Front propagation in periodic excitable media

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Abstract. This paper is devoted to the study of *pulsating* travelling fronts for reaction-diffusion-advection equations in a general class of periodic domains with underlying periodic diffusion and velocity fields. Such fronts move in some arbitrarily given direction with an unknown *effective* speed. The notion of pulsating travelling fronts generalizes that of travelling fronts for planar or shear flows. Various existence, uniqueness and monotonicity results are proved for two classes of reaction terms. Firstly, for a combustion-type nonlinearity, it is proved that the pulsating travelling front exists and that its speed is unique. Moreover, the front is increasing with respect to the time variable and unique up to translation in time. We also consider one class of monostable nonlinearity which arises either in combustion or biological models. Then, the set of possible speeds is a semi-infinite interval, closed and bounded from below. For each possible speed, there exists a pulsating travelling front which is increasing in time. This result extends the classical Kolmogorov-Petrovsky-Piskunov case. Our study covers in particular the case of flows in all of space with periodic advections such as periodic shear flows or a periodic array of vortical cells. These results are also obtained for cylinders with oscillating boundaries or domains with a periodic array of holes.

Résumé. Cet article concerne l'étude des solutions du type fronts progressifs *pulsatoires* pour des équations d'advection-diffusion-réaction dans une classe générale de domaines périodiques avec des coefficients d'advection et de diffusion périodiques. Ces fronts se propagent dans une direction arbitraire avec une certaine vitesse *effective* (inconnue). La notion de fronts progressifs pulsatoires généralise celle de fronts progressifs dans les flots uniformes le long de la direction de propagation (écoulements plans ou parallèles). Divers résultats d'existence, d'unicité et de monotonie sont établis pour deux classes de termes de réaction. La première est celle des non-linéarités de type combustion; on prouve que les fronts progressifs pulsatoires existent, que leur vitesse est unique et que ces fronts sont strictement croissants par rapport au temps et uniques à translation en temps près. Nous envisageons également les non-linéarités de type monostable intervenant aussi bien en combustion qu'en biologie. On prouve que l'ensemble des vitesses solutions est alors une demi-droite fermée bornée inférieurement et que pour chaque vitesse possible, il existe un front progressif pulsatoire croissant en temps. Ce dernier résultat étend le cas classique Kolmogorov-Petrovsky-Piskunov. Notre étude couvre en particulier le cas des flots dans tout l'espace avec des advections périodiques telles que les

flots parallèles ou un réseau périodique de cellules verticales. Ces résultats sont également obtenus pour des cylindres à bords périodiques ou des domaines avec trous périodiques.

1 Introduction and main results

This paper is devoted to the analysis of some front propagation phenomena for a class of reaction-diffusion-advection equations in various periodic domains.

In order to motivate our study, let us first recall in a very simple case the notion of travelling fronts : for the reaction-diffusion (with no advection) equation $\partial_t u - \Delta u = f(u)$ in all of space $\mathbb{R}^N$, a (planar) travelling front moving in an arbitrarily given direction $-e$ of the unit sphere S^{N-1} is a solution of the type $u(t,x) = \phi(x \cdot e + ct)$. For reaction-diffusion equations with periodic advection, of the type $\partial_t u + q(x) \cdot \nabla u - \Delta u = f(u)$, travelling fronts may not exist in general. In such *periodic* domains or media, the notion of travelling fronts has to be replaced by the more genral notion of *pulsating* (or periodic) travelling fronts (which we define precisely later) : a pulsating travelling front propagates in an arbitrarily given direction but its profile changes periodically with respect to time instead of being invariant as for a travelling front.

Pulsating travelling fronts appear in several physical contexts. As we will see, they may propagate in a variety of classes of periodic domains and media. These different geometrical configurations and the models are described in the next subsections. The precise mathematical results are also stated for each case. These results are all similar and can actually be stated in the unifying framework of a general class of periodic media and domains, which is described in section 1.5.

1.1 Pulsating travelling fronts in straight infinite cylinders

Let us first deal with the case of a straight infinite cylinder

$$\Omega = \{(x,y),\ x \in \mathbb{R},\ y \in \omega\}$$

where ω is a smooth (at least of class C^3) bounded and connected subset of $\mathbb{R}^{N-1}$ and let us consider the solutions $u(t,x,y)$ of the following reaction-diffusion-advection equation

$$\frac{\partial u}{\partial t} - \Delta u + q(x,y) \cdot \nabla_{x,y} u = f(u), \quad t \in \mathbb{R},\ (x,y) \in \Omega \tag{1.1}$$

together with Neumann boundary conditions on $\partial\Omega$

$$\partial_\nu u = 0, \quad (t,x,y) \in \mathbb{R} \times \partial\Omega \tag{1.2}$$

where $\nu = \nu(x,y) = \nu(y)$ is the outward unit normal to $\partial\Omega = \mathbb{R} \times \partial\omega$ and $\partial_\nu u = \frac{\partial u}{\partial \nu}$. These Neumann boundary conditions mean that there is no flux of u across the wall of the cylinder.

Such semilinear parabolic equations arise in particular in the modelling of thermodiffusive premixed flame propagation with unit a Lewis number and a simple chemistry. Then for a specific reaction term f, u represents an adimensionalized temperature. We refer the reader to [13], [72], [81], [101], [109] for a derivation and physical discussion of these equations.

The same equations also come from biological models of population dynamics where u stands for the relative concentration of some substance (*see e.g.* [2], [34], [84]).

The underlying velocity field $q(x,y) = (q_1(x,y),\cdots,q_N(x,y))$ is given in $\overline{\Omega}$, bounded in $C^{1,\delta}(\overline{\Omega})$ for some $\delta > 0$. It is assumed that

$$\left\{\begin{array}{rcl} \text{div } q & = & 0 \text{ in } \overline{\Omega}, \\ \forall (x,y) \in \overline{\Omega}, \quad q(x+L,y) & = & q(x,y), \\ \displaystyle\int_{(0,L)\times\omega} q_1(x,y)\, dx\, dy & = & 0, \\ q\cdot\nu & = & 0 \text{ on } \partial\Omega. \end{array}\right. \tag{1.3}$$

The second assertion in (1.3) means that q is periodic in the x variable with the period L being some given positive number. Furthermore, we assume that the field q is divergence-free (which corresponds to an incompressibility assumption for the underlying medium). The flow q may represent some turbulent fluctuations with respect to a mean field.

One of the goals of this paper is to analyze the influence of periodic advection, and of other periodic phenomena, on the propagation of flames. Related questions, including the analysis of flame extinction phenomema, have been treated in [3], [20], [27], [58], [108]. In dimension $N \geq 2$, equation (1.1) can then arise in turbulent combustion models to describe the propagation of a premixed flame in an array of vortical cells. Generally speaking, equation (1.1) is a transport equation for a passive quantity u in a periodic *excitable* medium.

We are interested here in particular solutions of (1.1-1.2), namely the pulsating travelling fronts, which propagate in a given direction, say to the left. There are only two possible directions of propagation in an infinite cylinder with bounded section : towards negative or positive x.

Definition 1.1 *A pulsating travelling front solution $u(t,x,y)$ of (1.1-1.2), propagating to the left, is a classical solution u defined for all $t \in I\!R$, $(x,y) \in \overline{\Omega}$, such that for some $c \neq 0$, the following holds*

$$u\left(t+\frac{L}{c},x,y\right) = u(t,x+L,y) \quad \textit{for all} \quad t \in I\!R,\ (x,y) \in \overline{\Omega}, \tag{1.4}$$

and such that

$$\forall t \in I\!R, \quad u(t,-\infty,y) = 0, \quad u(t,+\infty,y) = 1 \quad \textit{uniformly with respect to } y. \tag{1.5}$$

The number c is then called the effective speed of the front. Such a solution u is said to be classical if u is continuous, u is globally bounded in $C^{1,\mu}(I\!R \times \overline{\Omega})$ for all $\mu \in [0,1)$, and the derivatives of u with respect to the (x,y) variables up to the second order exist and are continuous in $I\!R \times \Omega$.

Such pulsating fronts correspond to flames with time-periodic shapes in combustion theory. Pulsating fronts can also be observed in other frameworks. For instance, some remarquable experiments carried out by P. Ronney and collaborators (*see* [90] and references therein) have shown the propagation of pulsating autocatalytic chemical waves in vertical cylinders with annular sections. More generally speaking, such fronts are of particular interest since, in periodic media, they can describe the behavior at large time of the solutions of the related Cauchy problem with front-like initial conditions. However, the question of the stability of the pulsating solutions constructed here is not treated in this paper. We hope to consider it in subsequent works.

Two main types of nonlinear reaction terms f are considered in this paper. The given function f is assumed to be Lipschitz-continuous in $[0,1]$ and to be of one of the following types. Either f satisfies

$$\left\{\begin{array}{l}\exists\theta\in(0,1),\ f(s)=0 \text{ for all } s\in[0,\theta],\ f(s)>0 \text{ for all } s\in(\theta,1),\ f(1)=0,\\ \exists\rho\in(0,1-\theta),\quad f \text{ is nonincreasing on } [1-\rho,1],\end{array}\right. \tag{1.6}$$

or it satisfies :

$$\left\{\begin{array}{l}f>0 \ \text{ on } (0,1),\quad f(0)=f(1)=0,\\ \exists\rho>0,\quad f \text{ is nonincreasing on } [1-\rho,1],\\ \exists\delta>0,\ \text{the restriction of } f \text{ to } [0,1] \text{ is } C^{1,\delta}([0,1]).\end{array}\right. \tag{1.7}$$

We refer to the class defined by (1.6) as the *combustion type* nonlinearity where $\theta>0$ is a threshold (or ignition) temperature at which the reaction starts (*see* [61]). The nonlinearities defined by (1.7) are classicaly referred to as "monostable" reaction terms. Case (1.7) can also be viewed as a combustion nonlinearity with ignition temperature equal to 0 [61], or can also be thought of as the production rate of a population in biological models [2], [36], [67], in which case the quantity u represents the density of a population. Furthermore, f is assumed to be extended by 0 outside $[0,1]$.

