambda})-q\cdot\vec{\lambda}+\vec{\lambda}\cdot A\vec{\lambda}+\zeta]$$

with L-periodicity condition (as a consequence, $w^*(\pm1)=c^*(\pm1)$ in dimension $N=1$). The speed $w^*(e)$ can then be viewed as a ray speed in the direction e. It follows from (1.8) that $w^*(e)\le c^*(e)$. Notice that the latter can also be easily obtained from (1.19) and the parabolic maximum principle, putting u_0 below a pulsating front moving with speed $c^*(e)$ in the direction e, even if it means changing f into a function $\tilde{f}$ such that $\tilde{f}\ge f$, $\tilde{f}'_u(z,0)=f'_u(z,0)$ and (1.7) holds for $\tilde{f}$).

Let us also mention that several works have dealt with the solutions of Cauchy problems for equations of the type (1.2), with small diffusion ε, together with large reaction $\varepsilon^{-1}f$, or with slowly varying flows of the type $q(\varepsilon z)$, or for equations involving more general spatio-temporal scales. Typically, the solutions of such Cauchy problems converge as $\varepsilon\to0^+$ to two-phase solutions of Hamilton-Jacobi type equations, separated by interfaces (*see e.g.* [33], [35], [36], [63], [64]). The determination of the asymptotic speed of propagation was also studied for nonlinear integral equation in dimension 1 (*see* [4], [27], [68], [84]), or for systems of reaction-diffusion equations in dimension 1 (*see* [81]).

Recently, Weinberger [89] extended the results of Gärtner and Freidlin to the general periodic framework described in [7] and here, with possible time-discrete equations. Under assumptions (1.1), (1.3), (1.4), (1.5) and (1.7), it is proved in [89] that, for any unit direction e of $\mathbb{R}^d$, there exists $w^*(e)>0$ such that, if $u(t,x,y)$ solves (1.2) with a nonnegative, continuous and compactly supported initial condition $u_0\not\equiv0$, then,

$$\left\{\begin{array}{ll} u(t,x+cte,y)\to1 & \text{if } 0\le c<w^*(e)\\ u(t,x+cte,y)\to0 & \text{if } c>w^*(e),\end{array}\right. \quad \text{as } t\to+\infty, \quad (1.21)$$

locallyin (x,y) with respect to the points (x,y) such that $(x+cte,y)\in\overline{\Omega}$. Furthermore,

$$\{\rho\xi,\ \xi\in S^{d-1},\ 0\le\rho\le w^*(\xi)\}:=\{x\in\mathbb{R}^d,\ x\cdot\xi\le c^*(\xi)\ \text{ for all } \xi\in S^{d-1}\}, \quad (1.22)$$

i.e. $w^*(e)=\min_{\xi\in\mathbb{R}^d,\ e\cdot\xi>0}\ c^*(\xi)/(e\cdot\xi)$, or

$$w^*(e)=\min_{\vec{\lambda}\cdot e>0}\frac{\tilde{k}(\vec{\lambda})}{\vec{\lambda}\cdot e}, \quad (1.23)$$

[4]The function $f=f(z,u)$ was actually assumed in [34] to be positive in $\mathbb{R}^N\times(0,1)$.

with $\vec{\lambda} \in \mathbb{R}^d$ and $\tilde{k}(\vec{\lambda})$ being the principal eigenvalue of the operator $L_{\vec{\lambda}} := \nabla \cdot (A\nabla) - 2\tilde{\lambda}A\nabla + q \cdot \nabla + [-\nabla \cdot (A\tilde{\lambda}) - q \cdot \tilde{\lambda} + \tilde{\lambda}A\tilde{\lambda} + \zeta]$ acting on the set $\tilde{E} = \{\psi \in C^2(\overline{\Omega}),\ \psi$ is L-periodic with respect to x and $\nu A\nabla\psi = (\nu A\tilde{\lambda})\psi$ on $\partial\Omega\}$ (we set $\tilde{\lambda} = (\vec{\lambda}, 0, \cdots, 0) \in \mathbb{R}^N$).

Remark 1.12 As already emphasized, it is clear from the parabolic maximum principle that $w^*(e) \le c^*(e)$ for all $e \in S^{d-1}$. The latter could also be viewed as a consequence of (1.8) and (1.20) in the case of equation (1.2) in $\mathbb{R}^N$, or from (1.22-1.23) in the general periodic case.

The equality $w^*(e) = c^*(e)$ holds for the homogeneous isotropic equation $u_t = \Delta u + f(u)$ in $\mathbb{R}^N$, for all direction e, but it does not hold in general. Indeed, consider the equation

$$u_t = a^2 u_{x_1x_1} + b^2 u_{x_2x_2} + f(u) \quad \text{in } \mathbb{R}^2,$$

where $a > 0$ and $b > 0$ are two given constants, and $f = f(u)$ satisfies (1.5) and (1.7). From the above formulas for $w^*(e)$ or $c^*(e)$, it is easy to see that, for all $\theta \in \mathbb{R}$ and $e = (\cos\theta, \sin\theta)$,

$$w^*(e) = 2\sqrt{f'(0)}\sqrt{\frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}}, \quad c^*(e) = 2\sqrt{f'(0)}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$$

(notice that the formula for $w^*(e)$ could also be deduced from the case of isotropic diffusion after scaling). Hence, the equality $w^*(e) = c^*(e)$ holds here if and only if $e = (\pm 1, 0)$ or $(0, \pm 1)$, or if $a = b$ (isotropic diffusion). In other words, in the case of anisotropic diffusion ($a \neq b$), the asymptotic spreading speed is less than the minimal speed of pulsating fronts in any direction which is not an eigenvector of the diffusion matrix. Notice also that the curve $r(\theta) = w^*(\cos\theta, \sin\theta)$ in polar coordinates is an ellipse, while the curve $r(\theta) = c^*(\cos\theta, \sin\theta)$ is not an ellipse in general (but the curve $r(\theta) = (c^*(\cos\theta, \sin\theta))^{-1}$ is an ellipse).

Some numerical simulations with isotropic but heterogeneous diffusion have been performed in [55], confirming that the radial speed $w^*(e)$ may be less that the minimal speed $c^*(e)$ of pulsating fronts. We conjecture that, by analogy, the strict inequality $w^*(e) < c^*(e)$ may also occur in some directions e in some domains with holes. However, a condition for the equality $w^*(e) = c^*(e)$ to hold or not is not known in general in the periodic setting.

In the sequel, we discuss some properties of the spreading speed $w^*(e)$ in periodic domains. As for the minimal speed of pulsating fronts, we study the influence on the speed $w^*(e)$ of all the phenomena involved in problem (1.2).

As in Theorem 1.3, let us first consider the case of the homogeneous equation (1.15) in a periodic domain Ω. Since $w^*(e) \le c^*(e)$ for any unit direction $e \in S^{d-1}$, it follows from Theorem 1.3 that $w^*(e) \le 2\sqrt{f'(0)}$ and that, if $w^*(e) = 2\sqrt{f'(0)}$, then Ω is a straight infinite cylinder in the direction $\tilde{e}$. Conversely, if Ω is a straight infinite cylinder in the direction $\tilde{e}$, then $c^*(e) = 2\sqrt{f'(0)}$ by Theorem 1.3; furthermore, the last equality holds for $w^*(e)$ as well, namely :

Theorem 1.13 *Under the assumptions (1.1) for Ω (with $d \ge 1$), and (1.5) and (1.7) for $f = f(u)$, let e be a unit direction of $\mathbb{R}^d$ and $u(t, x, y)$ be the solution of (1.15) with a given initial condition $u_0 \not\equiv 0$ which is nonnegative, continuous and compactly supported. Then $w^*(e) \le 2\sqrt{f'(0)}$ and equality holds if and only if Ω is invariant in the direction $\tilde{e}$.*

Theorem 1.13 rests on the following Liouville type result :

Proposition 1.14 *Let Ω satisfy (1.1). Let $g : [0, +\infty) \to \mathbb{R}$ be a C^1 function such that $g(0) = g(1) = 0$, $g'(0) > 0$, $g > 0$ in $(0, 1)$ and $g < 0$ in $(1, +\infty)$, and let $b \in \mathbb{R}^N$ be such that $|b| < 2\sqrt{g'(0)}$. Let u be a classical bounded solution of*

$$\begin{cases} \Delta u + b \cdot \nabla u + g(u) = 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.24}$$

Then $u \equiv 0$ or $u \equiv 1$.

This result, which is of independent interest, is a Liouville type result for some solutions of semi-linear elliptic equations in periodic domains. If u were assumed to be L-periodic and not identically equal to 0, then the conclusion $u \equiv 1$ would follow immediately from the strong maximum principle, since u would then be bounded from below by a positive constant (*see* case 1 of the proof of Proposition 1.14 in section 4). The difficultly here is that u is not assumed to be L-periodic a priori. Let us also mention that the conclusion of Proposition 1.14 was known in the case $\Omega = \mathbb{R}^N$, and was proved by Aronson and Weinberger [2], by using parabolic tools (*see* also Remark 4.3 below). The proof of Proposition 1.14 given in Section 4 rests on some sliding arguments and on the elliptic maximum principle.

The influence of all other phenomena (reaction, diffusion and advection) is summarized in the following propositions, most of which are consequences of the results stated in Section 1.2.

Let us start with the dependency on the reaction terms.

Proposition 1.15 *Under the assumptions (1.1), (1.3), (1.4), let e be a unit direction of $\mathbb{R}^d$ and let f and g be two functions satisfying (1.5) and (1.7). Call $w^*(e, f)$ and $w^*(e, g)$ the spreading speeds in the direction e for problem (1.2) with nonlinearities f and g respectively.*

If $f'_u(x, y, 0) \leq g'_u(x, y, 0)$ for all $(x, y) \in \overline{\Omega}$, then

$$w^*(e, f) \leq w^*(e, g).$$

If $f'_u(x, y, 0) \;\; \leq, \not\equiv \;\; g'_u(x, y, 0)$, then $w^*(e, f) < w^*(e, g)$. Hence, $w^*(e, Bf)$ is increasing in $B > 0$. Furthermore,

$$\limsup_{B \to +\infty} \frac{w^*(e, Bf)}{\sqrt{B}} < +\infty.$$

Lastly, if $\Omega = \mathbb{R}^N$, then $\liminf_{B \to +\infty} w^*(e, Bf)/\sqrt{B} > 0$.

The next result is about the influence of stirring on propagation.

Proposition 1.16 *Let $\Omega = \mathbb{R}^N$, $A = I$ and assume that $f = f(u)$ satisfies (1.5) and (1.7). For any unit vector e of $\mathbb{R}^N$, call $w_q^*(e)$ the spreading speed in direction e, with advection term q satisfying (1.4). Then,*

$$w_q^*(e) \geq w_0^*(e) = 2\sqrt{f'(0)},$$

and the equality $w_q^*(e) = 2\sqrt{f(0)}$ holds if and only if $q \cdot e \equiv 0$ in $\mathbb{R}^N$.

The last proposition is concerned with the influence of the diffusion on the asymptotic spreading speed.

Proposition 1.17 *Under the assumptions (1.1), (1.3), (1.5) and (1.7), let e be a unit direction of $\mathbb{R}^d$. Assume moreover that $q = 0$. Then,*

$$w^*(e) \leq 2\sqrt{M_0 M}.$$

Furthermore, if $\Omega = \mathbb{R}^N$, then

$$w^*(e) \geq \min(m_0\alpha_1/\tilde{b}, -\tilde{b} + 2\sqrt{m_0\alpha_1}),$$

where α_1 was given in (1.3), $m_0 = \min_{x\in\mathbb{R}^N}\ f'_u(x,0)$ and

$$\tilde{b} = \max_{x\in\mathbb{R}^N,\ \vec{\mu}\in\mathbb{R}^N,\ \vec{\mu}\neq 0} |\nabla\cdot(A(x)\vec{\mu})|/|\vec{\mu}|\ +\ \max_{x\in\mathbb{R}^N} |q(x)|.$$

Lastly, if $f = f(u)$ and $w^*_\gamma(e)$ denotes the spreading speed in the direction e, with diffusion matrix γA, then

$$w^*_\alpha(e) \leq w^*_\beta(e) \text{ if } 0 < \alpha \leq \beta.$$

The proofs of the above Propositions are sketched in Remarks 3.2, 3.3 and 3.4 in Section 3 below.

2 Variational formula for the minimal speed of pulsating travelling fronts

This section is devoted to the proof of formula (1.8) in Theorem 1.1. One assumes all the hypotheses in Theorem 1.1, and e denotes a unit vector of $\mathbb{R}^d$.

Let us first collect some useful properties of the first eigenvalue $k(\lambda)$ of the operator L_λ given in (1.9).

Lemma 2.1 *The function $\lambda \mapsto k(\lambda)$ is a convex function of λ. Furthermore, there exists a convex function k_0 such that $k_0(0) = k'_0(0) = 0$ and*

$$\forall\lambda\in\mathbb{R},\quad 0\ <\ \min_{\overline{\Omega}}\ \zeta\ \leq\ \min_{\overline{\Omega}}\ \zeta\ + k_0(\lambda)\ \leq\ k(\lambda)\ \leq\ \max_{\overline{\Omega}}\ \zeta\ + k_0(\lambda). \qquad (2.1)$$

Proof. Up to a change of notations (q into $-q$, and e into $-e$) in the equations in [7], the first eigenvalue $k(\lambda)$ of the operator L_λ corresponds to the eigenvalue $-\mu_{\gamma,\zeta}(\lambda) + \lambda\gamma$ of the operator $L_{\gamma,\lambda,\zeta} + \lambda\gamma$ in Proposition 5.7 of [7]. From parts (ii) and (iii) of Proposition 5.7 of [7], it follows that

$$\forall\lambda\in\mathbb{R},\quad \min_{\overline{\Omega}}\ \zeta\ + k_0(\lambda)\ \leq\ k(\lambda)\ \leq\ \max_{\overline{\Omega}}\ \zeta\ + k_0(\lambda),$$

where $k_0(\lambda)$ is the first eigenvalue of the operator $L_\lambda - \zeta$, and $k_0(0) = k'_0(0) = 0$ ($k_0(\lambda)$ corresponds to $-h(\lambda)$ in Proposition 5.7 of [7]). It follows from [7] that the function k_0 is convex. As a consequence, k_0 is a nonnegative function, and (2.1) follows.

Furthermore, as in [7], the first eigenvalue $k(\lambda)$ can be rewritten as

$$k(\lambda) = \min_{\psi \in F} \max_{\overline{\Omega}} \frac{L_\lambda \psi}{\psi} = \min_{\tilde{\psi} \in \tilde{F}} \max_{(x,y) \in \overline{\Omega}} \left(\frac{\nabla \cdot (A \nabla \tilde{\psi}) + q \cdot \nabla \tilde{\psi}}{\tilde{\psi}(x,y)} + \zeta(x,y) \right), \qquad (2.2)$$

where $F = \{\psi \in C^2(\overline{\Omega}),\ \psi$ is L-periodic with respect to x, $\nu A \nabla \psi = \lambda(\nu A \tilde{e})\psi$ on $\partial\Omega$ and $\psi > 0$ in $\overline{\Omega}\}$, and

$$\begin{aligned} \tilde{F} &= \{(x,y) \mapsto \psi(x,y) e^{-\lambda x \cdot e},\ \psi \in F\} \\ &= \{\tilde{\psi} \in C^2(\overline{\Omega}),\ \tilde{\psi} e^{\lambda x \cdot e} \text{ is } L\text{-periodic w.r.t. } x,\ \nu A \nabla \tilde{\psi} = 0 \text{ on } \partial\Omega \text{ and } \tilde{\psi} > 0 \text{ in } \overline{\Omega}\}. \end{aligned}$$

It follows from the last expression of $k(\lambda)$ in (2.2), as in Proposition 5.7 of [7], that the function k is convex with respect to λ. □

The main result of this section is the following

Proposition 2.2 *If $c \in \mathbb{R}$ satisfies*

$$c > \inf \{\gamma \in \mathbb{R};\ \exists \lambda > 0,\ k(\lambda) = \lambda \gamma\},$$

then $c > 0$ and there exists a solution $u(t,x,y)$ of (1.2) and (1.6), namely u is a pulsating travelling front propagating in the direction e with the effective speed c.

This proposition is proved at the end of this section. Let us now turn to the

Proof of Theorem 1.1. As already emphasized, it follows from Remark 1.16 and Section 6.4 in [7] that, for all pulsating travelling front propagating in the direction e, with speed $c \geq c^*(e)$, there exists $\lambda > 0$ such that $k(\lambda) = \lambda c$. Therefore,

$$c^*(e) \geq \inf \{c,\ \exists \lambda > 0,\ k(\lambda) = \lambda c\}. \qquad (2.3)$$

From Proposition 2.2 above, inequality (2.3) turns out to be an equality. Furthermore, the infimum is reached since for $c = c^*(e)$, there still exists $\lambda^* > 0$ such that $k(\lambda^*) = \lambda^* c^*(e)$.

Eventually, one concludes that

$$\inf_{\lambda > 0} \frac{k(\lambda)}{\lambda} = \inf \{c,\ \exists \lambda > 0,\ k(\lambda) = \lambda c\} = c^*(e) = \frac{k(\lambda^*)}{\lambda^*},$$

whence $c^*(e) = \min_{\lambda > 0} k(\lambda)/\lambda$. That completes the proof of Theorem 1.1. □

Remark 2.3 *Since $c^*(e) > 0$, it follows from the above proof and Lemma 2.1 that the function $\lambda \mapsto k(\lambda)/\lambda$ is continuous on $\mathbb{R}_+^*$ and*

$$\frac{k(\lambda)}{\lambda} \to +\infty \text{ as } \lambda \to 0^+, \text{ and } \liminf_{\lambda \to +\infty} \frac{k(\lambda)}{\lambda} > 0.$$

Let us now turn to the

Proof of Proposition 2.2. The proof follows the lines of those of [7] and [17], together with the additional assumption (1.7), and we just outline it.

Let c be as in Proposition 2.2 and let $c' < c$ and $\lambda' > 0$ be such that $k(\lambda') = \lambda' c'$. Let $\psi' \in E$ be the unique (up to multiplication) positive principal eigenfunction of

$$L_{\lambda'}\psi' = k(\lambda')\psi' \ \text{ in } \Omega.$$

Let us first observe that $k(\lambda')$ is positive from Lemma 2.1, whence c' and c are positive as well.

Finding a classical $C^2(\mathbb{R}\times\overline{\Omega})$ solution $u(t,x,y)$ of (1.2) and (1.6) is the same, up to the change of variables

$$u(t,x,y) = \phi(x\cdot e - ct, x, y), \quad \phi(s,x,y) = u\left(\frac{x\cdot e - s}{c}, x, y\right),$$

as proving the existence of a function $\phi \in C^2(\mathbb{R}\times\overline{\Omega})$ solving

$$\begin{cases} L\phi + f(x,y,\phi) := \nabla_{x,y}\cdot(A\nabla_{x,y}\phi) + (\tilde{e}A\tilde{e})\phi_{ss} + \nabla_{x,y}\cdot(A\tilde{e}\phi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi) \\ \qquad\qquad\qquad +q\cdot\nabla_{x,y}\phi + (q\cdot\tilde{e} + c)\phi_s + f(x,y,\phi) = 0 \ \text{ in } \mathbb{R}\times\overline{\Omega} \\ \phi(-\infty,\cdot,\cdot) = 1,\ \phi(+\infty,\cdot,\cdot) = 0 \text{ (uniform limits in } (x,y)\in\overline{\Omega}) \\ \phi \text{ is } L\text{-periodic with respect to } x \\ \nu A(\nabla_{x,y}\phi + \tilde{e}\phi_s) = 0 \ \text{ on } \mathbb{R}\times\partial\Omega. \end{cases}$$

The existence of a solution ϕ of the above problem shall be proved by solving regularized elliptic equations of the type

$$L^{\varepsilon}\phi + f(x,y,\phi) := L\phi + \varepsilon\phi_{ss} + f(x,y,\phi) = 0,$$

where $\varepsilon > 0$, in cylinders of the type $\Sigma_a = \{(s,x,y),\ -a < s < a,\ (x,y)\in\Omega\}$ which are bounded in the variable s. One shall then pass to the limits $a \to +\infty$ and $\varepsilon \to 0^+$.

To this end, let us first fix $a > 0$. The number $\varepsilon > 0$ shall be chosen later. Let us now extend the function f by $f(x,y,u) = 0$ for all $u \geq 1$ and $(x,y)\in\overline{\Omega}$. For $r\in\mathbb{R}$, let v_r be the function defined by

$$v_r(s,x,y) = e^{-\lambda'(s+r)}\psi'(x,y)$$

for all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$. This function v_r is a supersolution for $\varepsilon > 0$ small enough and for all $r\in\mathbb{R}$, in the sense that, from (1.7) and from the definition of λ' and ψ',

$$\begin{aligned} L^{\varepsilon}v_r + f(x,y,v_r) &\leq [\nabla\cdot(A\nabla\psi') + (\lambda')^2(\tilde{e}A\tilde{e})\psi' - 2\lambda'\tilde{e}A\nabla\psi' - \lambda'\nabla\cdot(A\tilde{e})\psi' \\ &\qquad +q\cdot\nabla\psi' - \lambda'(q\cdot\tilde{e}+c)\psi' + \varepsilon(\lambda')^2\psi']e^{-\lambda'(s+r)} \ + \ \zeta(x,y)v_r \\ &\leq [k(\lambda') - \lambda'c + \varepsilon(\lambda')^2]\psi' e^{-\lambda'(s+r)} \\ &\leq \lambda'(c' - c + \varepsilon\lambda')\psi' e^{-\lambda'(s+r)} \\ &\leq 0 \end{aligned}$$

as soon as $0 < \varepsilon \leq (c - c')/\lambda'$ (this is possible since $c' < c$ and $\lambda' > 0$). Furthermore, the function v_r satisfies

$$\nu A(\nabla_{x,y}v_r + \tilde{e}\partial_s v_r) = [\nu A\nabla\psi' - \lambda'(\nu A\tilde{e})\psi']e^{-\lambda'(s+r)} = 0 \ \text{ on } \mathbb{R}\times\partial\Omega$$

because of the definition of ψ'. Lastly, the function $v'_r := \min(v_r, 1)$ is therefore a supersolution in the above sense as well.

For all $r \in \mathbb{R}$, let h_r be the positive constant defined by

$$0 < h_r := \min_{(x,y)\in\overline{\Omega}} v'_r(a, x, y) \ \leq \ 1.$$

The constant function h_r clearly satisfies $L^\varepsilon h_r + f(x, y, h_r) = f(x, y, h_r) \geq 0$ in $\mathbb{R} \times \Omega$, together with $\nu A(\nabla_{x,y} h_r + \tilde{e}\partial_s h_r) = 0$ on $\mathbb{R} \times \partial\Omega$. Furthermore, $h_r \leq v'_r(s, x, y)$ for all $(s, x, y) \in \overline{\Sigma_a}$ since v'_r is nonincreasing with respect to s.