For such reaction-diffusion-advection equations, the first results on travelling fronts, namely the solutions u of the type $u(t,x,y)=\phi(x+ct,y)$, have been obtained in the one-dimensional case, with a zero velocity field $q=0$, in the celebrated pioneering paper of Kolmogorov, Petrovsky and Piskunov [67] for nonlinearities f of type (1.7) and later by Kanel' [61] for nonlinearities of type (1.6). These results have been generalized in the multidimensional case of straight infinite cylinders $\Omega=\mathbb{R}\times\omega$ with shear flows $q=(\alpha(y),0,\cdots,0)$ by Berestycki, Larrouturou, Lions [14] (existence for (1.6)) and by Berestycki, Nirenberg [18] (monotonicity, uniqueness for (1.6) and existence for (1.7)). The known results for travelling fronts in shear flows are the following : if f is of type (1.6), there exists a unique speed c and a unique travelling front $u(t,x,y)=\phi(x+ct,y)$ (ϕ is increasing in $s=x+ct$ and unique up to translation in s) whereas if f is of type (1.7), there exists a speed c^* such that travelling fronts $u(t,x,y)=\phi(x+ct,y)$ exist if and only if $c\geq c^*$ and, for each given $c\geq c^*$, the front ϕ is increasing and unique up to translation in s under the additional assumption $f'(0)>0$.

In the case of shear flows, the velocity field q is L-periodic in x for all period L and the equation (1.1) is invariant by translation in the variable x. Thus, travelling fronts solutions, which satisfy (1.4) for all $L\in\mathbb{R}$, are a particular class of pulsating travelling fronts. Such fronts move with constant instantaneous speed c to the left and their profile does not change as time runs (as opposed to the pulsating travelling fronts). Note that the problem for travelling fronts can then be reduced to a semilinear elliptic equation for the function ϕ.

Our first result is to generalize the above results for pulsating travelling fronts in straight infinite cylinders with periodic velocity fields q :

Theorem 1.2 *Let q be a velocity field satisfying (1.3).*

1) If f satisfies (1.6), then there exists a unique classical solution (c, u) of (1.1)-(1.2) and (1.4)-(1.5). The function u is increasing in t and unique up to translation in t. Moreover, $0 < u < 1$ and $c > 0$.

2) If f satisfies (1.7), there exists a positive real number c^* such that : if $c < c^*$, there is no classical solution (c, u) of (1.1)-(1.2) and (1.4)-(1.5). For all $c \geq c^*$, there exists a classical solution (c, u), such that $0 < u < 1$ and u is increasing in t; if $f'(0) > 0$ and $c \geq c^*$, then any solution u of (1.1)-(1.2) and (1.4)-(1.5) is increasing in t.

Remark 1.3 In the case of a function f satisfying (1.7) and under the additional assumption $f'(0) > 0$, we conjecture that, for each speed $c \geq c^*$, the solution u is unique up to translation in t. This question remains open.

Remark 1.4 As is easily seen and observed on computations, contrarily to the case of shear flows, the function u will not be increasing in general in the variable x.

We refer to [28], [29] and [65] for some recent *a priori* bounds of the speeds of propagation of the solutions of the associated Cauchy problem with front-like initial conditions. Constantin, Kiselev, Oberman and Ryzhik [28] have defined the notion of bulk burning rate as follows : $V(t) = |\omega|^{-1} \int_{\mathbb{R}\times\omega} u_t(t, x, y)\ dxdy$ where $|\omega|$ is the Lebesgue-measure of ω. By analyzing a decomposition of the velocity field q into positive and negative parts, they have obtained some lower bounds for $V(t)$ –or for the time-average of $V(t)$– in the case where u is a solution of the corresponding Cauchy problem with front-like initial conditions [28], [65]. These estimates lead to lower bounds for the effective speed c (defined here) of any pulsating travelling front solving (1.1-1.2) and (1.4-1.5). Indeed, for such a solution u, one has $T^{-1} \int_{t_0}^{t_0+T} V(t) dt = c$ with $T = L/c$, for any $t_0 \in \mathbb{R}$. Kiselev and Ryzhik [66] have also recently proved an upper bound for the behavior at large time of this bulk burning rate for thermodiffusive systems of two equations. To be more precise, under some assumptions at the initial time, the burning rate for a system of two equations is asymptotically smaller than the minimal speed of propagation of the pulsating travelling fronts for the corresponding single equation.

1.2 Cylinder type domains with periodic boundaries

The periodicity of the velocity field can actually derive directly from the periodicity of the domain. That is the case when, instead of a straight infinite cylinder, one considers an infinite cylinder Ω with a smooth and oscillating boundary :

$$\Omega = \{(x, y) \in \mathbb{R}^N,\ x \in \mathbb{R},\ y \in \omega(x)\}, \tag{1.8}$$

where the function $x \mapsto \omega(x)$ is periodic with period $L > 0$. Straight infinite cylinders correspond to the case where $\omega = constant$. Let now q be a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) velocity field satisfying

$$\begin{cases} \text{div } q &= 0 \text{ in } \overline{\Omega}, \\ \forall (x,y) \in \overline{\Omega}, \quad q(x+L,y) &= q(x,y), \\ \displaystyle\int_{\{x\in(0,L),\ y\in\omega(x)\}} q_1(x,y)\,dx\,dy &= 0, \\ q\cdot\nu &= 0 \text{ on } \partial\Omega. \end{cases} \tag{1.9}$$

Note here that, as soon as $x \mapsto \omega(x)$ is not constant, the usual notion of travelling front is not sufficient to describe the propagation of a front, even if the velocity field q is equal to 0. On the other hand, pulsating travelling fronts can be defined in this framework. Namely, the notion of pulsating travelling fronts is the same as in Definition 1.1.

In the case where f is of the "bistable" type and where $q = 0$, some conditions for the existence or non-existence of pulsating travelling fronts have been given by Matano [80].

In the cases where f is of the types (1.6) or (1.7), the same result as Theorem 1.2 holds for infinite cylinders with periodic boundaries :

Theorem 1.5 *Under the assumptions (1.8) and (1.9), the conclusions 1) and 2) of Theorem 1.2 hold.*

1.3 Fronts in the whole space with periodic flows

A natural question about pulsating travelling fronts concerns the case where the domain Ω is the whole space $\mathbb{R}^N$. Let us consider the reaction-diffusion-advection equation

$$\frac{\partial u}{\partial t} - \Delta u + q(x)\cdot\nabla_x u = f(u), \quad t \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{1.10}$$

If the velocity field q in (1.10) is equal to a constant vector q_0, then planar travelling fronts of the type $u(t,x) = \phi(x\cdot e + ct)$, propagating in a given direction $-e \in S^{N-1}$, exist in both cases (1.6) or (1.7), and the set of possible speeds is equal to the set of planar speeds for the equation with $q \equiv 0$, translated with the shift $q_0 \cdot e$.

Similarly, consider the case of a shear flow $q = \alpha(x)e$ where $e\cdot\nabla\alpha = 0$ (*i.e.* $\text{div}(q) = 0$) and α is periodic with respect to the variables orthogonal to e. Let P be the orthogonal projection to the hyperplane orthogonal to e. Wrinkled travelling fronts of the type $u(t,x) = \phi(x\cdot e + ct, P(x))$, moving in the direction $-e$, exist. Planar travelling fronts of the type $u(t,x) = \phi_0(x\cdot e' + c_0 t)$ also exist for any direction $e' \in S^{N-1}$ such that $e' \perp e$; moreover, the couple (c_0, ϕ_0) does not depend on q and is the unique solution of $\phi_0'' - c_0\phi_0' + f(\phi_0) = 0$ with $\phi_0(-\infty) = 0$, $\phi_0(+\infty) = 1$.

On the other hand, it can easily be checked in that case that, provided that the shear flow $q = \alpha(x)e$ is not constant, there exists no travelling front, with nonzero speed, propagating in a direction $-e'$ which is neither equal to $\pm e$ nor perpendicular to e. Indeed, suppose by contradiction that such a travelling front u exists. Even if it means rotating the frame, one can assume that $e = (1, 0, \cdots, 0)$. Therefore, the function u solves $u_t - \Delta u + \alpha(y)u_{x_1} = f(u)$ where $y = (x_2, \cdots, x_N)$. Let $P\ : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the orthogonal projection on the hyperplane perpendicular to e'. Since u is a travelling front in the direction $-e'$, there exists a function $\phi\ : \mathbb{R}^N \to \mathbb{R}$ such that $u(t,x) = \phi(x\cdot e' + ct, P(x))$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and $\phi(s,z) \to 0$

(resp. $\phi(s,z)\to 1$) as $s\to-\infty$ (resp. $s\to+\infty$) uniformly in $z\in\mathbb{R}^{N-1}$. Then, the function ϕ satisfies

$$\begin{array}{l} c\phi_1(x\cdot e'+ct,P(x))-\Delta\phi(x\cdot e'+ct,P(x)) \\ +\alpha(y)\ [\ e'_1\phi_1(x\cdot e'+ct,P(x))+b\cdot\nabla_{2,\cdots,N}\phi(x\cdot e'+ct,P(x))\ ] \ =\ f(\phi(x\cdot e'+ct,P(x))) \end{array}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, where e'_1 denotes the first component of e' and $b\in\mathbb{R}^{N-1}$ denotes the (constant) vector $b=\partial_{x_1}P$. Since e' is not parallel to $(1,0,\cdots,0)$, it follows that, for each $(y,s,z)\in\mathbb{R}^{N-1}\times\mathbb{R}\times\mathbb{R}^{N-1}$, there exists $x_1\in\mathbb{R}$ and $t\in\mathbb{R}$ such that $x=(x_1,y)$ satisfies $x\cdot e'+ct=s$ and $P(x)=z$. Therefore, the function $\phi=\phi(s,z)$ satisfies

$$c\phi_s-\Delta_{s,z}\phi+\alpha(y)(e'_1\phi_s+b\cdot\nabla_z\phi)=f(\phi)\ \text{ for all } (s,z)\in\mathbb{R}^N \text{ and for all } y\in\mathbb{R}^{N-1}.$$

Since α is not constant, one has $e'_1\phi_s+b\cdot\nabla_z\phi\equiv 0$. In other words, $u_{x_1}\equiv 0$. Moreover, since e' is not orthogonal to $(1,0,\cdots,0)$, e'_1 is not zero. Assume first that $e'_1>0$. It follows then from the definition of ϕ that, for each $t\in\mathbb{R}$ and $y\in\mathbb{R}^{N-1}$, $u(t,x_1,y)=\phi(x\cdot e'+ct,P(x))\to 0$ as $x_1\to-\infty$ while $u(t,x_1,y)\to 1$ as $x_1\to+\infty$. That contradicts the fact that u does not depend on x_1. The case $e'_1<0$ leads to a similar contradiction.

This example shows that, even for shear flows, the notion of travelling fronts is not sufficient to describe the propagation of fronts in most of the directions of S^{N-1}.