From the general results of Berestycki and Nirenberg [16] (*see* also Lemma 5.1 in [7]), there exists a solution $w_r \in C(\overline{\Sigma_a}) \cap C^2(\overline{\Sigma_a} \backslash \{\pm a\} \times \partial\Omega)$ of

$$\begin{cases} L^\varepsilon w_r + f(x, y, w_r) = 0 \ \text{ in } \Sigma_a \\ \nu A(\nabla_{x,y} w_r + \tilde{e}\partial_s w_r) = 0 \ \text{ on } (-a, a) \times \partial\Omega \\ w_r \text{ is } L\text{-periodic with respect to } x \\ w_r(-a, x, y) = v'_r(-a, x, y) \ \text{ for all } (x, y) \in \overline{\Omega} \\ w_r(a, \cdot, \cdot) = h_r \\ 0 < h_r \leq w_r(s, x, y) \leq v'_r(s, x, y) \ \text{ for all } (s, x, y) \in \overline{\Sigma_a}, \end{cases} \tag{2.4}$$

as soon as $0 < \varepsilon \leq (c - c')/\lambda'$. Furthermore, since the function v'_r is nonincreasing with respect to s and since the coefficients of $L^\varepsilon \cdot + f(x, y, \cdot)$ do not depend on the variable s, it follows that the function w_r is actually unique and it is nonincreasing with respect to s. This can be done as in Lemma 5.2 in [7], by using the same sliding method as in [16]. Lastly, the same device as in Lemma 5.3 in [7] yields that w_r is nonincreasing with respect to r, and that the function $r \mapsto w_r$ is continous with respect to r in $C^{2,\alpha}_{loc}(\overline{\Sigma_a} \backslash \{\pm a\} \times \partial\Omega)$ (for all $0 < \alpha < 1$) and in $C(\overline{\Sigma_a})$.

Since $0 \leq h_r \leq w_r \leq v'_r \leq 1$ in $\overline{\Sigma_a}$ and $h_r \to 1$ (resp. $v'_r \to 0$) uniformly in $\overline{\Sigma_a}$ as $r \to -\infty$ (resp. $r \to +\infty$), one finally concludes that, for each $\varepsilon \in (0, (c - c')/\lambda']$ and for all $a > 0$, there exists a unique $r_{\varepsilon,a} \in \mathbb{R}$ such that the function $w^{\varepsilon,a} := w_{r_{\varepsilon,a}}$ satisfies (2.4) and

$$\max_{(x,y)\in\overline{\Omega}} w^{\varepsilon,a}(0, x, y) = 1/2.$$

Let $\varepsilon \in (0, (c - c')/\lambda']$ be fixed and consider a sequence $a_n \to +\infty$. From the standard elliptic estimates up to the boundary, the functions w^{ε,a_n} converge, up to extraction of some subsequence, in $C^{2,\alpha}_{loc}(\mathbb{R} \times \overline{\Omega})$ (for all $0 < \alpha < 1$) to a function w^ε solving

$$\begin{cases} L^\varepsilon w^\varepsilon + f(x, y, w^\varepsilon) = 0 \ \text{ in } \mathbb{R} \times \overline{\Omega} \\ \nu A(\nabla_{x,y} w^\varepsilon + \tilde{e}\partial_s w^\varepsilon) = 0 \ \text{ on } \mathbb{R} \times \partial\Omega \\ w^\varepsilon \text{ is } L\text{-periodic with respect to } x \\ 0 \leq w^\varepsilon \leq 1, \quad \max_{(x,y)\in\overline{\Omega}} w^\varepsilon(0, x, y) = 1/2. \end{cases} \tag{2.5}$$

Furthermore, the function w^ε is nonincreasing with respect to s.

From the monotonicity of w^ε with respect to s and from the standard elliptic estimates, it follows that $w^\varepsilon(s,x,y) \to \phi_\pm(x,y)$ in $C^{2,\alpha}(\overline{\Omega})$ as $s \to \pm\infty$, where the functions $\phi_\pm$ satisfy

$$\begin{cases} \nabla\cdot(A\nabla\phi_\pm) + q\cdot\nabla\phi_\pm + f(x,y,\phi_\pm) = 0 \ \text{ in } \overline{\Omega} \\ \nu A\nabla\phi_\pm = 0 \ \text{ on } \partial\Omega \\ \phi_\pm \text{ is } L\text{-periodic with respect to } x \\ 0 \le \phi_+ \le \phi_- \le 1. \end{cases} \tag{2.6}$$

Integrating by parts over the cell C leads to

$$\int_C f(x,y,\phi_\pm(x,y))\ dx\ dy = 0,$$

whence $f(x,y,\phi_\pm(x,y)) \equiv 0$ in $\overline{\Omega}$ by continuity. Now multiply equation (2.6) by $\phi_\pm$ and integrate by parts over C. It follows that

$$\int_C \nabla\phi_\pm A\nabla\phi_\pm = 0$$

and that $\phi_\pm$ are constants. From the monotonicity of w^ε and the normalization of w^ε on the section $\{0\}\times\overline{\Omega}$, together with assumption (1.7), one concludes that

$$\phi_+ = 0 \ \text{ and } \ \phi_- = 1.$$

Let us now come back to the variables (t,x,y). For $\varepsilon \in (0,(c-c')/\lambda')$, the functions u^ε defined by

$$u^\varepsilon(t,x,y) = w^\varepsilon(x\cdot e - ct, x, y) \ \text{ for all } (t,x,y)\in\mathbb{R}\times\overline{\Omega}$$

satisfy

$$\begin{cases} u^\varepsilon_t = \nabla\cdot(A\nabla_{x,y}u^\varepsilon) + \dfrac{\varepsilon}{c^2}u^\varepsilon_{tt} + q\cdot\nabla_{x,y}u^\varepsilon + f(x,y,u^\varepsilon) \ \text{ in } \mathbb{R}\times\overline{\Omega} \\ \nu A\nabla_{x,y}u^\varepsilon = 0 \ \text{ on } \mathbb{R}\times\partial\Omega \\ \forall k\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z},\ \forall (t,x,y)\in\mathbb{R}\times\overline{\Omega}, \quad u^\varepsilon\left(t-\dfrac{k\cdot e}{c},x,y\right) = u^\varepsilon(t,x+k,y) \\ \displaystyle\max_{x\cdot e=ct,\ (t,x,y)\in\mathbb{R}\times\overline{\Omega}} u^\varepsilon(t,x,y) = 1/2. \end{cases} \tag{2.7}$$

Furthermore, each function u^ε is nondecreasing in the variable t and $u^\varepsilon(t,x,y)\to 1$ (resp. $\to 0$) as $t\to+\infty$ (resp. $t\to-\infty$) in $C^2_{loc}(\overline{\Omega})$.

As in Lemma 5.11 in [7], by multiplying the equation (2.5) by 1, w^ε and $\partial_s w^\varepsilon$ and integrating by parts over $\mathbb{R}\times C$, it follows that, for every compact set $K\subset\overline{\Omega}$, there exists a constant $C(K)$ independent of ε such that

$$\int_{\mathbb{R}\times K} [(u^\varepsilon_t)^2 + |\nabla_{x,y}u^\varepsilon|^2]\ dt\ dx\ dy \le C(K)\left(\frac{1+N\|q\|^2_\infty}{2\alpha_1} + 2\max_{(x,y)\in\overline{\Omega}} F(x,y,1)\right),$$

where $F(x,y,t) = \displaystyle\int_0^t f(x,y,\tau)d\tau$ and α_1 is given from (1.3).

Let $(\varepsilon_n)_n \in (0,(c-c')/\lambda']$ be a sequence converging to 0^+. There exists a function $u \in H^1_{loc}(\mathbb{R}\times\Omega)$ such that, up to extraction of some subsequence, the functions u^{ε_n} converge, in $L^2_{loc}(\mathbb{R}\times\Omega)$ strong, $H^1_{loc}(\mathbb{R}\times\Omega)$ weak and almost everywhere in $\mathbb{R}\times\Omega$, to a function u. From parabolic regularity, the function u is then a classical solution of

$$\begin{cases} u_t = \nabla\cdot(A\nabla_{x,y}u) + q\cdot\nabla_{x,y}u + f(x,y,u) \ \text{ in } \mathbb{R}\times\overline{\Omega} \\ \nu A\nabla_{x,y}u = 0 \ \text{ on } \mathbb{R}\times\partial\Omega \\ \forall k\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z},\ \forall(t,x,y)\in\mathbb{R}\times\overline{\Omega}, \quad u\left(t-\dfrac{k\cdot e}{c},x,y\right) = u(t,x+k,y) \\ 0\le u\le 1 \ \text{ and } u_t\ge 0 \ \text{ in } \mathbb{R}\times\overline{\Omega}. \end{cases}$$

Furthermore, from the normalization of u^ε on the set $\{x\cdot e = ct\}$ and from the monotonicity of u^ε in t, one has

$$u(t,x,y)\le 1/2 \ \text{ for all } (t,x,y) \text{ such that } x\cdot e\le ct. \tag{2.8}$$

On the other hand, equation (2.7) is an elliptic regularization of a parabolic equation. From Theorem A.1 in [7][5] (it is easy to check that assumptions are satisfied, especially the functions u^ε are of class $C^3(\mathbb{R}\times\overline{\Omega})$ from the regularity assumptions and from the standard elliptic estimates), the following gradient estimates hold :

$$\|\nabla_{x,y}u^\varepsilon\|_{L^\infty(\mathbb{R}\times\overline{\Omega})}\le C, \tag{2.9}$$

where C is independent of ε.

Since $\max_{x\cdot e=ct} u^\varepsilon(t,x,y) = 1/2$ and $u^\varepsilon(t-k\cdot e/c,x,y) = u^\varepsilon(t,x+k,y)$ in $\mathbb{R}\times\overline{\Omega}$ for all $k\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}$, there exists a sequence of points $(t_n,x_n,y_n)\in\mathbb{R}\times\overline{C}$ such that $x_n\cdot e = ct_n$ and $u^{\varepsilon_n}(t_n,x_n,y_n) = 1/2$. Therefore, the sequence $(t_n,x_n,y_n)_n$ is bounded and converges, up to extraction of some subsequence, to a point $(\overline{t},\overline{x},\overline{y})\in\mathbb{R}\times\overline{C}$ such that $\overline{x}\cdot e = c\overline{t}$. Choose any $\eta>0$. From the uniform gradient estimates (2.9), there exists $r>0$ such that $u^{\varepsilon_n}(t_n,x,y)\ge 1/2-\eta$ for all n and for all $(x,y)\in B_r(x_n,y_n)\cap\overline{\Omega}$, where $B_r(x_n,y_n)$ denotes the euclidian closed ball in $\mathbb{R}^N$ of radius r and center (x_n,y_n). Since each u^ε is nondecreasing in t, it follows that, for n large enough,

$$u^{\varepsilon_n}(t,x,y)\ge 1/2-\eta$$

for all $t\ge t_n$ and $(x,y)\in B_{r/2}(\overline{x},\overline{y})\cap\overline{\Omega}$. Since u^{ε_n} converges to the continous function u almost everywhere, one gets that

$$u(t,x,y)\ge 1/2-\eta$$

for all $t\ge\overline{t}$ and for all $(x,y)\in B_{r/2}(\overline{x},\overline{y})\cap\overline{\Omega}$. Since $\eta>0$ was arbitrary, it follows that $u(\overline{t},\overline{x},\overline{y})\ge 1/2$. From (2.8) and the (t,x) periodicity of u, one concludes that

$$\max_{x\cdot e=ct,\ (t,x,y)\in\mathbb{R}\times\overline{\Omega}} u(t,x,y) = 1/2. \tag{2.10}$$

[5]We also refer to Theorem 1.6 in [8] for more general estimates of a class of elliptic regularizations of degenerate equations.