Let us now consider the case of a divergence-free velocity field q, of class $C^{1,\delta}(\mathbb{R}^N)$ (with $\delta>0$), which is L-periodic with respect to the space variables, in the sense that there exists an N-uple $(L_i)\in(\mathbb{R}^*_+)^N$ such that

$$\left\{\begin{array}{rcl} \text{div}\ q & = & 0\ \text{ in } \mathbb{R}^N, \\ \forall k\in\prod_{i=1}^N L_i\mathbb{Z},\quad \forall x\in\mathbb{R}^N,\quad q(x+k) & = & q(x), \\ \int_{\prod_{i=1}^N(0,L_i)} q(x)\ dx & = & 0. \end{array}\right. \tag{1.11}$$

Definition 1.6 *Under the above assumption (1.11), a pulsating travelling front solution $u(t,x)$ of (1.10), propagating in an arbitrarily given direction $-e\in S^{N-1}$, is a solution u defined for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^N$ such that, for some $c\neq 0$, the following holds :*

$$\left\{\begin{array}{rcl} \forall k\in\prod_{i=1}^N L_i\mathbb{Z},\quad \forall x\in\mathbb{R}^N,\quad u\left(t+\dfrac{k\cdot e}{c},x\right) & = & u(t,x+k), \\ \forall t\in\mathbb{R},\quad u(t,x)\underset{x\cdot e\to-\infty}{\longrightarrow} 0,\quad u(t,x) & \underset{x\cdot e\to+\infty}{\longrightarrow} & 1, \end{array}\right. \tag{1.12}$$

where the above limits hold locally in t and uniformly with respect to the variables orthogonal to e. The number c is then called the effective speed of the front.

Remark 1.7 If it is just assumed that the first assertion in (1.12) holds and that the limits in the second assertion are satisfied locally in t and in the variable orthogonal to e, then it still follows that these limits are actually uniform in the variable orthogonal to e. Note that this uniformity condition means that the front can be viewed as *asymptotically* planar far away from any arbitrary origin in $\mathbb{R}^N$. This uniformity condition is also discussed in Remark 1.9 below.

The questions of the existence and uniqueness of pulsating travelling fronts have been solved by Xin [102], [104] in the case of a function f satisfying (1.6) (positive ignition temperature θ), under the additional assumption that $f'(1) < 0$: for each given unit vector $e \in S^{N-1}$, there exists a unique solution $u(t,x)$ of (1.10) and (1.12), and u is increasing in t and unique up to translation in t. This result, which actually holds for more general equations involving space-dependent diffusion terms ([104], *see* also section 1.5) has been proved through a continuation method based on some invertibility properties of linearized operators around solutions in the variables $(ct + x \cdot e, x)$.

The method used by Xin does not seem to easily extend to the case of a nonlinearity f satisfying (1.7), whereas the method used in the present paper allows for the following result, which is similar to Theorems 1.2 and 1.5 :

Theorem 1.8 *Let q be a $C^{1,\delta}(\mathbb{R}^N)$ (with $\delta > 0$) velocity field satisfying (1.11) and let $e \in S^{N-1}$ be a unit vector of $\mathbb{R}^N$. If f is of the type (1.6), then there exists a unique solution $(c,u) = (c(e), u(e))$ of (1.10) and (1.12), the function u being increasing in t and unique up to translation in t. If f is of the type (1.7), then there exists $c^* = c^*(e) > 0$ such that no solution (c,u) exists if $c < c^*$, and a time-increasing solution u exists for each $c \geq c^*$; lastly, all solutions u are increasing in t if $f'(0) > 0$.*

Remark 1.9 The uniformity of the limits in (1.12) with respect to the variables which are orthogonal to e is necessary for the uniqueness result in case (1.6) to hold. If this uniformity condition is not satisfied, then the uniqueness of the speed and of the front (up to translation in t) may not hold anymore. In order to see that, it is enough to consider the case of the equation

$$\partial_t u - \Delta u = f(u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N \tag{1.13}$$

and to consider the travelling fronts moving in direction $-e_N = (0, \cdots, 0, -1)$. If f is of the type (1.6), a planar travelling front $u_0(t,x) = \phi_0(x_N + c_0 t)$ (in the sense of Definition 1.6, where the k can be any arbitrary vector in $\mathbb{R}^N$) exists, and the speed c_0 and the profile ϕ_0 are unique. Such a front satisfies $u_0(t,x',x_N) \to 0$ as $x_N \to -\infty$ and $u_0(t,x',x_N) \to 1$ as $x_N \to +\infty$ locally in t and uniformly in $x' \in \mathbb{R}^{N-1}$. If these uniform limits with respect to x' are replaced with the simple limits, then there exist many other planar fronts $u(t,x)$ which satisfy (1.13) and simple limits as $x' \to \pm\infty$: for instance, if $e \in S^{N-1}$ is such that its N-th component σ is positive, the function $u(t,x) = \phi_0(x \cdot e + c_0 t)$ solves (1.13) and $u(t,x',x_N) \to 0$ (resp. $\to 1$) as $x_N \to -\infty$ (resp. $x_N \to +\infty$) locally in t and in $x' \in \mathbb{R}^{N-1}$. But these limits are not uniform in x', unless $e = e_N$. Furthermore, this solution could be viewed as a planar front moving in direction $-e_N$ with speed $c = c_0/\sigma > 0$ in the sense that

$$\forall t \in \mathbb{R}, \quad \forall (x',x_N) \in \mathbb{R}^N, \quad \forall h \in \mathbb{R}, \quad u(t + h/c, x', x_N) = u(t, x', x_N + h). \tag{1.14}$$

Note also that even for this homogeneous equation (1.13), *nonplanar* fronts satisfying (1.14) and the simple limits $u(t,x',x_N) \to 0$ (resp. $\to 1$) as $x_N \to -\infty$ (resp. $x_N \to +\infty$) locally in t and $x' \in \mathbb{R}^{N-1}$, exist. *Conical-shaped* fronts have been built in [22] and [48] in the case where f satisfies (1.6), and fronts with more general shapes have been given in [49] in the case where f satisfies (1.7) and is moreover concave in $[0,1]$.

1.4 Periodic domains with holes

Another class of periodic domains and media is the case where the domains have periodic holes with a velocity field having the same periodicity. For instance, consider first the case of the whole space with periodic holes; namely, let Ω be a domain with a smooth boundary and such that

$$\forall k \in \prod_{i=1}^{N} L_i \mathbb{Z}, \qquad \Omega + k = \Omega \tag{1.15}$$

for some $(L_i)_{1 \leq i \leq N} \in (\mathbb{R}_+^*)^N$. Let $\nu = \nu(x)$ be the outward unit normal to Ω. Let q be a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) velocity field such that

$$\begin{cases} \text{div } q = 0 \text{ in } \overline{\Omega}, \\ \forall k \in \prod_{i=1}^{N} L_i \mathbb{Z}, \quad \forall x \in \overline{\Omega}, \quad q(x+k) = q(x), \\ \displaystyle\int_{\prod_{i=1}^{N}(0,L_i) \,\cap\, \Omega} q(x)\, dx = 0, \\ q \cdot \nu = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.16}$$

Definition 1.10 *Given any direction $e \in S^{N-1}$, a pulsating travelling front in the direction $-e$ is a solution $u(t,x)$ defined for all $t \in \mathbb{R}$ and $x \in \overline{\Omega}$, and such that, for some $c \neq 0$, the following holds :*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + q(x) \cdot \nabla_x u = f(u), \quad t \in \mathbb{R}, \; x \in \Omega, \\ \partial_\nu u = 0, \quad t \in \mathbb{R}, \; x \in \partial\Omega, \\ \forall k \in \prod_{i=1}^{N} L_i \mathbb{Z}, \quad \forall x \in \overline{\Omega}, \quad u\left(t + \dfrac{k \cdot e}{c}, x\right) = u(t, x+k), \\ \forall t \in \mathbb{R}, \quad u(t,x) \underset{x \cdot e \to -\infty}{\longrightarrow} 0, \quad u(t,x) \underset{x \cdot e \to +\infty}{\longrightarrow} 1, \end{cases} \tag{1.17}$$

where the above limits hold locally in t and uniformly in the directions orthogonal to e. The speed c is then called the effective speed of the front.

For a nonlinearity f satisfying (1.6), the existence of pulsating travelling fronts has been proved by Heinze [52] in the limit of asymptotically small holes, by using a perturbation technique around the homogenized equation.

With the method used in this paper, the same result as in the case of the whole space can be obtained for the case of the space with periodic holes :

Theorem 1.11 *Let Ω be a domain satisfying (1.15) and let q satisfy (1.16). Then the same results as in Theorem 1.8 hold as far as the pulsating travelling fronts solving (1.17) are concerned.*

1.5 General periodic domains and periodic excitable media

The results presented in the previous subsections can all be written in a more general framework which we whall describe below.

Let Ω be a connected unbounded open set, with a smooth boundary (at least of class C^3), and such that

$$\left\{\begin{array}{l}\exists 1\le d\le N,\ \exists L_1,\cdots,L_d>0,\ \forall k=(k_i)_{1\le i\le d}\in\prod_{i=1}^{d}L_i\mathbb{Z},\qquad \Omega+\sum_{i=1}^{d}k_ie_i=\Omega\\ \text{and } \Omega \text{ is bounded with respect to the variables } x_{d+1},\cdots,x_N,\end{array}\right. \tag{1.18}$$

where $(e_i)_{1\le i\le N}$ is the canonical basis of $\mathbb{R}^N$. Let us denote by $x=(x_1,\cdots,x_d)$ the first d coordinates and by $y=(x_{d+1},\cdots,x_N)$ the last $N-d$ ones. Let $\nu=\nu(x,y)$ be the outward unit normal to Ω. Let C be the period cell defined by

$$C=\{(x,y)\in\Omega,\ x\in(0,L_1)\times\cdots\times(0,L_d)\}.$$

We say that a field $v(x,y)$ defined in $\overline{\Omega}$ is L-periodic with respect to the variable x if $v(x+k,y)=v(x,y)$ for all $k\in\prod_{i=1}^{d}L_i\mathbb{Z}$ and for all $(x,y)\in\overline{\Omega}$.

Before going any further, let us observe that that class of domains contains all domains described above : the infinite cylinders with straight or oscillating boundaries, the whole space with or without periodic holes. Domains of the class (1.18) also include infinite cylinders or slabs with periodic holes.

From now on and throughout the paper, $q=(q_1,\cdots,q_N)$ denotes a globally $C^{1,\delta}$ vector field defined in $\overline{\Omega}$, where $\delta>0$.

In some results below, one will assume that

$$\left\{\begin{array}{rcl}\text{div } q&=&0 \text{ in } \overline{\Omega},\\ q \text{ is } L\text{-periodic w.r.t. } x,&&\\ \forall 1\le i\le d,\ \displaystyle\int_C q_i\,dx\,dy&=&0,\\ q\cdot\nu&=&0 \text{ on } \partial\Omega,\end{array}\right. \tag{1.19}$$

or that q satisfies the following weaker assumption :

$$\left\{\begin{array}{rcl}\text{div } q&=&0 \text{ in } \overline{\Omega},\\ \forall 1\le i\le d,\ \ q_i \text{ is } L\text{-periodic w.r.t. } x,&&\\ \forall 1\le i\le d,\ \displaystyle\int_C q_i\,dx\,dy&=&0,\\ q\cdot\nu&=&0 \text{ on } \partial\Omega.\end{array}\right. \tag{1.20}$$

In (1.20), only the first d components of q are L-periodic with respect to x.

Furthermore, throughout the paper, $A(x,y)=(A_{ij}(x,y))_{1\le i,j\le N}$ denotes a globally C^3 matrix field defined in $\overline{\Omega}$, and such that

$$\exists 0<c_1\le c_2,\quad \forall\xi\in\mathbb{R}^N,\quad \forall(x,y)\in\overline{\Omega},\quad c_1|\xi|^2\le\sum_{1\le i,j\le N}A_{ij}(x,y)\xi_i\xi_j\le c_2|\xi|^2, \tag{1.21}$$

where, for any $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$, $|\xi|^2 = \xi_1^2 + \cdots + \xi_N^2$. In some results below, one will assume that

$$A \text{ is symmetric and } L\text{-periodic w.r.t. } x, \tag{1.22}$$

or that A satisfies the following weaker assumption :

$$\forall 1 \le i \le d,\ \forall 1 \le j \le N, \quad A_{ij} \text{ is } L\text{-periodic w.r.t. } x. \tag{1.23}$$

Lastly, let $f(x, y, u)$ be a function defined in $\overline{\Omega} \times \mathbb{R}$ such that

$$\begin{cases} f \text{ is globally Lipschitz-continuous in } \overline{\Omega} \times \mathbb{R}, \\ \forall (x,y) \in \overline{\Omega}, \quad \forall s \in (-\infty, 0] \cup [1, +\infty), \quad f(x,y,s) = 0, \\ \exists \rho \in (0,1), \quad \forall (x,y) \in \overline{\Omega}, \quad \forall 1-\rho \le s \le s' \le 1, \quad f(x,y,s) \ge f(x,y,s'). \end{cases} \tag{1.24}$$

One assumes that

$$f \text{ is } L\text{-periodic w.r.t. } x. \tag{1.25}$$

The function f is assumed to be of one of the following two types : either

$$\begin{cases} \exists \theta \in (0,1), \quad \forall (x,y,s) \in \overline{\Omega} \times [0,\theta], \quad f(x,y,s) = 0, \\ \forall s \in (\theta, 1), \quad \exists (x,y) \in \overline{\Omega}, \quad f(x,y,s) > 0, \\ \exists \rho \in (0, 1-\theta), \quad \forall (x,y) \in \overline{\Omega}, \quad \forall 1-\rho \le s \le s' < 1, \quad f(x,y,s) \ge f(x,y,s') > 0, \end{cases} \tag{1.26}$$

or

$$\begin{cases} \forall s \in (0,1), \quad \exists (x,y) \in \overline{\Omega}, \quad f(x,y,s) > 0, \\ \exists \delta > 0, \text{ the restriction of } f \text{ to } \overline{\Omega} \times [0,1] \text{ is } C^{1,\delta} \text{ with respect to } u. \end{cases} \tag{1.27}$$

Nonlinear source terms of the types (1.26) or (1.27) generalize those of the types (1.6) or (1.7). Typical examples of functions $f(x, y, u)$ satisfying (1.24-1.25) and either (1.26) or (1.27) are the functions of the type $f(x, y, u) = h(x, y)\tilde{f}(u)$, where h is a globally Lipschitz-continuous, positive, bounded and L-periodic w.r.t. x function defined in $\overline{\Omega}$, and $\tilde{f}$ is a Lipschitz-continuous function satisfying (1.6) or (1.7).

Throughout the paper, if z and z' are two vectors in $\mathbb{R}^N$ and B is an $N \times N$-matrix, then zBz' denotes the number $zBz' := \sum_{1 \le i,j \le N} z_i B_{ij} z'_j$.

Definition 1.12 *Let e be an arbitrarily given unit vector in $\mathbb{R}^d$. We are interested in the functions $u(t, x, y)$, which we call* pulsating travelling fronts *propagating in direction $-e$ with a so-called effective speed $c \neq 0$, and which are classical solutions of*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) + q \cdot \nabla u = f(x,y,u), \quad t \in \mathbb{R},\ (x,y) \in \Omega, \\ \nu A \nabla u = 0, \quad t \in \mathbb{R},\ (x,y) \in \partial\Omega, \\ \forall k \in \prod_{i=1}^{d} L_i \mathbb{Z}, \quad u\left(t + \dfrac{k \cdot e}{c}, x, y\right) = u(t, x+k, y) \text{ for all } (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \\ u(t,x,y) \underset{x \cdot e \to -\infty}{\longrightarrow} 0, \quad u(t,x,y) \underset{x \cdot e \to +\infty}{\longrightarrow} 1 \text{ for each } (t,y), \end{cases} \tag{1.28}$$

where the above limits hold locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ orthogonal to e.

That framework for the propagation of pulsating travelling fronts contains all situations described in the previous subsections. Note here that the Laplace operator has been replaced with a general nonhomogenous diffusion operator $\mathrm{div}(A\nabla u)$. Such diffusion operators have also been considered in several papers in the onedimensional case or in the case of the whole space (*see* [87], [103], [104], [105], [107]), and also in similar problems modelling the propagation of fronts in periodic solid media (*see* [24], [86]).

For that general framework, we have the following results, which generalize the results of Theorems 1.2, 1.5, 1.8 and 1.11 :

Theorem 1.13 *Let Ω be a domain satisfying (1.18), let e be a unit vector in $I\!R^d$ and let f be a nonlinearity satisfying (1.24-1.25) and (1.26). Let q (resp. A) be a globally $C^{1,\delta}(\overline{\Omega})$ (resp. $C^3(\overline{\Omega})$) vector field (resp. matrix field), where $\delta > 0$, and assume that A satisfies (1.21). Then,*

a) if q and A satisfy (1.19) and (1.22), there exists a classical solution $(c, u) = (c(e), u(e))$ of (1.28) such that $c(e) > 0$,

b) without any additional assumption, for any classical solution (c, u) of (1.28), the speed c is unique, the function cu is increasing in t and the function u is unique up to translation in the variable t,

c) if q and A satisfy (1.20) and (1.23), then, for any classical solution (c, u) of (1.28), the speed c is positive and the function u is increasing in t.

Theorem 1.14 *Let Ω be a domain satisfying (1.18), let e be a unit vector in $I\!R^d$ and let f be a nonlinearity satisfying (1.24-1.25) and (1.27). Let q (resp. A) be a globally $C^{1,\delta}(\overline{\Omega})$ (resp. $C^3(\overline{\Omega})$) vector field (resp. matrix field), where $\delta > 0$, and assume that (1.19), (1.21) and (1.22) are satisfied. Then,*

a) there exists $c^*(e) > 0$ such that problem (1.28) has no solution (c, u) if $c < c^*(e)$ while, for each $c \geq c^*(e)$, it has a solution (c, u) such that u is increasing in t,

b) if $f_u^+(x, y, 0) := \lim_{u\to 0^+} f(x, y, u)/u > 0$ for all $(x, y) \in \overline{\Omega}$, then any solution u of (1.28) is increasing in t.

Remark 1.15 Theorems 1.2, 1.5, 1.8 and 1.11 hold in the general case where the Laplace operator is replaced with a divergence type operator $\mathrm{div}(A\nabla u)$ together with Neumann type boundary conditions $\nu A\nabla u = 0$ on $\partial\Omega$. These theorems also hold when the source term $f(u)$ is replaced with a function $f(x, y, u)$ satisfying (1.24-1.25) and either (1.26) or (1.27).

At this stage, the question of the uniqueness of the pulsating travelling fronts for each speed $c \geq c^*$, in the case where f satisfies (1.27), remains open, even under the assumption $f_u^+(x, y, 0) > 0$.

Another related open problem concerns the case where the function f is of the bistable type, namely, there exists $\theta \in (0, 1)$ such that $f(0) = f(\theta) = f(1)$, $f < 0$ on $(0, \theta)$, $f > 0$ on $(\theta, 1)$ and f is nonincreasing in a right neighborhood of 0 and in a left neighborhood of 1. Some conditions for the existence or nonexistence of pulsating travelling fronts in infinite cylinders with periodic boundary have been given by Matano [80]. Other existence, nonexistence or stability results have been obtained by Xin [103], [105] and Papanicolaou and Xin [87] in the case of the whole space with almost uniform diffusion and advection coefficients.

Lastly, let us mention here that the methods used in sections 3 and 4 of this paper to prove the uniqueness and monotonicity properties of the pulsating travelling fronts in the case of a nonlinearity f with positive ignition temperature (1.26) actually work and lead to the same uniqueness and monotonicity results in the case of a bistable nonlinearity f, or for more general bistable-like nonlinearities $f(x,y,u)$ which are nonincreasing in u in a right neighbourhood of 0 and in a left neighbourhood of 1, uniformly with respect to $(x,y)\in\overline{\Omega}$.

1.6 Further results

In a forthcoming paper [12] written in collaboration with N. Nadirashvili, the question of the propagation of pulsating travelling waves solving (1.28) in a domain of the class (1.18) is considered under the additional assumption that the function f satisfies the following assumption, which is a particular case of (1.27) :

$$\left\{\begin{array}{l} f \text{ is globally Lipschitz-continuous in } \overline{\Omega}\times I\!\!R \text{ and } L\text{-periodic with respect to } x,\\ \forall (x,y)\in\overline{\Omega},\quad \forall s\in(-\infty,0]\cup[1,+\infty),\quad f(x,y,s)=0,\\ \exists\delta>0,\ \text{the restriction of } f \text{ to } \overline{\Omega}\times[0,1] \text{ is of class } C^{1,\delta} \text{ with respect to } u,\\ \forall (x,y,s)\in\overline{\Omega}\times(0,1),\quad 0<f(x,y,s)\le f_u^+(x,y,0)s\\ \qquad\qquad\qquad \text{where } f_u^+(x,y,0):=\lim_{u\to0^+}f(x,y,u)/u,\\ \exists\rho\in(0,1),\quad \forall(x,y)\in\overline{\Omega},\quad \forall 1-\rho\le s\le s'\le 1,\quad f(x,y,s)\ge f(x,y,s'). \end{array}\right. \tag{1.29}$$

The simplest case of a function $f(x,y,u)$ satisfying (1.29) is when $f(x,y,u)=\tilde{f}(u)$ and the Lipschitz-continuous function $\tilde{f}$ satisfies : $\tilde{f}=0$ outside $(0,1)$, $\tilde{f}_u^+(0)=\lim_{u\to0^+}\tilde{f}(u)/u>0$, $0<\tilde{f}(s)\le\tilde{f}'(0)s$ in $(0,1)$ and $\tilde{f}$ is nonincreasing in a left neighbourhood of 1. This last case corresponds to the nonlinearity considered in the classical paper of Kolmogorov, Petrovsky and Piskunov [67] and it arises especially in biological models [2], [36], [84] (the quantity u then represents the density of a population).