Lastly, the standard parabolic estimates together with the monotonicity of u with respect to t imply that $u(t,x,y) \to u_\pm(x,y)$ in $C^2_{loc}(\overline{\Omega})$ as $t \to \pm\infty$, where the functions $u_\pm$ satisfy

$$\begin{cases} \nabla \cdot (A\nabla u_\pm) + q \cdot \nabla u_\pm + f(x,y,u_\pm) = 0 \ \text{ in } \overline{\Omega} \\ \nu A \nabla u_\pm = 0 \ \text{ on } \partial\Omega \\ u_\pm \text{ are } L\text{-periodic with respect to } x \end{cases}$$

and are such that $0 \le u_- \le u_+ \le 1$. As explained earlier for the functions $\phi_\pm$ solving (2.6), one can easily prove that the functions $u_\pm$ are actually constant and satisfy $f(x,y,u_\pm) = 0$ for all $(x,y) \in \overline{\Omega}$. Furthermore, $0 \le u_- \le 1/2 \le u_+ \le 1$ from (2.10) and $u_t \ge 0$. One concludes from (1.7) that $u_- = 0$ and $u_+ = 1$.

Eventually, the function u is a classical solution of (1.2) and (1.6). Indeed, because of the (t,x) periodicity of u, the limits $u(t,x,y) \to 0$ (resp. $\to 1$) as $x \cdot e \to +\infty$ (resp. $x \cdot e \to -\infty$) hold locally in (t,y) and uniformly in the x variables which are orthogonal to e.

That completes the proof of Proposition 2.2. $\square$

3 Influence of the geometry of the domain and of the coefficients of the medium

3.1 Influence of the geometry of the domain : proofs of Theorems 1.3 and 1.4

This subsection deals with the influence of the geometry of the domain on the speed of propagation of pulsating fronts for the homogeneous equation (1.15) in periodic domains Ω. Namely, one shall prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. One first recalls that the minimal speed $c^*(e)$ of the pulsating fronts solving (1.15) and (1.6) is positive (*see* [7]). Furthermore, from Theorem 1.1, $c^*(e)$ is given by the formula

$$c^*(e) = \min_{\lambda > 0} \frac{k(\lambda)}{\lambda},$$

where $k(\lambda)$ is the first eigenvalue of the problem

$$\Delta\psi_\lambda - 2\lambda\tilde{e} \cdot \nabla\psi_\lambda + (\lambda^2 + f'(0))\psi_\lambda = k(\lambda)\psi_\lambda \ \text{ in } \Omega \tag{3.1}$$

and ψ_λ is positive in $\overline{\Omega}$, L-periodic with respect to x, and satisfies $\partial_\nu\psi_\lambda = \lambda(\nu \cdot \tilde{e})\psi_\lambda$ on $\partial\Omega$.

Multiply the above equation by ψ_λ and integrate by parts over the cell C. It follows from the boundary and periodicity conditions that

$$-\int_C |\nabla\psi_\lambda|^2 \ + \ (\lambda^2 + f'(0)) \int_C \psi_\lambda^2 = k(\lambda) \int_C \psi_\lambda^2. \tag{3.2}$$

Therefore,

$$\forall \lambda > 0, \quad k(\lambda) \le \lambda^2 + f'(0) \tag{3.3}$$

and $c^*(e) = \min_{\lambda>0} k(\lambda)/\lambda \le 2\sqrt{f'(0)}$.

Assume now that the domain Ω is invariant in the direction $\tilde{e}$. Then $\nu \cdot \tilde{e} = 0$ on $\partial\Omega$ and a (unique up to multiplication) solution ψ_λ of the eigenvalue problem (3.1) is $\psi_\lambda = 1$. Therefore, $k(\lambda) = \lambda^2 + f'(0)$ for all $\lambda > 0$. Thus, $c^*(e) = 2\sqrt{f'(0)}$.

Conversely, assume that $c^*(e) = 2\sqrt{f'(0)}$. Set $\lambda^* = \sqrt{f'(0)}$. One claims that $k(\lambda^*) = (\lambda^*)^2 + f'(0)$. If not, then $k(\lambda^*) < (\lambda^*)^2 + f'(0)$ from (3.3) and

$$c^*(e) \le \frac{k(\lambda^*)}{\lambda^*} < \frac{(\lambda^*)^2 + f'(0)}{\lambda^*} = 2\sqrt{f'(0)},$$

which contradicts our assumption. Therefore, $k(\lambda^*) = (\lambda^*)^2 + f'(0)$ and it follows from (3.2) that ψ_{λ^*} is constant. As a consequence, $\nu \cdot \tilde{e} \equiv 0$ on $\partial\Omega$. Hence, Ω is invariant in the direction $\tilde{e}$. □

Let us now turn to the

Proof of Theorem 1.4. Up to a rotation of the frame, one can assume without loss of generality that $e = e_1 = (1, 0, \cdots, 0)$. Furthermore, if there is a family of domains $(\Omega_\alpha)_{0\le\alpha<1}$ of $\mathbb{R}^2$ such that the conclusion of Theorem 1.4 holds with $N = 2$ and $e = e_1$, then the family of domains $(\Omega'_\alpha)_{0\le\alpha<1} = (\Omega_\alpha \times \mathbb{R}^{N-2})_{0\le\alpha<1}$ satisfies the conclusion of Theorem 1.4 in higher dimensions N with $e = e_1$.

Therefore, it is enough to deal with the case $N = 2$ and $e = e_1 = (1, 0)$. Fix $L_1 = L_2 = 1$, and $0 < \beta < 1/2$. Let now $(\Omega_\alpha)_{0\le\alpha<1}$ be a family of smooth open connected subsets of $\mathbb{R}^2$ satisfying (1.1) with $L_1 = L_2 = 1$, satisfying (1.16) and such that

$$\forall\ 0 \le \alpha \le \frac{1}{3}, \quad \mathbb{R}^2 \backslash \Omega_\alpha \ \subset\ \mathbb{Z}^2 + \left(\frac{1}{2} - \alpha, \frac{1}{2} + \alpha\right) \times \left(\frac{1}{2} - \alpha, \frac{1}{2} + \alpha\right)$$

and

$$\forall \frac{2}{3} \le \alpha < 1, \quad \mathbb{Z}^2 + (1-\alpha, \alpha) \times [\beta, 1-\beta] \ \subset\ \mathbb{R}^2 \backslash \Omega_\alpha \ \subset\ \mathbb{Z}^2 + \left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right) \times [\beta, 1-\beta].$$

One also assumes that, for each $\alpha_0 \in (0,1)$, there exists $r > 0$ such that the sets Ω_α are C^3 uniformly with respect to $\alpha \in (\alpha_0 - r, \alpha_0 + r)$.

Let us prove that this family of domains fulfills the conclusion of Theorem 1.4 with $N = 2$ and $e = e_1$.

One first observes that, for each $\alpha \in (0,1)$, the domain Ω_α is not invariant in the direction e_1, whence $c_\alpha < 2\sqrt{f'(0)}$ from Theorem 1.3.

The other parts facts in Theorem 1.4 are proved in Steps 2, 3 and 4 below. Step 1 is concerned with the derivation of inequality (3.5) below.

Step 1. Let first $\alpha \in [0,1)$ be fixed. The minimal speed $c_\alpha = c^*(e_1, \Omega_\alpha)$ of the pulsating fronts satisfying (1.15) and (1.6) in Ω_α with Neumann boundary conditions on $\partial\Omega_\alpha$, is given by the formula

$$c_\alpha = \min_{\lambda>0} \frac{k_\alpha(\lambda)}{\lambda},$$

where $k_\alpha(\lambda)$ is the first eigenvalue of the problem

$$\Delta\psi_{\alpha,\lambda} - 2\lambda\partial_1\psi_{\alpha,\lambda} + (\lambda^2 + f'(0))\psi_{\alpha,\lambda} = k_\alpha(\lambda)\psi_{\alpha,\lambda} \text{ in } \Omega_\alpha \tag{3.4}$$

and $\psi_{\alpha,\lambda}$ is positive in $\overline{\Omega_\alpha}$, $(1,1)$-periodic with respect to (x_1, x_2), and satisfies $\partial_\nu\psi_{\alpha,\lambda} = \lambda(\nu \cdot e_1)\psi_{\alpha,\lambda}$ on $\partial\Omega_\alpha$, where ν stands for the unit exterior normal to Ω_α. Observe now that, from the monotonicity of the domains (Ω_α), one has

$$\Omega_\alpha \supset \mathbb{R} \times (-\beta, \beta).$$

Therefore, it follows from the maximum principle (*see* [18] for more details) that $k_\alpha(\lambda) \geq \kappa(\lambda)$ for all $\lambda > 0$, where $\kappa(\lambda)$ (resp. ψ_λ) is the first eigenvalue (resp. eigenfunction) of

$$\begin{cases} \Delta\psi_\lambda - 2\lambda\partial_1\psi_\lambda + (\lambda^2 + f'(0))\psi_\lambda = \kappa(\lambda)\psi_\lambda \text{ in } \mathbb{R} \times (-\beta, \beta), \\ \psi_\lambda > 0 \text{ in } \mathbb{R} \times (-\beta, \beta), \\ \psi_\lambda = 0 \text{ on } \mathbb{R} \times \{\pm\beta\}, \\ \psi_\lambda \text{ is 1-periodic with respect to } x_1. \end{cases}$$

By uniqueness, the function ψ_λ does not depend on the variable x_1, whence $\kappa(\lambda) = \lambda^2 + f'(0) - (\pi/(2\beta))^2$. It follows that

$$\forall \alpha \in [0,1), \quad \forall \lambda > 0, \quad k_{\alpha,}(\lambda) \geq \lambda^2 + f'(0) - \left(\frac{\pi}{2\beta}\right)^2. \tag{3.5}$$

Step 2. Let α be fixed in $(0,1)$ and let us now prove that the function $t \mapsto c_t$ is continuous at α. If not, there exists $\varepsilon > 0$ and a sequence $(\alpha_n)_{n\in\mathbb{N}} \to \alpha$ such that $|c_{\alpha_n} - c_\alpha| \geq \varepsilon$ for all n. Up to extraction of some subsequence, two cases may occur :

Case 1 : $c_{\alpha_n} \leq c_\alpha - \varepsilon$ for all n. For each n, let $\lambda_n > 0$ be such that $c_{\alpha_n} = k_{\alpha_n}(\lambda_n)/\lambda_n$ (the existence of such λ_n follows from Theorem 1.1), and let $\psi_n = \psi_{\alpha_n,\lambda_n}$ solve (3.4) in Ω_{α_n}. The functions ψ_n are positive in $\overline{\Omega_{\alpha_n}}$, $(1,1)$-periodic in (x_1, x_2) and satisfy $\partial_\nu\psi_n = \lambda_n(\nu \cdot e_1)\psi_n$ on $\partial\Omega_{\alpha_n}$. Up to normalization, one can assume that $\psi_n(0,0) = 1$.

Since $0 \leq c_{\alpha_n} = k_{\alpha_n}(\lambda_n)/\lambda_n \leq c_\alpha - \varepsilon$, it follows from (3.5) that the sequence (λ_n) is bounded. On the other hand, $k_{\alpha_n}(\lambda_n) \geq f'(0)$ from Lemma 2.1. Therefore, the sequence (λ_n) is bounded from below by a positive constant. Up to extraction of some subsequence, one can then assume that $\lambda_n \to \lambda \in (0, +\infty)$ as $n \to +\infty$. On the other hand, one can also assume that $c_{\alpha_n} \to c \in [0, c_\alpha - \varepsilon]$ as $n \to +\infty$.

Furthermore, since the domains (Ω_{α_n}) are uniformly C^3, the functions ψ_n satisfy uniform $C^{2,\delta}$ bounds in $\overline{\Omega_{\alpha_n}}$ up to the boundary. Up to extraction of some subsequence, the functions ψ_n converge in $C^2_{loc}(\Omega_\alpha)$ to a solution ψ of

$$\Delta\psi - 2\lambda\partial_1\psi + (\lambda^2 + f'(0))\psi = c\lambda\psi_{\alpha,\lambda} \text{ in } \Omega_\alpha,$$

which can be extended as a C^2 function in $\overline{\Omega_\alpha}$ such that $\partial_\nu\psi = \lambda(\nu \cdot e_1)\psi$ on $\partial\Omega_\alpha$. Furthermore, ψ is nonnegative, $(1,1)$-periodic, and satisfies $\psi(0,0) = 1$. From the strong maximum principle, the function ψ is positive. It is therefore the first eigenfunction of problem (3.4) with the above periodicity and boundary condition. Hence, $c\lambda = k_\alpha(\lambda)$. Formula (1.13) implies that $c \geq c_\alpha$, which contradicts the fact that $c \leq c_\alpha - \varepsilon$. In other words, case 1 is ruled out.

Case 2 : $c_{\alpha_n} \geq c_\alpha - \varepsilon$ for all n. Let now $\lambda > 0$ be such that $c_\alpha = k_\alpha(\lambda)/\lambda$. From (3.3) and (1.8), one has

$$\lambda^2 + f'(0) \geq k_{\alpha_n}(\lambda) \geq \lambda c_{\alpha_n} \geq \lambda(c_\alpha + \varepsilon). \tag{3.6}$$

Up to extraction of some subsequence, one can assume that $k_{\alpha_n}(\lambda) \to k > 0$ as $n \to +\infty$, and that the functions $\psi_{\alpha_n,\lambda}$, normalized by $\psi_{\alpha_n,\lambda}(0,0) = 1$, converge locally in Ω_α to a positive $(1,1)$-periodic $C^2(\overline{\Omega_\alpha})$ solution ψ of

$$\begin{cases} \Delta\psi - 2\lambda\partial_1\psi + (\lambda^2 + f'(0))\psi = k\psi_{\alpha,\lambda} \ \text{ in } \Omega_\alpha \\ \partial_\nu\psi = \lambda(\nu \cdot e_1)\psi \ \text{ on } \partial\Omega_\alpha. \end{cases}$$

One concludes that $k = k_\alpha(\lambda)$, whence $k_\alpha(\lambda) \geq \lambda(c_\alpha + \varepsilon)$ from (3.6). This contradicts the definition of λ. Therefore, case 2 is ruled out too.

That proves the continuity of the map $\alpha \mapsto c_\alpha$ in $(0,1)$.

Step 3. Let us now prove that $c_\alpha \to 2\sqrt{f'(0)}$ as $\alpha \to 0^+$. Assume not. Since $0 \leq c_\alpha \leq 2\sqrt{f'(0)}$ for all $\alpha \in [0,1)$ because of Theorem 1.3, there exists then a sequence $(\alpha_n) \to 0^+$ such that $c_{\alpha_n} \to c \in [0, 2\sqrt{f'(0)})$ as $n \to +\infty$. On the other hand, there exists a sequence (λ_n) such that $c_{\alpha_n} = k_{\alpha_n}(\lambda_n)/\lambda_n$ for each n. As in Case 1 of Step 2 above, one can prove that the sequence (λ_n) is bounded from below and above by two positive constants. From (3.3) and Lemma 2.1, it follows that the sequence $(k_{\alpha_n}(\lambda_n))$ is itself bounded from below and above by two positive constants. Therefore, $c > 0$.

For each n, let $u_n(t, x_1, x_2)$ be a pulsating travelling front solving (1.6) with the speed c_{α_n} and such that

$$\begin{cases} (u_n)_t = \Delta u_n + f(u_n) \ \text{ in } \mathbb{R} \times \Omega_{\alpha_n} \\ \partial_\nu u_n = 0 \ \text{ on } \mathbb{R} \times \partial\Omega_{\alpha_n}. \end{cases} \tag{3.7}$$

Furthermore, each u_n satisfies $0 \leq u_n \leq 1$ and $(u_n)_t \geq 0$ in $\mathbb{R} \times \Omega_{\alpha_n}$. Up to normalization, one can assume that $u_n(0,0,0) = 1/2$.

Owing to the construction of the domains Ω_α, and from standard parabolic estimates, the functions u_n converge, up to extraction of some subsequence, to a classical solution $u = u(t, x_1, x_2)$ of

$$u_t = \Delta u + f(u) \ \text{ in } \mathbb{R} \times (\mathbb{R}^2 \backslash (\mathbb{Z}^2 + (1/2, 1/2)))$$

such that $0 \leq u \leq 1$. The singularities on the lines $\mathbb{R} \times (\mathbb{Z}^2 + (1/2, 1/2))$ in (t, x_1, x_2) variables are then removable and the function u can be extended to a classical solution u of $u_t = \Delta u + f(u)$ in $\mathbb{R} \times \mathbb{R}^2$. On the other hand, the function u satisfies

$$u\left(t - \frac{k_1}{c}, x_1, x_2\right) = u(t, x_1 + k_1, x_2 + k_2)$$

for all $(t, x_1, x_2) \in \mathbb{R} \times \mathbb{R}^2$ and $(k_1, k_2) \in \mathbb{Z}^2$. Furthermore, $u_t \geq 0$ in $\mathbb{R} \times \mathbb{R}^2$ and $u(0,0,0) = 1/2$. By passing to the limit $t \to \pm\infty$, one can prove as in [7] that $u(t, x_1, x_2) \to 0$ (resp. 1) as $t \to -\infty$ (resp. $t \to +\infty$) locally in (x_1, x_2).

Eventually, the function u is a pulsating travelling front, solving (1.6) with $e = e_1$, and (1.15) in $R \times \mathbb{R}^2$, with the speed c. But the minimal speed for this problem is equal to

$2\sqrt{f'(0)}$ (from Theorem 1.3 –the domain $\mathbb{R}^2$ is invariant in the direction e_1). Therefore, $c \geq 2\sqrt{f'(0)}$, which is contradiction with our assumption.

As a consequence, the function $\alpha \mapsto c_\alpha$ is continuous at 0.

Step 4. Let us now prove that $c_\alpha \to 2\sqrt{f'(0)}$ as $\alpha \to 1^-$. Assume not. As above, there exists then a sequence $(\alpha_n) \to 1^-$ such that $c_{\alpha_n} \to c \in (0, 2\sqrt{f'(0)})$ as $n \to +\infty$. Let $u_n = u_n(t, x_1, x_2)$ be a pulsating travelling front solving (3.7) and (1.6) with speed c_{α_n}. Up to normalization, one can assume that $u_n(0,0,0) = 1/2$. Consider now the restrictions, still called u_n, of the functions u_n to $\mathbb{R} \times \mathbb{R} \times [-\beta, \beta]$. Owing to the construction of the domains Ω_α, the functions u_n converge, up to extraction of some subsequence, to a classical solution $u(t, x_1, x_2)$ of

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \mathbb{R} \times (\mathbb{R} \times [-\beta, \beta] \backslash \mathbb{Z} \times \{\pm\beta\}) \\ \partial_\nu u = 0 & \text{on } \mathbb{R} \times \mathbb{R} \backslash \mathbb{Z} \times \{\pm\beta\} \end{cases}$$

such that $0 \leq u \leq 1$. The singularities on the lines $\mathbb{R} \times \mathbb{Z} \times \{\pm\beta\}$ in (t, x_1, x_2) variables are removable and the function u can then be extended to a classical solution u of (1.15) in $\mathbb{R} \times \mathbb{R} \times [-\beta, \beta]$ such that $\partial_\nu u = 0$ on $\mathbb{R} \times \mathbb{R} \times \{\pm\beta\}$. On the other hand, the function u satisfies

$$u\left(t - \frac{k_1}{c}, x_1, x_2\right) = u(t, x_1 + k_1, x_2)$$

for all $(t, x_1, x_2) \in \mathbb{R} \times [-\beta, \beta]$ and $k_1 \in \mathbb{Z}$. Furthermore, $u_t \geq 0$ in $\mathbb{R} \times \mathbb{R}^2$ and $u(0,0,0) = 1/2$. By passing to the limit $t \to \pm\infty$, one can prove that $u(t, x_1, x_2) \to 0$ (resp. 1) as $t \to -\infty$ (resp. $t \to +\infty$) locally in (x_1, x_2).

Eventually, the function u is a pulsating travelling front, solving (1.15) in $\mathbb{R} \times \mathbb{R} \times [-\beta, \beta]$, together with Neumann boundary conditions on $\mathbb{R} \times \mathbb{R} \times \{\pm\beta\}$. The function u satisfies (1.6) with $e = e_1$ and speed c. Since the domain $\mathbb{R} \times [-\beta, \beta]$ is invariant in the direction e_1, the minimal speed of such travelling fronts is $2\sqrt{f'(0)}$. One then gets a contradiction with our assumption that $c < 2\sqrt{f'(0)}$.

Therefore, $c_\alpha \to 2\sqrt{f'(0)}$ as $\alpha \to 1^-$ and the proof of Theorem 1.4 is complete. □

Remark 3.1 *Let the family $(\Omega_\alpha)_{0 \leq \alpha < 1}$ of domains of $\mathbb{R}^2$ be given as above. Let e be a unit direction of $\mathbb{R}^2$ such that $e \neq \pm e_1$, and let $c^*_\alpha(e)$ be the minimal speed of pulsating fronts solving (1.6) and (1.15) in Ω_α with Neumann boundary conditions on $\partial\Omega_\alpha$. As above, the function $\alpha \mapsto c^*_\alpha(e)$ is continuous on $[0,1)$, with $c^*_0(e) = 2\sqrt{f'(0)}$ and $c^*_\alpha(e) < 2\sqrt{f'(0)}$ for all $\alpha \in (0,1)$. With the same arguments as above, it easily follows than $\liminf_{\alpha \to 1^-} c^*_\alpha(e) \geq 2\sqrt{f'(0)}|e^1|$, where e^1 is the x_1-component of the direction e. Neverthless, the determination of the limit, if any, of $c^*_\alpha(e)$ as $\alpha \to 1^-$ is still open.*

3.2 Influence of the coefficients of the medium : proofs of Theorems 1.6, 1.8 and 1.10

This subsection is devoted to the study of the influence of the coefficients of the medium (reaction, advection and diffusion terms) on the speed of propagation of pulsating travelling fronts.

Let us first investigate the dependance on the reaction term f.

Proof of Theorem 1.6. a). Let f and g satisfy (1.5) and (1.7) and assume that $f'_u(x,y,0) \le g'_u(x,y,0)$ for all $(x,y) \in \overline{\Omega}$. For any $\lambda > 0$, let $k(\lambda, f)$ (resp. $k(\lambda, g)$) be the first eigenvalue of (1.9) with $\zeta(x,y) = f'_u(x,y,0)$ (resp. $\zeta(x,y) = g'_u(x,y,0)$). It follows from monotonicity properties of the first eigenvalue of elliptic problems (*see* [18]) that $k(\lambda, f) \le k(\lambda, g)$. Hence, Theorem 1.1 yields $c^*(e,f) \le c^*(e,g)$.