From Theorem 1.14, under the assumptions (1.19), (1.21) and (1.22), for each given unit direction e of $I\!\!R^d$, there exists a minimal speed $c^*(e)$ for the pulsating travelling fronts in the sense of Definition 1.12. One of the goals of the paper [12] is to find an explicit formula for the minimal speed $c^*(e)$.

We have obtained the following three equivalent variational formulas for $c^*(e)$:

$$c^*(e)=\min\left\{c,\ \exists\lambda>0,\ \mu_{c,\zeta}(\lambda)=0\right\} \tag{1.30}$$

where $\zeta(x,y):=f_u^+(x,y,0)$ and $\mu_{c,\zeta}(\lambda)$ is the principal eigenvalue of the elliptic operator

$$-L_{c,\lambda,\zeta}\psi:=-\text{div}(A\nabla\psi)-\lambda[\text{div}(A\tilde{e}\ \psi)+\tilde{e}A\nabla\psi]+q\cdot\nabla\psi+(\lambda q\cdot\tilde{e}+\lambda c-\lambda^2\tilde{e}A\tilde{e}-\zeta)\psi$$

acting on the set E of L-periodic with respect to x functions $\psi(x,y)$ such that $\nu A(\tilde{e}\lambda\psi+\nabla\psi)=0$ on $\partial\Omega$. Here, $\tilde{e}$ denotes the vector $\tilde{e}=(e_1,\cdots,e_d,0,\cdots,0)$. Thus, under the KPP assumption (1.29), the minimal speed $c^*(e)$ can be explicitely given in terms of e, the domain Ω, the coefficients q and A and of $f_u^+(\cdot,\cdot,0)$. In the general case where f satisfies (1.27) and $f_u^+(\cdot,\cdot,0)>0$, it actually follows from section 6.4 (*see* Lemma 5.5) that the minimal speed $c^*(e)$ is always greater than or equal to the right hand side of (1.30). Note also that the formula (1.30) is similar to that of Berestycki and Nirenberg [18] for travelling waves in infinite cylinders with shear flows and nonlinear source terms $f(u)$ which do not depend on (x, y).

As observed by Xin in [107] for pulsating fronts in $I\!R^N$, the above formula (1.30) is equivalent to the following one :

$$c^*(e) = \min_{\lambda>0} \frac{-k_\zeta(\lambda)}{\lambda} \tag{1.31}$$

where $k_\zeta(\lambda)$ is the principal eigenvalue of the operator

$$-\tilde{L}_{\lambda,\zeta}\psi := -\text{div}(A\nabla\psi) - \lambda[\text{div}(A\tilde{e}\ \psi) + \tilde{e}A\nabla\psi] + q \cdot \nabla\psi + (\lambda q \cdot \tilde{e} - \lambda^2 \tilde{e}A\tilde{e} - \zeta)\psi$$

on the same set E of functions ψ as above. Note that the formula (1.31) is similar to those of Gärtner and Freidlin [41] or Majda and Souganidis [75] for the asymptotic speed of propagation of solutions of Cauchy problem in $I\!R^N$ with periodic coefficients and compactly supported initial conditions (see [12] for a further study of the asymptotic speeds of propagation). Note also that when $\Omega = I\!R^N$, $A = I$, $q = 0$ and $f = f(u)$ (with $f(u) \leq f'(0)u$ in $[0, 1]$), this formula (1.31) gives the well-known KPP formula $c^*(e) = 2\sqrt{f'(0)}$ for the minimal speed of planar fronts. Let us also mention here that a formula similar to (1.31) for a nonlinear source term $f(u)$ of the KPP type (1.29) has recently been obtained by Schwetlick for a similar *hyperbolic* transport equation [94].

Lastly, the following equivalent formula also holds

$$c^*(e) = \min_{\lambda>0} \min_{\psi\in F} \max_{(x,y)\in\overline{\Omega}} \frac{\tilde{L}_{\lambda,\zeta}\psi}{\lambda\psi} \tag{1.32}$$

where $F = \{\psi \in E, \psi \in C^2(\overline{\Omega}),\ \psi > 0 \text{ in } \overline{\Omega}\}$. This formula is obtained from (1.31) and from some characterizations of principal eigenvalues of elliptic operators [19], [88].

Remark 1.16 Formula (1.32) for the minimal speed of multidimensional pulsating fronts is similar to that of Hudson and Zinner [55] for the minimal speed of pulsating travelling fronts in the case of one-dimensional equations of the type $u_t = u_{xx} + f(x, u)$, where f is 1-periodic in x, $f(x, u) > 0$ for $u \in (0, \overline{u}(x))$, $f(x, 0) = f(x, \overline{u}(x)) = 0$ and $\mu(x) = f'_u(x, 0) = \sup_{u\in(0,\overline{u}(x))} f(x, u)/u$. Namely, Hudson and Zinner have obtained the following formula for the minimal speed c^* of pulsating travelling fronts moving to the left :

$$c^* = \min_{r>0} \min_{\{\psi=\psi(x)\in C^2(I\!R),\ \psi>0,\ \psi\ 1-\text{periodic}\}} \max_{x\in[0,1]} \frac{\psi'' + 2r\psi' + (r^2 + \mu(x))\psi}{r\psi}. \tag{1.33}$$

1.7 Organization of the paper

The next sections are organized as follows : in section 2, a short overview of related results on travelling fronts for reaction-diffusion-advection equations is done. Sections 3, 4 and 5 are respectively devoted to the proof of the monotonicity properties, of the uniqueness properties and of the existence result of Theorem 1.13 in the case where the function f satisfies (1.26). The monotonicity and uniqueness results are based on a sliding method in another set of variables $(s, x, y) = (x \cdot \tilde{e} + ct, x, y)$, where the equation is *elliptic degenerate*, and on the parabolic maximum principle in the original variables (t,x,y) (remember that for travelling fronts with constant speed c, the equation of the profile of the front is *elliptic* in some variables, say $(x+ct,y)$ in the case of an infinite straight cylinder). The existence of a solution (c,u) in Theorem 1.13 is obtained as a limit of solutions of regularized elliptic equations in approximated bounded domains. The main difficulty is to deal with the degeneracy of the equations and to prove that the solution obtained at the limit is not trivial. We especially use some exponentially decaying upper solutions in some semi-infinite domains and solve some eigenvalue problems in the cell of periodicity. We also use some Bernstein-type gradient estimates independent of the regularization parameter. These estimates are proved in section 7.

Section 6 is devoted to the proof of Theorem 1.14, in the case where the function f satisfies (1.27). The existence of a solution for the minimal speed $c^*(e)$ is obtained as a limit of solutions for nonlinearities f_θ of the type (1.26) and approximating f (with small ignition temperatures θ). The existence of solutions for any speed $c\geq c^*(e)$ is obtained through a method using sub- and super-solutions, and the non-existence of solutions with speeds $c<c^*(e)$ follows from a sliding method and from a comparison with suitable sub-solutions.

Although parts of some proofs of the present paper follow the lines of some cited papers, especially [18], they are both more technical because of the generality of the framework which is considered here and more delicate because, roughly speaking, we will have to deal with degenerate elliptic equations, instead of elliptic equations in [18].

2 A brief overview of the related literature

Since this is the first in a series of papers, we indicate here most of the relevant references of works which are related to our program.

2.1 One-dimensional results

The first analyses of the propagation phenomena for reaction-diffusion-advection equations like (1.1) dealt with the study of planar travelling fronts, for one-dimensional equations with zero velocity field. In 1937, Kolmogorov, Petrovsky and Piskunov [67] proved that, for a nonlinearity f of the type (1.7) and such that $f(s)\leq f'(0)s$ for all $s\in[0,1]$, travelling waves of the type $u(t,x)=\phi(x+ct)$ for the equation $u_t=u_{xx}+f(u)$ exist whenever $c\geq c^*=2\sqrt{f'(0)}$. Since this celebrated pioneering paper, there has been a great amount of work on the questions of existence, uniqueness or stability properties of planar travelling fronts for different types of reaction terms $f(u)$ arising in combustion or biological models, see *e.g.* Aronson and Weinberger [2], Fife and McLeod [35], Johnson and Nachbar [57], Kanel' [61].

Many papers have also been devoted to the study of planar travelling fronts for systems of one-dimensional diffusion-reaction equations [9], [16], [21], [30], [33], [39], [62], [79], [83], [89], [99]. The results have shown either some differences from the case of single equations or some analogies. For instance, for some systems of diffusion-reaction equations, the set of possible speeds of propagation may be the same size (a singleton or a semi-infinite interval bounded from below, depending on the nonlinearities) as for single equations.

As far as pulsating one-dimensional travelling fronts are concerned, and as already mentionned in Remark 1.16, Hudson and Zinner [55] have proved the existence of a semi-infinite interval $[c^*, +\infty)$ of possible speeds of pulsating travelling fronts for one-dimensional KPP type diffusion-reaction (with no advection) equation $u_t = u_{xx} + f(x, u)$. The variational formula (1.33, similar to (1.32) has also been obtained for the minimal speed c^*.

2.2 Shear-like flows

The works on *wrinkled* travelling fronts for multidimensional reaction-diffusion-advection equations with heterogeneous coefficients have first been devoted to the case of shear flows in infinite straight cylinders with Neumann boundary conditions (*see* Berestycki, Larrouturou and Lions [14], Berestycki and Nirenberg [18]). Some existence, uniqueness, monotonicity properties similar to the case of planar fronts have been obtained. These results for multidimensional travelling fronts have generalized those for onedimensional fronts and Theorem 1.2 generalizes for the most part these results to the case of periodic plows instead of shear flows.