Assume furthermore that $f'_u(x,y,0) \underset{\not\equiv}{\le} g'_u(x,y,0)$. Let $\lambda_0 > 0$ be chosen so that $c^*(e,g) = k(\lambda_0, g)/\lambda_0$. We claim that $k(\lambda_0, f) < k(\lambda_0, g)$. If this holds, then

$$c^*(e,f) \le \frac{k(\lambda_0, f)}{\lambda_0} < \frac{k(\lambda_0, g)}{\lambda_0} = c^*(e,g)$$

and we are done. Assume then that $k(\lambda_0, f) \ge k(\lambda_0, g)$. Let ψ_f (resp. ψ_g) be a positive first eigenvalue of problem (1.9) with $\lambda = \lambda_0$ and $\zeta(x,y) = f'_u(x,y,0)$ (resp. $\zeta(x,y) = g'_u(x,y,0)$). Let $\tau > 0$ be such that $\psi_f \le \tau\psi_g$ in $\overline{\Omega}$ with equality somewhere (such a $\tau > 0$ exists since both ψ_f and ψ_g are continuous, positive and L-periodic with respect to x in $\overline{\Omega}$). The function $z := \psi_f - \tau\psi_g$ satisfies

$$\begin{array}{rcl} \nabla\cdot(A\nabla z) - 2\lambda_0\tilde{e}A\nabla z + q\cdot\nabla z + [-\lambda_0\nabla\cdot(A\tilde{e}) & & \\ -\lambda_0 q\cdot\tilde{e} + \lambda_0^2\tilde{e}A\tilde{e} + f'_u(x,y,0) - k(\lambda_0, f)]z & = & (k(\lambda_0,f) - k(\lambda_0,g))\tau\psi_g \\ & & +(g'_u(x,y,0) - f'_u(x,y,0))\tau\psi_g \\ & \ge, \not\equiv & 0 \ \text{ in } \Omega \end{array} \tag{3.8}$$

from our assumptions. On the other hand, $z \le 0$ in $\overline{\Omega}$ with equality somewhere, and $\nu A\nabla z = \lambda_0(\nu A\tilde{e})z$ on $\partial\Omega$. The strong maximum principle and Hopf lemma imply that $z \equiv 0$ in $\overline{\Omega}$. But the right-hand side of (3.8) is not identically equal to 0. One has then obtained a contradiction, whence $k(\lambda_0, f) < k(\lambda_0, g)$.

b) Let f satisfy (1.5) and (1.7). First, it follows from part a) that the function $B \mapsto c^*(e, Bf)$ is increasing with respect to $B > 0$. For any $\lambda > 0$ and $B > 0$, let $k(\lambda, B)$ and $\psi_{\lambda,B}$ be the first eigenvalue and eigenfunction of problem (1.9) with $\zeta(x,y) = Bf'_u(x,y,0)$. Multiply equation (1.9) by $\psi_{\lambda,B}$ and integrate by parts over C. One gets

$$k(\lambda,B)\int_C \psi_{\lambda,B}^2 = -\int_C \nabla\psi_{\lambda,B}A\nabla\psi_{\lambda,B} - \lambda\int_C (q\cdot\tilde{e})\psi_{\lambda,B}^2 + \lambda^2\int_C (\tilde{e}A\tilde{e})\psi_{\lambda,B}^2 + \int_C Bf'_u(x,y,0)\psi_{\lambda,B}^2.$$

It follows that

$$k(\lambda,B) \le \lambda\|q\cdot\tilde{e}\|_\infty + \lambda^2\|\tilde{e}A\tilde{e}\|_\infty + B\|f'_u(\cdot,\cdot,0)\|_\infty.$$

Hence,

$$c^*(e,Bf) = \min_{\lambda>0}\frac{k(\lambda,B)}{\lambda} = O(\sqrt{B}) \ \text{ as } B \to +\infty.$$

Assume now that $\Omega = \mathbb{R}^N$ or $\nu A\cdot\tilde{e} \equiv 0$ on $\partial\Omega$. In both cases, integrating over C the equation (1.9) satisfied by $\psi_{\lambda,B}$ with $\zeta(x,y) = Bf'_u(x,y,0)$ leads to

$$k(\lambda,B)\int_C \psi_{\lambda,B} = \lambda\int_C \nabla\cdot(A\tilde{e})\psi_{\lambda,B} - \lambda\int_C (q\cdot\tilde{e})\psi_{\lambda,B} + \lambda^2\int_C (\tilde{e}A\tilde{e})\psi_{\lambda,B} + \int_C Bf'_u(x,y,0)\psi_{\lambda,B}.$$

Hence,

$$k(\lambda, B) \geq -\lambda\|\nabla\cdot(A\tilde{e})\|_\infty - \lambda\|q\cdot\tilde{e}\|_\infty + \lambda^2\alpha_1 + B \min_{(x,y)\in\overline{\Omega}} f'_u(x,y,0),$$

where $\alpha_1 > 0$ is given in (1.3). Since $\min_{(x,y)\in\overline{\Omega}} f'_u(x,y,0) > 0$, it follows that there exists $\gamma > 0$ such that

$$c^*(e, Bf) = \min_{\lambda>0} \frac{k(\lambda, B)}{\lambda} \geq \gamma\sqrt{B}$$

for B large enough. That completes the proof of Theorem 1.6. □

Remark 3.2 *With the same tools as above, it can easily be seen that part a) of Theorem 1.6 extends to the ray speed $w^*(e)$, as defined in section 1.3 and in (1.23). Similarly, with obvious notations, $w^*(e,Bf) \leq c^*(e,Bf) = O(\sqrt{B})$ as $B \to +\infty$, and $\liminf_{B\to+\infty} w^*(e,Bf)/\sqrt{B} > 0$ if $\Omega = \mathbb{R}^N$.*

That corresponds to Proposition 1.15.

Proof of Theorem 1.8. Under the assumptions of Theorem 1.8, one has $\nu\cdot\tilde{e} = 0$ on $\partial\Omega$, and $c^*_q(e)$ is given by the formula

$$c^*_q(e) = \min_{\lambda>0} \frac{k_q(\lambda)}{\lambda},$$

where $k_q(\lambda)$ and $\psi_{\lambda,q}$ denote the unique eigenvalue and positive L-periodic eigenfunction of

$$\Delta\psi_{\lambda,q} - 2\lambda\tilde{e}\cdot\nabla\psi_{\lambda,q} + q\cdot\nabla\psi_{\lambda,q} + [-\lambda q\cdot\tilde{e} + \lambda^2 + f'(0)]\psi_{\lambda,q} = k_q(\lambda)\psi_{\lambda,q} \text{ in } \Omega \tag{3.9}$$

with $\nu\cdot\nabla\psi_{\lambda,q} = 0$ on $\partial\Omega$. Divide the following formula by $\psi_{\lambda,q}$ and integrate by parts over C. It follows from (1.4) and the L-periodicity of q and $\psi_{\lambda,q}$ that

$$\int_C \frac{|\nabla\psi_{\lambda,q}|^2}{\psi^2_{\lambda,q}} + (\lambda^2 + f'(0))|C| = k_q(\lambda)|C|, \tag{3.10}$$

whence

$$k_q(\lambda) \geq \lambda^2 + f'(0) = k_0(\lambda). \tag{3.11}$$

Therefore, $c^*_q(e) \geq 2\sqrt{f'(0)} = c^*_0(e)$.

If $q\cdot\tilde{e} \equiv 0$, then $\psi_{\lambda,q}$ is constant for each $\lambda > 0$, whence $k_q(\lambda) = \lambda^2 + f'(0)$ and $c^*_q(e) = 2\sqrt{f'(0)} = c^*_0(e)$.

Assume now that $c^*_q(e) = c^*_0(e) = 2\sqrt{f'(0)}$. Let $\lambda^* > 0$ be such that

$$c^*_q(e) = \frac{k_q(\lambda^*)}{\lambda^*}.$$

Then $k_q(\lambda^*) = 2\lambda^*\sqrt{f'(0)}$, whereas (3.11) yields $k_q(\lambda^*) \geq (\lambda^*)^2 + f'(0)$. Therefore, $\lambda^* = \sqrt{f'(0)}$ and $k_q(\lambda^*) = (\lambda^*)^2 + f'(0)$. From (3.10), one gets that $\psi_{\lambda^*,q}$ is constant, and from (3.9) one concludes that $q\cdot\tilde{e} \equiv 0$ in $\overline{\Omega}$. □

Remark 3.3 *With the same arguments as above and with obvious notations, one can deduce from (1.20) that, if $\Omega = \mathbb{R}^N$, $A = I$ and $f = f(u)$ satisfies (1.5) and (1.7), then $w_q^*(e) \geq w_0^*(e) = 2\sqrt{f'(0)}$ for all unit vector e of $\mathbb{R}^N$. Furthermore, the equality $w_q^*(e) = 2\sqrt{f'(0)}$ holds if and only if $q \cdot e \equiv 0$.*

That corresponds to Proposition 1.16.

Proof of Theorem 1.10. Under the assumptions of Theorem 1.10, let $k(\lambda)$ and ψ_λ be the first eigenvalue and eigenfunctions of the operator L_λ defined in (1.9). Multiply the equation $L_\lambda \psi_\lambda = k(\lambda)\psi_\lambda$ by ψ_λ and integrate over C. One gets

$$k(\lambda) \int_C \psi_\lambda^2 = -\int_C \nabla\psi_\lambda A \nabla\psi_\lambda + \lambda^2 \int_C \tilde{e}A\tilde{e}\ \psi_\lambda^2 + \int_C \zeta(x,y)\psi_\lambda^2. \tag{3.12}$$

Hence,

$$k(\lambda) \leq \max_{(x,y)\in\overline{\Omega}} \zeta(x,y) \ + \ \lambda^2 \max_{(x,y)\in\overline{\Omega}} \tilde{e}A(x,y)\tilde{e}, \tag{3.13}$$

and (1.17) follows from (1.8).

Assume now that $\tilde{e}A(x,y)\tilde{e} = M$ and $\zeta(x,y) = M_0$ are constant in $\overline{\Omega}$, and that $\nabla\cdot(A\tilde{e}) \equiv 0$ in $\overline{\Omega}$ and $\nu A\tilde{e} = 0$ on $\partial\Omega$ (if $\partial\Omega \neq \emptyset$). Then, ψ_λ is constant for each $\lambda > 0$, whence $k(\lambda) = M_0 + \lambda^2 M$ and $c^*(e) = 2\sqrt{M_0 M}$.

Assume now that $c^*(e) = 2\sqrt{M_0 M}$, where $M_0 = \max_{(x,y)\in\overline{\Omega}} \zeta(x,y)$ and $M = \max_{(x,y)\in\overline{\Omega}} \tilde{e}A(x,y)\tilde{e}$. Let $\lambda^* = \sqrt{M_0/M} > 0$. It follows from (1.8) that $k(\lambda^*) \geq c^*(e)\lambda^* = 2M_0$. On the other hand, $k(\lambda^*) \leq M_0 + (\lambda^*)^2 M = 2M_0$ from (3.13). Therefore, $k(\lambda^*) = 2M_0 = M_0 + (\lambda^*)^2 M$. One deduces from (3.12) and the equation $L_{\lambda^*}\psi_{\lambda^*} = k(\lambda^*)\psi_{\lambda^*}$ that ψ_{λ^*} is constant, $\zeta(x,y) \equiv M_0$, $\tilde{e}A(x,y)\tilde{e} \equiv M$, $\nabla\cdot(A\tilde{e}) \equiv 0$, and $\nu A\tilde{e} = 0$ on $\partial\Omega$.