Similar existence or uniqueness results for travelling fronts in straight infinite cylinders with Dirichlet boundary conditions have been obtained by Gardner [40] and Vega [98].

Papanicolaou and Xin [87] have considered the case of almost uniform velocity fields and they have derived asymptotic expansions (with respect to uniform advection) for the unique speed of propagation, or for the minimal speed in the case of KPP type nonlinearities.

Asymptotic formulas for the speeds in the case of shear flows with large amplitude have been derived by Audoly, Berestycki and Pomeau [4]. Lower bounds for the speeds have been obtained by Constantin, Kiselev, Oberman and Ryzhik [28], and Kiselev and Ryzhik [65] for a more general class of shear-like percolating flows.

Lastly, the cases of monotone shear flows along the main direction x of the cylinder, and of almost shear flows with more general reaction terms have been considered by Hamel [45], [46]. For instance, in the case of a monotone advection, with a nonlinear source term of the type (1.6), the speed of propagation is not unique and the set of possible speeds is in general an interval with nonempty interior.

2.3 Flows with an array of vortical cells and more general flows - cylinders and whole space

As mentionned earlier, several works have been devoted to the study of pulsating travelling fronts in all of space $I\!R^N$ with periodic velocity fields (1.10), either with nonlinearities of the type (1.6) with positive ignition temperature (Xin [102], [104], [107]) or with bistable nonlinearities (Xin [103], [105], see also Nakamura [85] for the one-dimensional case with periodic diffusion coefficient and Alikakos, Bates and Chen [1] for the case of a bistable time-periodic source term). Similar existence results have been obtained for other classes of equations arising in different models [24], [32], [106].

As for shear flows, bounds for the effective speeds of propagation have been derived in the case of flows with vortical cells in straight infinite cylinders or in all of space $I\!R^N$ (*see* [4], [28], [65]).

In the present paper, the width of the front is of the same scale as the advection and the diffusion. This is not the case for velocity fields $q(x/\varepsilon)$ or diffusion matrices $A(x/\varepsilon)$ involving very small scales ε. The homogenization limit $\varepsilon \to 0$ has been carried out by Freidlin [37], [38], Heinze [51] and Xin [107].

On the other hand, the cases of slowly varying flows $q(\varepsilon x)$ or diffusion matrices $A(\varepsilon x)$, for which the width of the front is very small compared to the convecting and diffusing lengths, have been studied by Freidlin [37], Majda and Souganidis [75] and Xin [107]. As the parameter ε goes to 0, such problems lead to Hamilton-Jacobi equations which describe the large-scale and large-time front propagation. Similar Hamilton-Jacobi equations have been obtained from more general turbulent reaction-diffusion-advection equations with Gaussian random velocity fields [76].

The question of front propagation in general random media has also been considered by Freidlin [37], [38] and Xin [107].

Lastly, we refer to the papers of Avellaneda and Majda [5], [6], [7] and Majda and McLaughlin [74] for a study of turbulent advection-diffusion equations (with no reaction) involving statistical velocity fields with arbitrary many spatial and temporal scales. Renormalized equations for averaged passive quantities at large spatio-temporal scales and bounds for the effective diffusivities have been derived. Fannjiang and Papanicolaou [31] have also studied the influence of advection on the effective diffusivity for advection-diffusion equations.

2.4 Stability analysis

Many works have been devoted to the behavior at large time of solutions of Cauchy problems for reaction-diffusion-advection equations like (1.1) under a large class of initial conditions. Many results have especially dealt with the convergence to travelling fronts. These works have been initiated by Kolmogorov, Petrovsky and Piskunov [67] in the one-dimensional case with no advection (see also [2], [23], [35], [59], [60], [70], [82], [93], [96]) and followed by the study of the asymptotic and global stability of travelling waves in infinite cylinders with shear flows, by Berestycki, Larrouturou and Roquejoffre [15], Mallordy and Roquejoffre [78] and Roquejoffre [91], [92].

So far, few papers (*see* [71], [86], [103]) have dealt with the question of the stability of pulsating travelling fronts in periodic media like the real line or the whole space.

2.5 Discrete diffusion

Similar existence results or propagation failure phenomena for the propagation of fronts have also been obtained for nonlinear evolution equations with shear-like flows, for which the Laplace operator is replaced with a discrete diffusion operator. Such equations arise for instance in models of cellular networks. Let us mention the papers of Cahn, Mallet-Paret and Van Vleck [25], Chow, Mallet-Paret and Shen [26], Keener [64], Mallet-Paret [77], Zinner [110] on discrete equations or systems with bistable-type nonlinearities, and that of Harris, Hudson and Zinner [50], [54] for KPP-type positive nonlinearities in the one- or two-dimensional cases.

Let us also mention the work of Weinberger [100] on a general framework including especially the above continuous equations and the time or space discrete equations which are invariant by translation with respect to the space variables. Weinberger has obtained the existence of planar fronts in this general framework as well as a characterization of the asymptotic speed of propagation for initially compactly supported solutions in the case of KPP-type nonlinearities.

2.6 Formulas for the speeds

One of the most important questions related to the front propagation phenomena is the determination of the speed of propagation of the travelling fronts, or of the pulsating travelling fronts in the periodic framework. In the theory of combustion for instance, the determination of the burning velocity of a deflagration flame is a fundamental question.

Many works have been devoted to finding some formulas for the speeds of propagation of travelling waves for reaction-diffusion-advection equations, more general than those arising in combustion models. The first formula comes back to the paper of Kolmogorov, Petrovsky and Piskunov [67] and concerns the minimal speed $c^*=2\sqrt{f'(0)}$ of planar travelling fronts for the equation $u_t=u_{xx}+f(u)$ with nonlinearities $f(u)$ satisfying (1.29).

Other formulas of the variational type have been derived for such one-dimensional equations. Let us for instance mention the formula

$$c^*=\min_{\eta:[0,1]\to I\!R,\ \eta(0)=0,\ \eta'(0)>0,\ \eta>0\text{ in }(0,1]}\ \sup_{u\in(0,1]}\left(\eta'(u)+\frac{f(u)}{\eta(u)}\right)$$

of Hadeler and Rothe [44] for nonlinearities of the type (1.7). The latter implies $2\sqrt{f'(0)}\le c^*\le 2\sqrt{\sup_{(0,1]}f(u)/u}$ and gives $c^*=2\sqrt{f'(0)}$ in the case (1.29). Integral formulations have been given by Benguria and Depassier [8]. Variational formulas have also been obtained for systems of one-dimensional equations (*see* Kan-On [63], Mischaikow and Hutson [83], Takase and Sleeman [95], Volpert, Volpert and Volpert [99]), or for equations with discrete diffusion (Harris, Hudson and Zinner [50]).

Some of those formulas have been generalized in the multidimensional case with shear flows (*see* Hamel [47], Heinze, Papanicolaou and Stevens [53], and Hudson and Zinner [54] for discrete diffusion operators). For instance, in the case (1.6), the unique speed c of travelling fronts $\phi(x+ct,y)$ solving (1.1) in a cylinder $\Omega=I\!R\times\omega$ with a shear flow $q=(\alpha(y),0,\cdots,0)$, is given by

$$c=\min_{w\in\mathcal{E}}\ \sup_{(x_1,y)\in\overline{\Omega}}\left(\frac{\Delta w+f(w)}{\partial_x w}-\alpha(y)\right)=\max_{w\in\mathcal{E}}\ \inf_{(x_1,y)\in\overline{\Omega}}\left(\frac{\Delta w+f(w)}{\partial_x w}-\alpha(y)\right)$$

where $\mathcal{E}=\{w\in W^{2,p}_{loc}(\Omega),\ \Delta w\in C(\overline{\Omega}),\ 0<w<1,\ \partial_x w>0\text{ in }\overline{\Omega},\ \partial_\nu w=0\text{ on }\partial\Omega,\ w(-\infty,\cdot)=0,\ w(+\infty,\cdot)=1\}$ and $p>N$ (*see* [47]). In the case (1.7) with $f'(0)>0$, the minimal speed c^* for travelling fronts is equal to

$$c^*=\min_{w\in\mathcal{E}}\ \sup_{(x_1,y)\in\overline{\Omega}}\left(\frac{\Delta w+f(w)}{\partial_x w}-\alpha(y)\right).$$

Explicit formulas for the speeds of propagation of travelling waves have been obtained in some asymptotic cases, for instance in the limit of high activation energies (*see* Berestycki, Nicolaenko and Scheurer [16], Giovangigli [42] in the one-dimensional case, and Berestycki, Caffarelli and Nirenberg [10] in the multi-dimensional case). Formal asymptotics in the case of shear flows with large amplitude have been derived by Audoly, Berestycki and Pomeau [4].

As already underlined at the end of section 1.1, we also refer to the papers of Constantin, Kiselev, Oberman and Ryzhik [28], [29], [65] and [66] for some *a priori* bounds of the speeds of propagation of the solutions of the Cauchy problem associated to (1.1) with front-like initial conditions. The estimates they have obtained especially lead to some bounds for the effective speeds of propagation of any pulsating travelling fronts solving (1.1-1.2) and (1.4-1.5) in the case of periodic advection.

For pulsating travelling fronts in periodic media, as already said, the only formula -(1.33)- derived by Hudson and Zinner [55], concerns the minimal speed of propagation in the one-dimensional case $u_t = u_{xx} + f(x, u)$ with KPP-type nonlinearities.

Lastly, let us mention the formulas of the type (1.31) obtained by Gärtner and Freidlin [41] for the asymptotic speed of propagation of solutions of Cauchy problems with compactly supported initial conditions, for equations of the type (1.10) in the whole space with periodic coefficients. These formulas have been used by Papanicolaou and Xin [87] in some perturbating cases. A further study of the asymptotic speeds of propagation is done in [12].

3 Case with ignition temperature : monotonicity properties

This section is devoted to the proof of Theorem 1.13, part c), about monotonicity properties of any solution u of (1.28) in the case where f satisfies (1.24-1.25) and (1.26). The proof is based on a sliding method in a new system of coordinates.

Throughout the paper, Ω is a smooth domain satisfying (1.18).

3.1 Change of variables and proof of the positivity of the speed c

In this subsection, $f(x, y, u)$ denotes a globally Lipschitz-continuous function defined in $\overline{\Omega} \times \mathbb{R}$ and satisfying (1.25), namely, f is L-periodic with respect to x. In this subsection, the fields $q(x, y)$ and $A(x, y)$ satisfy (1.20) and (1.23). We recall that throughout the paper q and A are assumed to be respectively globally $C^{1,\delta}(\overline{\Omega})$ and $C^3(\overline{\Omega})$ (with $\delta > 0$), and that A satisfies (1.21).