Assume now that $0 < \alpha \leq \beta$, and let $c_\alpha^*(e)$ (resp. $c_\beta^*(e)$) denote the minimal speed of pulsating fronts in the direction e with diffusion αA (resp. βA). By (1.8), one has

$$c_\alpha^*(e) = \min_{\lambda>0} \frac{k^\alpha(\lambda)}{\lambda} \ \text{ and } c_\beta^*(e) = \min_{\lambda>0} \frac{k^\beta(\lambda)}{\lambda}, \tag{3.14}$$

where $k^\alpha(\lambda)$ (resp. $k^\beta(\lambda)$) is the first eigenvalue of the operator $\alpha\tilde{L}_\lambda + f'(0)$ (resp. $\beta\tilde{L}_\lambda + f'(0)$) and $\tilde{L}_\lambda$ is the operator $\tilde{L}_\lambda = \nabla\cdot(A\nabla) - 2\lambda\tilde{e}A\nabla - \lambda\nabla\cdot(A\tilde{e}) + \lambda^2\tilde{e}A\tilde{e}$ acting on the set E (E has been defined in Theorem 1.1). Under the notations of Lemma 2.1, $k_0(\lambda)$ is the first eigenvalue of $\tilde{L}_\lambda$, whence

$$k^\alpha(\lambda) = \alpha k_0(\lambda) + f'(0) \ \text{ and } k^\beta(\lambda) = \beta k_0(\lambda) + f'(0)$$

for all $\lambda > 0$. On the other hand, it follows from Lemma 2.1 that the function k_0 is nonnegative. Therefore, $k^\alpha(\lambda) \leq k^\beta(\lambda)$ and (3.14) yields $c_\alpha^*(e) \leq c_\beta^*(e)$. That completes the proof of Theorem 1.10. □

Let us now turn to the

Proof of the lower bound (1.18). Under the notations in Remark 1.11, integrate the equation $L_\lambda\psi_\lambda = k(\lambda)\psi_\lambda$ over C. It follows that

$$k(\lambda)\int_C \psi_\lambda = \lambda\int_C \nabla\cdot(A\tilde{e})\psi_\lambda - \lambda\int_C q\cdot\tilde{e}\ \psi_\lambda + \lambda^2\int_C \tilde{e}A\tilde{e}\ \psi_\lambda + \int_C \zeta(x,y)\psi_\lambda.$$

Therefore, $k(\lambda) \geq -\lambda b + \lambda^2 m + m_0$. On the other hand, Lemma 2.1 yields $k(\lambda) \geq m_0$. Formula (1.18) easily follows from (1.8). $\square$

Remark 3.4 *Under the assumptions and notations of Theorem 1.10, then $w^*(e) \leq c^*(e) \leq 2\sqrt{M_0 M}$ for all unit vector e of $\mathbb{R}^d$. Furthermore, under the additional assumption that $f = f(u)$, then the ray speed $w^*_\gamma(e)$ in the unit direction e of $\mathbb{R}^d$ for problem (1.2) with diffusion matrix γA ($\gamma > 0$) is given by*

$$w^*_\gamma(e) = \min_{\vec{\lambda} \in \mathbb{R}^d,\ \vec{\lambda}\cdot e > 0} \frac{\tilde{k}^\gamma(\vec{\lambda})}{\vec{\lambda}\cdot e},$$

where $\tilde{k}^\gamma(\vec{\lambda}) = \gamma k_{0,\vec{\lambda}/|\vec{\lambda}|}(|\vec{\lambda}|) + f'(0)$ and, for all unit vector e' of $\mathbb{R}^d$ and all $\mu > 0$, $k_{0,e'}(\mu)$ denotes the first eigenvalue of the operator $\nabla\cdot(A\nabla\psi) - 2\mu\, e'A\nabla\psi - \mu\nabla\cdot(Ae')\psi + \mu^2 e'Ae'\psi$ acting on the space of L-periodic functions ψ such that $\nu A\nabla\psi = \mu(\nu Ae')\psi$ on $\partial\Omega$. Lemma 2.1 yields $k_{0,e'}(\mu) \geq 0$. Therefore, $w^*_\alpha(e) \leq w^*_\beta(e)$ as soon as $0 < \alpha \leq \beta$.

Lastly, if $\Omega = \mathbb{R}^N$, with the same arguments as above, it easily follows from (1.20) that $w^*(e) \geq \min(m_0\alpha_1/\tilde{b}, -\tilde{b} + 2\sqrt{m_0\alpha_1})$, where α_1 was given in (1.3), $m_0 = \min_{x\in\mathbb{R}^N} f'_u(x,0)$ and $\tilde{b} = \max_{x\in\mathbb{R}^N,\ \vec{\mu}\in\mathbb{R}^N,\ \vec{\mu}\neq 0} |\nabla\cdot(A(x)\vec{\mu})|/|\vec{\mu}| + \max_{x\in\mathbb{R}^N} |q(x)|$.

That corresponds to Proposition 1.17.

4 Spreading speed

This section is devoted to the proof of Theorem 1.13. It is based on the following auxiliary Lemmas 4.1 and 4.2, and on Proposition 1.14.

Lemma 4.1 *Let Ω satisfy (1.1) with $d \geq 1$. Let λ_R be the first eigenvalue, and ψ_R be the first eigenfunction of*

$$\left\{\begin{array}{rcll} -\Delta\psi_R &=& \lambda_R\psi_R & \text{in } \Omega\cap B_R \\ \psi_R &>& 0 & \text{in } \overline{\Omega}\cap B_R \\ \psi_R &=& 0 & \text{on } \overline{\Omega}\cap\partial B_R \\ \partial_\nu\psi_R &=& 0 & \text{on } \partial\Omega\cap B_R \\ \|\psi_R\|_{L^\infty(\Omega\cap B_R)} &=& 1, & \end{array}\right. \tag{4.1}$$

where B_R is the open euclidean ball of radius $R > 0$ and centre 0. Then $\lambda_R \to 0$ as $R \to +\infty$.

Proof. It follows from the maximum principle that λ_R is decreasing with respect to R. Furthermore, λ_R has the following variational representation :

$$\lambda_R = \min_{\psi\in H^1(\Omega\cap B_R)\setminus\{0\},\ \psi_{|\overline{\Omega}\cap\partial B_R} = 0} \frac{\displaystyle\int_{\Omega\cap B_R} |\nabla\psi|^2}{\displaystyle\int_{\Omega\cap B_R} \psi^2} \geq 0.$$

Let ξ be a given $C^\infty(\mathbb{R}^N)$ function such that $\xi = 1$ in $B_{1/2}$ and $\xi = 0$ in $\mathbb{R}^N \backslash B_1$. Taking the function $\psi(z) = \xi(z/R)$ as a test function leads to

$$\lambda_R \leq \frac{R^{-2}\|\nabla \xi\|_\infty^2 |\Omega \cap B_R|}{|\Omega \cap B_{R/2}|}.$$

But $|\Omega \cap B_R|/|\Omega \cap B_{R/2}|$ is bounded as $R \to +\infty$ because of (1.1), whence $\lambda_R \to 0^+$ as $R \to +\infty$. □

Lemma 4.2 *Let Ω satisfy (1.1) with $d \geq 1$, let e be in S^{d-1}, and assume that Ω is invariant in the direction $\tilde{e}$. Let $f : [0, +\infty) \to \mathbb{R}$ be a function of class C^1 such that $f(0) = 0$ and $f'(0) > 0$, and let c be such that $|c| < 2\sqrt{f'(0)}$. Then there exist $R > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there is a function w satisfying*

$$\left\{\begin{array}{rcll} \Delta_{x,y} w + c\tilde{e} \cdot \nabla_{x,y} w + f(w) & \geq & 0 & \text{in } \Omega \cap B_R \\ w & > & 0 & \text{in } \overline{\Omega} \cap B_R \\ w & = & 0 & \text{on } \overline{\Omega} \cap \partial B_R \\ \partial_\nu w & = & 0 & \text{on } \partial\Omega \cap B_R \\ \|w\|_{L^\infty(\Omega \cap B_R)} & \leq & \varepsilon. & \end{array}\right. \tag{4.2}$$

Proof. Let R be fixed large enough so that the first eigenvalue λ_R of (4.1) satisfies

$$\lambda_R < f'(0) - c^2/4.$$

The latter is possible by Lemma 4.1 and since $|c| < 2\sqrt{f'(0)}$. It then follows that the function $w(x,y) = \varepsilon e^{-ce\cdot x/2}\psi_R(x,y)$ satisfies

$$\Delta w + c\tilde{e} \cdot \nabla w + f(w) = f(\varepsilon e^{-ce\cdot x/2}\psi_R(x,y)) - \left(\frac{c^2}{4} + \lambda_R\right) \varepsilon e^{-ce\cdot x/2}\psi_R(x,y) \geq 0 \ \text{ in } \Omega \cap B_R$$

for $\varepsilon > 0$ small engouh. On the other hand, the function w is positive on $\overline{\Omega} \cap B_R$, vanishes on $\overline{\Omega} \cap \partial B_R$, has small $L^\infty(\Omega \cap B_R)$ norm for ε small enough, and it satisfies the Neumann boundary condition $\partial_\nu w = 0$ on $\partial\Omega \cap B_R$ because so does ψ_R and Ω is invariant in the direction $\tilde{e}$. That completes the proof of Lemma 4.2. □

Proof of Proposition 1.14. First of all, one can assume without loss of generality that $u \not\equiv 0$, whence the strong maximum principle yields $u > 0$ in $\overline{\Omega}$. If Ω is bounded (this corresponds to the case $d = 0$), then the minimum m of u in $\overline{\Omega}$ is reached and positive, and, since g is positive in $(0, 1)$, the strong maximum principle and Hopf lemma yield $m \geq 1$. Similarly, since g is negative in $(1, +\infty)$, the maximum M of u satisfies $M \leq 1$. Therefore, $u \equiv 1$.

Let us now consider the general case of a domain Ω which is unbounded, i.e. $d \geq 1$. Two cases may occur :

Case 1 : $m = \inf_{\overline{\Omega}} u > 0$. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence of points in $\overline{\Omega}$ such that $u(x_n, y_n) \to m$ as $n \to +\infty$. If m is reached, then the points (x_n, y_n) may be assumed to be bounded. In the general case, there exist some points $\tilde{x}_n \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}$ such that

$(x_n-\tilde{x}_n,y_n)\in\overline{C}$ (remember that C is the cell of periodicity of Ω). Up to extraction of some subsequence, one can assume that $(x_n-\tilde{x}_n,y_n)\to(\overline{x},\overline{y})\in\overline{C}$ as $n\to+\infty$.

Call $u_n(x,y)=u(x+\tilde{x}_n,y)$. The functions u_n are defined in Ω, by L-periodicity of Ω, and they satisfy the same equation (1.24) as u. From standard elliptic estimates and Sobolev injections, the functions u_n converge, up to extraction of some subsequence, in $C^{2,\delta}_{loc}(\overline{\Omega})$ (for all $0\le\delta<1$) to a function u_∞ solving (1.24). Furthermore, $u_\infty\ge m$ and $u_\infty(\overline{x},\overline{y})=m$. If $m<1$, then $g(m)>0$ and one gets a contradiction with the strong maximum principle and Hopf lemma. Therefore, $m\ge 1$. Similarly, one can prove that $M=\sup_{\overline{\Omega}}u\le 1$. Hence, $u\equiv 1$.

<u>Case 2</u> : $m=\inf_{\overline{\Omega}}u=0$. Remember that $(e_i)_{1\le i\le d}$ denotes the canonic basis of $\mathbb{R}^d$. Since Ω satisfies (1.1), the function $(x,y)\mapsto u(x+L_1e_1,y)$ is defined in $\overline{\Omega}$. On the other hand, Harnack type inequalities imply that the function

$$v(x,y)=\frac{u(x+L_1e_1,y)}{u(x,y)}$$

is globally bounded in $\overline{\Omega}$. Call $M_1=\limsup_{u(x,y)\to 0,\ (x,y)\in\overline{\Omega}}v$ and let $(x_n,y_n)\in\overline{\Omega}$ be such that $u(x_n,y_n)\to 0$ and $v(x_n,y_n)\to M_1$ as $n\to+\infty$. Let $\tilde{x}_n\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}$ be such that $(x_n-\tilde{x}_n,y_n)\in\overline{C}$. Up to extraction of some subsequence, one can assume that $(x_n-\tilde{x}_n,y_n)\to(\overline{x},\overline{y})\in\overline{C}$ as $n\to+\infty$.

For each $n\in\mathbb{N}$, let u_n be the function defined in $\overline{\Omega}$ by

$$u_n(x,y)=\frac{u(x+\tilde{x}_n,y)}{u(x_n,y_n)}.$$

From Harnack inequalities, the functions u_n are locally bounded in $\overline{\Omega}$. On the other hand, the functions $(x,y)\mapsto u(x+\tilde{x}_n,y)$ satisfy the same equation as u and $u(x_n,y_n)\to 0$ as $n\to+\infty$. From standard elliptic estimates, the functions u_n converge in $C^{2,\delta}_{loc}(\overline{\Omega})$ (for all $0\le\alpha<1$), up to extraction of some subsequence, to a nonnegative function u_∞ solving

$$\left\{\begin{array}{rcl}\Delta u_\infty+b\cdot\nabla u_\infty+g'(0)u_\infty&=&0\text{ in }\Omega\\ \partial_\nu u_\infty&=&0\text{ on }\partial\Omega.\end{array}\right.$$

Furthermore, $u_\infty(\overline{x},\overline{y})=1$, whence $u_\infty>0$ in $\overline{\Omega}$ from the strong maximum principle. Owing to the definitions of M_1 and of the sequence (x_n,y_n), one has that $0<u_\infty(x+L_1e_1,y)/u_\infty(x,y)\le M_1$ and $u_\infty(\overline{x}+L_1e_1,\overline{y})/u_\infty(\overline{x},\overline{y})=M_1$. Notice then that $M_1>0$ since u_∞ is positive in $\overline{\Omega}$.

The function

$$\xi(x,y):=\frac{u_\infty(x+L_1e_1,y)}{u_\infty(x,y)}$$

satisfies

$$\left\{\begin{array}{rcl}\Delta\xi+2\dfrac{\nabla u_\infty}{u_\infty}\cdot\nabla\xi+b\cdot\nabla\xi&=&0\text{ in }\Omega\\ \partial_\nu\xi&=&0\text{ on }\partial\Omega,\end{array}\right.$$

together with $\xi\le M_1$ in $\overline{\Omega}$ and equality at $(\overline{x},\overline{y})$. It follows from the strong maximum principle that $\xi\equiv M_1$ in $\overline{\Omega}$, whence $u_\infty(x+L_1e_1,y)\equiv M_1u_\infty(x,y)$.

In other words, calling $\alpha_1 = (\ln M_1)/L_1$, the function $\varphi_1(x,y) := e^{-\alpha_1 x_1} u_\infty(x,y)$ is positive in $\overline{\Omega}$ and it satisfies

$$\left\{\begin{array}{rcl} \Delta\varphi_1 + 2\alpha_1\partial_{x_1}\varphi_1 + \alpha_1^2\varphi_1 + b\cdot\nabla\varphi_1 + b_1\alpha_1\varphi_1 + g'(0)\varphi_1 & = & 0 \text{ in } \Omega \\ \partial_\nu\varphi_1 + \alpha_1\nu\cdot e_1\ \varphi_1 & = & 0 \text{ on } \partial\Omega \\ \varphi_1(x+L_1e_1,y) & = & \varphi_1(x,y) \text{ in } \Omega, \end{array}\right.$$

where $b = (b_1, \cdots, b_N)$.

Notice now that the function $\varphi_1(x+L_2e_2,y)/\varphi_1(x,y)$ is globally bounded in $\overline{\Omega}$, once again from Harnack type inequalities. Then call

$$M_2 := \sup_{(x,y)\in\overline{\Omega}} \frac{\varphi_1(x+L_2e_2,y)}{\varphi_1(x,y)}$$

and do the same procedure as before, and so on d times. One then gets the existence of a positive L-periodic function φ in $\overline{\Omega}$ satisfying

$$\left\{\begin{array}{rcl} \Delta\varphi + 2\alpha\cdot\nabla_x\varphi + |\alpha|^2\varphi + b\cdot\nabla\varphi + b\cdot\tilde{\alpha}\ \varphi + g'(0)\varphi & = & 0 \text{ in } \Omega \\ \partial_\nu\varphi + \nu\cdot\alpha\ \varphi & = & 0 \text{ on } \partial\Omega, \end{array}\right.$$

for some $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$, where $|\alpha|^2 = \alpha_1^2 + \cdots + \alpha_d^2$, $\tilde{\alpha} = (\alpha_1, \cdots, \alpha_d, 0, \cdots, 0) \in \mathbb{R}^N$ and ∇ means the gradient with respect to both (x,y) variables.

Divide the above equation by φ and integrate by parts over the cell C. By periodicity, it follows that

$$\int_C \left\{ \frac{|\nabla\varphi|^2}{\varphi^2} + 2\alpha\cdot\frac{\nabla_x\varphi}{\varphi} + |\alpha|^2 + b\cdot\frac{\nabla\varphi}{\varphi} + b\cdot\tilde{\alpha} + g'(0) \right\} = 0.$$

In other words,

$$\int_C \left\{ \left|\frac{\nabla\varphi}{\varphi} + \tilde{\alpha} + \frac{b}{2}\right|^2 + \left(g'(0) - \frac{|b|^2}{4}\right) \right\} = 0.$$

One then gets a contradiction with the assumption $|b| < 2\sqrt{g'(0)}$.

As a conclusion, case 2 is ruled out, whence $\inf_{\overline{\Omega}} u > 0$ and $u \equiv 1$. □

Remark 4.3 *In the case where $\Omega = \mathbb{R}^N$, the above proof can be slightly simplified. Indeed, using Lemma 4.2, there is $R > 0$ and a function w such that $w > 0$ in B_R, $w = 0$ on ∂B_R, $w < u$ in $\overline{B_R}$ and $\Delta w + c\tilde{e}\cdot\nabla w + f(w) \geq 0$ in B_R. Since the equation (1.24) satisfied by u is invariant by translation, and since $u > 0$ in $\mathbb{R}^N$, one can slide u in any direction and prove that, for all $x_0 \in \mathbb{R}^N$, $w < u(\cdot + x_0)$ in $\overline{B_R}$. Therefore, $\inf_{\mathbb{R}^N} u > 0$ and one concludes as in Case 1 of the above proof that $u \equiv 1$.*

This result in the case $\Omega = \mathbb{R}^N$ has been known since the paper of Aronson and Weinberger [2], who used parabolic tools. The above arguments actually provide a simpler proof using elliptic arguments.

Let us now turn to the

Proof of Theorem 1.13. As already emphasized, it only remains to prove that, if Ω is a straight cylinder in the direction $\tilde{e}$, then $w^*(e) = 2\sqrt{f'(0)}$ for any nonnegative continuous and compactly supported initial condition $u_0 \not\equiv 0$.

Let u_0 be such a function. Since $w^*(e) \le c^*(e) \le 2\sqrt{f'(0)}$, one only has to prove that $w^*(e) \ge 2\sqrt{f'(0)}$.

Let $0 \le c < 2\sqrt{f'(0)}$ and let us actually prove that $u(t, x + ct\, e, y) \to 1$ as $t \to +\infty$ locally in $(x, y) \in \overline{\Omega}$. Let us mention that, since Ω is invariant in the direction $\tilde{e}$, the functions $(x, y) \mapsto v(t, x, y) = u(t, x + ct\, e, y)$ is defined in $\overline{\Omega}$ for all $t \ge 0$. The function v actually satisfies

$$v_t = \Delta v + c\tilde{e} \cdot \nabla v + f(v)$$

for all $t > 0$ and $(x, y) \in \overline{\Omega}$, together with Neumann boundary conditions on $\partial\Omega$, and initial condition u_0. Notice that $v(t, x, y) > 0$ for all $t > 0$ and $(x, y) \in \overline{\Omega}$ from the strong maximum principle.

Call $M = \sup_{\mathbb{R}^N} u_0$. One has $0 < M < +\infty$ by assumption. Let $\xi(t)$ be the function solving $\dot{\xi} = f(\xi)$ and $\xi(0) = M$. From the assumptions on f, one has $\xi(t) \to 1$ as $t \to +\infty$. Furthermore, $v(t, x, y) \le \xi(t)$ for all $t \ge 0$ and $(x, y) \in \overline{\Omega}$ by the parabolic maximum principle. Therefore,

$$\limsup_{t\to+\infty} \sup_{(x,y)\in\overline{\Omega}} v(t, x, y) \le 1. \tag{4.3}$$

On the other hand, from Lemma 4.2, there exists $R > 0$ (large enough so that $B_R \cap \Omega \ne \emptyset$) and a function w solving (4.2) and such that, say, $v(1, x, y) \ge w(x, y)$ for all $(x, y) \in \overline{B_R} \cap \overline{\Omega}$ (remember that $\min_{(x,y)\in\overline{B_R}\cap\overline{\Omega}} v(1, x, y) > 0$). Let $\tilde{w}$ be the function defined in $\overline{\Omega}$ by $\tilde{w} = w$ in $B_R \cap \overline{\Omega}$ and $\tilde{w} = 0$ in $\overline{\Omega} \backslash B_R$. The function $\tilde{w}$ is then a subsolution for the equation satisfied by v. Therefore, the function θ solving

$$\begin{cases} \theta_t &= \Delta\theta + c\tilde{e} \cdot \nabla\theta + f(\theta),\ t > 0,\ (x, y) \in \Omega, \\ \partial_\nu\theta &= 0,\ t > 0,\ (x, y) \in \partial\Omega, \\ \theta(0, x, y) &= \tilde{w}(x, y), \end{cases}$$

is nondecreasing with respect to t and, since $\tilde{w} \le v(1, \cdot, \cdot)$ in $\overline{\Omega}$, the function θ satisfies

$$\forall t \ge 0,\ \forall (x, y) \in \overline{\Omega},\quad \theta(t, x, y) \le v(t + 1, x, y). \tag{4.4}$$

Without loss of generality, one can assume that $\tilde{w} \le 1$ in $\overline{\Omega}$, whence $\theta(t, x, y) \le 1$ for all $(t, x, y) \in \mathbb{R}^+ \times \overline{\Omega}$. By monotonicity in t, the function $\theta(t, x, y)$ converges as $t \to +\infty$ to a function $\psi(x, y)$. From standard parabolic estimates, the convergence $\theta(t, x, y) \to \psi(x, y)$ as $t \to +\infty$ holds locally uniformly in $\overline{\Omega}$ and the function ψ is a classical solution of (1.24) with $g = f$, $b = c\tilde{e}$ and $|b| < 2\sqrt{f'(0)}$. Furthermore, $0 \le \tilde{w} \le \psi \le 1$ in $\overline{\Omega}$ (thus, $\psi \not\equiv 0$ since $w > 0$ in $B_R \cap \Omega \ne \emptyset$). Proposition 1.14 yields $\psi \equiv 1$.

One deduces from (4.4) that $\liminf_{t\to+\infty} \min_{(x,y)\in K} v(t, x, y) \ge 1$ for all compact set $K \subset \overline{\Omega}$.

One concludes from (4.3) that $v(t, x, y) \to 1$ as $t \to +\infty$ locally in $(x, y) \in \overline{\Omega}$. That completes the proof of Theorem 1.13. □

Remark 4.4 *If Ω is invariant in the direction $\tilde{e}$ and satisfies (1.1), then $u(t, x+ct\,e, y) \to 1$ as $t \to +\infty$ locally in $(x,y) \in \overline{\Omega}$ for $0 \le c < 2\sqrt{f'(0)}$ and for all solution u of (1.15) with continuous bounded nonnegative initial condition $u_0 \not\equiv 0$. The latter indeed holds from the maximum principle even if u_0 is not compactly supported. Furthermore, under the additional assumption that u_0 is compactly supported, then $u(t, x+ct\,e, y) \to 0$ as $t \to +\infty$ locally in $(x,y) \in \overline{\Omega}$ for all $c > 2\sqrt{f'(0)}$.*

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