Let (c, u) be a bounded classical solution of (1.28). Remember that c is assumed to be not zero and that e is a unit vector in $\mathbb{R}^d$. Let $\tilde{e}$ be the vector in $\mathbb{R}^N$ defined by

$$\tilde{e} = (e_1, \cdots, e_d, 0, \cdots, 0).$$

Let us now make the same change of variables as Xin [102], [104]; namely, let $\phi(s, x, y)$ be the function defined by :

$$\phi(s, x, y) = u\left(\frac{s - x \cdot e}{c}, x, y\right) \quad \text{for all } s \in \mathbb{R} \text{ and } (x, y) \in \overline{\Omega}. \tag{3.1}$$

The function ϕ, which is globally bounded in $C^{1,\mu}(\mathbb{R}\times\overline{\Omega})$ for each $\mu\in[0,1)$, satisfies the following degenerate elliptic equation

$$\begin{array}{r}\mathrm{div}_{x,y}(A\nabla_{x,y}\phi)+(\tilde{e}A\tilde{e})\phi_{ss}+\mathrm{div}_{x,y}(A\tilde{e}\phi_s)+\partial_s(\tilde{e}A\nabla_{x,y}\phi)\\ -q\cdot\nabla_{x,y}\phi-(q\cdot\tilde{e}+c)\phi_s+f(x,y,\phi)\ =0\ \text{ in } \mathcal{D}'_L(\mathbb{R}\times\overline{\Omega})\end{array}\tag{3.2}$$

in the sense that, for all $\psi\in\mathcal{D}(\mathbb{R}\times\overline{\Omega})$ L-periodic with respect to x,

$$\begin{array}{l}-\int_{\mathbb{R}\times C}(\nabla_{x,y}\psi)A(\nabla_{x,y}\phi)-\int_{\mathbb{R}\times C}(\tilde{e}A\tilde{e})\phi_s\psi_s\\ -\int_{\mathbb{R}\times C}((\nabla_{x,y}\psi)A\tilde{e})\phi_s-\int_{\mathbb{R}\times C}(\tilde{e}A(\nabla_{x,y}\phi))\psi_s\\ -\int_{\mathbb{R}\times C}(q\cdot\nabla_{x,y}\phi)\psi-\int_{\mathbb{R}\times C}(q\cdot\tilde{e}+c)\phi_s\psi+\int_{\mathbb{R}\times C}f(x,y,\phi)\psi\ =0,\end{array}\tag{3.3}$$

together with boundary and periodicity conditions

$$\left\{\begin{array}{l}\nu A(\tilde{e}\phi_s+\nabla_{x,y}\phi)=0\text{ on }\mathbb{R}\times\partial\Omega\\ \phi\text{ is }L\text{-periodic with respect to }x.\end{array}\right.\tag{3.4}$$

Note that in the case where Ω is a straight infinite cylinder $\Omega=\mathbb{R}\times\omega$ with $e=1$ and A is the identity matrix, then the boundary condition for ϕ on $\mathbb{R}\times\partial\Omega$ reduces to $\partial_\nu\phi=0$.

On the other hand, in the general case, since $u(t,x,y)\to 0$ as $x\cdot e\to-\infty$ and $u(t,x,y)\to 1$ as $x\cdot e\to+\infty$ locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e, and since ϕ is L-periodic with respect to x, the change of variables $s=ct+x\cdot e$ guarantees that

$$\phi(-\infty,\cdot,\cdot)=0,\quad \phi(+\infty,\cdot,\cdot)=1\ \text{ uniformly in } (x,y)\in\overline{\Omega}.\tag{3.5}$$

Notice that the latter holds whatever the sign of c is.

The following lemma answers the question of the sign of c :

Lemma 3.1 *Assume that q and A satisfy (1.20) and (1.23). Assume that f is globally Lipschitz-continuous in $\overline{\Omega}\times\mathbb{R}$, L-periodic with respect to x, nonnegative and not identically equal to* 0. *If (c,u) is a classical solution of (1.28), then $c>0$.*

Proof. Choose any $a>0$ and take in (3.3) a sequence of functions $\psi_n(s,x,y)=\psi_n(s)\in\mathcal{D}(\mathbb{R}\times\overline{\Omega})$ such that $0\le\psi_n\le 1$, $\psi_n=1$ for $|s|\le a$ and $\psi_n=0$ for $|s|\ge a+1/n$. The passage to the limit $n\to+\infty$ leads to

$$\begin{array}{r}\int_C(\tilde{e}A\tilde{e})[\phi_s(\cdot,x,y)]_{-a}^a+\int_C[\tilde{e}A\nabla_{x,y}\phi(\cdot,x,y)]_{-a}^a-\int_{(-a,a)\times C}q\cdot\nabla_{x,y}\phi\\ -\int_C(q\cdot\tilde{e}+c)[\phi(\cdot,x,y)]_{-a}^a+\int_{(-a,a)\times C}f(x,y,\phi)\ =0,\end{array}\tag{3.6}$$

where $[\varphi(\cdot)]_\alpha^\beta=\varphi(\beta)-\varphi(\alpha)$ for any function φ on the interval $[\alpha,\beta]$.

From standard parabolic estimates, the partial derivatives of the function u with respect to (t,x,y) approach 0 as $x\cdot e\to\pm\infty$, locally in (t,y). Because of the L-periodicity of ϕ with respect to x, it follows in particular that ϕ_s and $\nabla_{x,y}\phi\to 0$ as $s\to\pm\infty$, uniformly in $(x,y)\in\overline{\Omega}$. Therefore, the first and second terms in (3.6) approach 0 as $a\to+\infty$. By integration by parts and because of (1.20), the third term $\int_{(-a,a)\times C} q\cdot\nabla_{x,y}\phi$ is actually equal to 0 for all $a>0$. Lastly, since the first d components of q have zero ensemble mean, one gets

$$\int_C (q\cdot\tilde{e}+c)[\phi(\cdot,x,y)]_{-a}^{a}\to c|C|$$

as $a\to+\infty$, where $|C|$ is the Lebesgue measure of the period cell C.

Eventually, one concludes that the function $f(x,y,\phi(s,x,y))$ is integrable over $I\!R\times C$ and that

$$c|C| = \int_{I\!R\times C} f(x,y,\phi(s,x,y))\,ds\,dx\,dy. \tag{3.7}$$

Since the functions f and ϕ are L-periodic with respect to x and since $\phi(-\infty,x,y)=0$, $\phi(+\infty,x,y)=1$ uniformly in $(x,y)\in\overline{\Omega}$, it easily follows from the assumptions of Lemma 3.1 that the function $f(x,y,\phi(s,x,y))$ is continuous, L-periodic with respect to x, nonnegative and not identically equal to 0 in $I\!R\times\overline{C}$. Therefore, $c>0$. □

3.2 Maximum principles

The proofs of the monotonicity and uniqueness properties for c and u use some versions of the maximum principle in unbounded domains for some problems related to (3.2)-(3.5). In this subsection, we state these maximum principles under a slightly more general framework.

Lemma 3.2 *Let $g(x,y,u)$ be a globally bounded and globally Lipschitz-continuous function defined in $\overline{\Omega}\times I\!R$ and assume that g is nonincreasing with respect to u in $\overline{\Omega}\times(-\infty,\delta]$ for some $\delta>0$. Let $h\in I\!R$ and $\Sigma_h^-:=(-\infty,h)\times\Omega$. Let $c\neq 0$ and $\phi^1(s,x,y)$, $\phi^2(s,x,y)$ be two globally $C^{1,\mu}(\overline{\Sigma_h^-})$ functions (for some $\mu>0$) such that*

$$\left\{\begin{array}{rcll} L\,\phi^1+g(x,y,\phi^1) & \geq & 0 & \text{in } \mathcal{D}'(\Sigma_h^-),\\ L\,\phi^2+g(x,y,\phi^2) & \leq & 0 & \text{in } \mathcal{D}'(\Sigma_h^-),\\ \nu A[\tilde{e}(\phi^1_s-\phi^2_s)+\nabla_{x,y}(\phi^1-\phi^2)] & \leq & 0 & \text{on } (-\infty,h]\times\partial\Omega,\\ \lim\limits_{s_0\to-\infty}\ \sup\limits_{\{s\leq s_0,\ (x,y)\in\overline{\Omega}\}}\ [\phi^1(s,x,y)-\phi^2(s,x,y)] & \leq & 0, & \end{array}\right. \tag{3.8}$$

where

$$\begin{array}{rl} L\,\phi := & \mathrm{div}_{x,y}(A\nabla_{x,y}\phi)+(\tilde{e}A\tilde{e})\phi_{ss}+\mathrm{div}_{x,y}(\tilde{e}A\phi_s)+\partial_s(\tilde{e}A\nabla_{x,y}\phi)\\ & -q\cdot\nabla_{x,y}\phi-(q\cdot\tilde{e}+c)\phi_s. \end{array} \tag{3.9}$$

If $\phi^1\leq\delta$ in $\overline{\Sigma_h^-}$ and $\phi^1(h,x,y)\leq\phi^2(h,x,y)$ for all $(x,y)\in\overline{\Omega}$, then

$$\phi^1\leq\phi^2\ \text{ in } \overline{\Sigma_h^-}.$$

Remark 3.3 Note here that ϕ^1, ϕ^2, q, A and g are not assumed to be L-periodic with respect to x and that q is not assumed to satisfy (1.19) or (1.20).

Proof. We use here a method similar to that of Li [73] or Vega [97] for strictly elliptic problems (*see* other applications of this method in [48]).

Since ϕ^1 and ϕ^2 are globally bounded, one has $\phi^1 - \varepsilon \le \phi^2$ in $\overline{\Sigma_h^-}$ for $\varepsilon > 0$ large enough. Let us set

$$\varepsilon^* = \inf \left\{ \varepsilon > 0,\ \phi^1 - \varepsilon \le \phi^2 \ \text{ in } \overline{\Sigma_h^-} \right\} \ \ge 0.$$

By continuity, one has $\phi^1 - \varepsilon^* \le \phi^2$ in $\overline{\Sigma_h^-}$. In order to complete the proof of Lemma 3.2, one then only has to prove that $\varepsilon^* = 0$.

Assume $\varepsilon^* > 0$. Take a sequence $\varepsilon_n \overset{<}{\to} \varepsilon^*$. There exists a sequence of points $(s_n, x_n, y_n) \in \overline{\Sigma_h^-}$ such that

$$\phi^1(s_n, x_n, y_n) - \varepsilon_n \ge \phi^2(s_n, x_n, y_n).$$

Since $\lim_{s_0 \to -\infty} \sup_{\{s \le s_0,\ (x,y) \in \overline{\Omega}\}} \left(\phi^1(s,x,y) - \phi^2(s,x,y)\right) \ \le 0$, the sequence (s_n) is bounded from below. Since it is also bounded from above ($s_n \le h$), one can assume, up to extraction of some subsequence, that $s_n \to \overline{s} \in (-\infty, h]$ as $n \to +\infty$. On the other hand, let $\tilde{x}_n$ be in $\prod_{i=1}^d L_i \mathbb{Z}$ such that $(x_n - \tilde{x}_n, y_n) \in \overline{C}$. Up to extraction of some subsequence, one can also assume that $(x_n - \tilde{x}_n, y_n) \to (\overline{x}, \overline{y}) \in \overline{C}$ as $n \to +\infty$.

Set

$$\phi_n^i(s,x,y) = \phi^i(s, x + \tilde{x}_n, y) \ \text{ for all } (s,x,y) \in \overline{\Sigma_h^-},\ \ i = 1, 2.$$

The above functions are defined in the same set $\overline{\Sigma_h^-} = (-\infty, h] \times \overline{\Omega}$ because of the choice of $\tilde{x}_n$ and the L-periodicity of Ω with respect to x. From the regularity assumptions for ϕ^1 and ϕ^2 and up to extraction of some subsequence, the functions ϕ_n^i converge in C^1_{loc} to two functions $\phi_\infty^i \in C^{1,\mu}(\overline{\Sigma_h^-})$. Similarly, since q and A are globally $C^1(\overline{\Omega})$ (q and A are even in $C^{1,\delta}(\overline{\Omega})$ and $C^3(\overline{\Omega})$ respectively), one can assume that the fields $q_n(x,y) = q(x + \tilde{x}_n, y)$ and $A_n(x,y) = A(x + \tilde{x}_n, y)$ converge locally in $\overline{\Omega}$ to two globally bounded and Lipschitz fields q_∞ and A_∞ as $n \to +\infty$. The matrix field $A_\infty(x,y)$ satisfies the same ellipticity condition (1.21) as A.

The functions ϕ_n^i satisfy

$$L_n \phi_n^1 - L_n \phi_n^2 \ge -g(x + \tilde{x}_n, y, \phi_n^1(s,x,y)) + g(x + \tilde{x}_n, y, \phi_n^2(s,x,y)) \quad \text{in } \mathcal{D}'(\Sigma_h^-)$$

where

$$L_n\, \phi := \text{div}_{x,y}(A_n \nabla_{x,y} \phi) + (\tilde{e} A_n \tilde{e}) \phi_{ss} + \text{div}_{x,y}(\tilde{e} A_n \phi_s) + \partial_s(\tilde{e} A_n \nabla_{x,y} \phi) - q_n \cdot \nabla_{x,y} \phi - (q_n \cdot \tilde{e} + c)\phi_s.$$

Since $\phi^1 \le \delta$ in $\overline{\Sigma_h^-}$ and $g(x,y,u)$ is nonincreasing with respect to u in $\overline{\Omega} \times (-\infty, \delta]$, one gets

$$L_n \phi_n^1 - L_n \phi_n^2 \ge -g(x + \tilde{x}_n, y, \phi_n^1(s,x,y) - \varepsilon^*) + g(x + \tilde{x}_n, y, \phi_n^2(s,x,y)) \quad \text{in } \mathcal{D}'(\Sigma_h^-). \quad (3.10)$$

From the assumptions of Lemma 3.2, one can also assume, up to extraction of some subsequence, that the functions

$$R_n(s,x,y) := -g(x + \tilde{x}_n, y, \phi_n^1(s,x,y) - \varepsilon^*) + g(x + \tilde{x}_n, y, \phi_n^2(s,x,y))$$

converge to a function $R_\infty(s,x,y)$ locally in $\overline{\Sigma_h^-}$. Since $|R_n| \le \|g\|_{Lip} |\phi_n^1 - \varepsilon^* - \phi_n^2|$ for all n, one gets $|R_\infty| \le \|g\|_{Lip} |\phi_\infty^1 - \varepsilon^* - \phi_\infty^2|$ at the limit. In other words, there exists a globally bounded function $B(s,x,y)$ defined in $\overline{\Sigma_h^-}$ such that

$$R_\infty(s,x,y) = B(s,x,y)\ [\phi_\infty^1(s,x,y) - \varepsilon^* - \phi_\infty^2(s,x,y)] \quad \text{for all } (s,x,y) \in \overline{\Sigma_h^-}.$$

By passage to the limit in (3.10), it follows that the functions ϕ^1_∞ and ϕ^2_∞ satisfy

$$L_\infty\phi^1_\infty - L_\infty\phi^2_\infty \geq B(s,x,y)\ (\phi^1_\infty - \varepsilon^* - \phi^2_\infty) \quad \text{in } \mathcal{D}'(\Sigma_h^-)$$

where

$$\begin{aligned} L_\infty\ \phi := \ & \mathrm{div}_{x,y}(A_\infty\nabla_{x,y}\phi) + (\tilde{e}A_\infty\tilde{e})\phi_{ss} + \mathrm{div}_{x,y}(\tilde{e}A_\infty\phi_s) + \partial_s(\tilde{e}A_\infty\nabla_{x,y}\phi) \\ & -q_\infty\cdot\nabla_{x,y}\phi - (q_\infty\cdot\tilde{e}+c)\phi_s. \end{aligned}$$

Moreover, the inequalities $\phi^1_\infty \leq \delta$ in $\overline{\Sigma_h^-}$ and $\phi^1_\infty(h,\cdot,\cdot) \leq \phi^2_\infty(h,\cdot,\cdot)$ in $\overline{\Omega}$ hold as well. Furthermore, $\phi^1_\infty - \varepsilon^* \leq \phi^2_\infty$ in $\overline{\Sigma_h^-}$ and $\phi^1_\infty(\overline{s},\overline{x},\overline{y}) - \varepsilon^* \geq \phi^2_\infty(\overline{s},\overline{x},\overline{y})$ by passage to the limit. Therefore,

$$\phi^1_\infty(\overline{s},\overline{x},\overline{y}) - \varepsilon^* = \phi^2_\infty(\overline{s},\overline{x},\overline{y}),$$

whence $\overline{s} < h$.

Coming back to the variables (t,x,y), let us define

$$E_h = \{(t,x,y) \in I\!R \times \Omega,\ ct + x\cdot e < h\}$$

and set

$$u^i(t,x,y) = \phi^i_\infty(ct + x\cdot e, x, y) \ \text{ for all } (t,x,y) \in \overline{E_h},\ i = 1,2.$$

The function $z := u^1 - \varepsilon^* - u^2$, which is defined and globally C^1 in $\overline{E_h}$, satisfies

$$\mathrm{div}_{x,y}(A_\infty\nabla_{x,y}z) - q_\infty(x,y)\cdot\nabla_{x,y}z - \partial_t z \geq b(t,x,y)z \ \text{ in } \mathcal{D}'(E_h)$$

where the function $b(t,x,y) := B(ct + x\cdot e, x, y)$ is globally bounded in $\overline{E_h}$. Moreover,

$$\nu A_\infty\nabla_{x,y}z \leq 0 \ \text{ on } \{ct + x\cdot e \leq h,\ (x,y) \in \partial\Omega\}.$$

On the other hand, the function z is nonpositive and it vanishes at the point $(\overline{t},\overline{x},\overline{y}) = (\frac{\overline{s}-\overline{x}\cdot e}{c},\overline{x},\overline{y})$, which is such that $c\overline{t} + \overline{x}\cdot e\ (= \overline{s}) < h$. Therefore, it follows from the maximum principle that $z \equiv 0$ in $\overline{E_h}\cap\{t \leq \overline{t}\}$. In other words, $u^1 - \varepsilon^* \equiv u^2$ in $\overline{E_h}\cap\{t \leq \overline{t}\}$. In particular, one has $\phi^1_\infty - \varepsilon^* \equiv \phi^2_\infty$ in $\overline{\Sigma_h^-}\cap\{\frac{s-x\cdot e}{c} \leq \overline{t}\}$. Since the set $\{x\cdot e\}$ is not bounded from above or below, there exists a point $(h,x_0,y_0) \in \overline{\Sigma_h^-}\cap\{\frac{s-x\cdot e}{c} \leq \overline{t}\}$. At that point, one has $\phi^1_\infty(h,x_0,y_0) - \varepsilon^* = \phi^2_\infty(h,x_0,y_0)$. But the latter is impossible because $\phi^1_\infty \leq \phi^2_\infty$ for $s = h$.

Hence, the assumption $\varepsilon^* > 0$ is ruled out and the proof of Lemma 3.2 is complete. □

Changing s into $-s$ in Lemma 3.2 leads to the following

Lemma 3.4 *Let $g(x,y,u)$ be a globally bounded and globally Lipschitz-continuous function defined in $\overline{\Omega}\times I\!R$ and assume that g is nonincreasing with respect to u in $\overline{\Omega}\times[1-\delta,\infty)$ for some $\delta > 0$. Let $h \in I\!R$ and $\Sigma_h^+ := (h,+\infty)\times\Omega$. Let $c \neq 0$ and $\phi^1(s,x,y)$, $\phi^2(s,x,y)$ be two bounded and globally $C^{1,\mu}(\overline{\Sigma_h^+})$ functions (for some $\mu > 0$) such that*

$$\left\{\begin{array}{rcll} L\ \phi^1 + g(x,y,\phi^1) & \geq & 0 & \text{in } \mathcal{D}'(\Sigma_h^+), \\ L\ \phi^2 + g(x,y,\phi^2) & \leq & 0 & \text{in } \mathcal{D}'(\Sigma_h^+), \\ \nu A[\tilde{e}(\phi^1_s - \phi^2_s) + \nabla_{x,y}(\phi^1 - \phi^2)] & \leq & 0 & \text{on } [h,+\infty)\times\partial\Omega, \\ \displaystyle\lim_{s_0\to+\infty}\ \sup_{\{s\geq s_0,\ (x,y)\in\overline{\Omega}\}} [\phi^1(s,x,y) - \phi^2(s,x,y)] & \leq & 0, & \end{array}\right.$$

where L is the same operator as in Lemma 3.2.

If $\phi^2 \geq 1-\delta$ in $\overline{\Sigma_h^+}$ and $\phi^1(h,x,y) \leq \phi^2(h,x,y)$ for all $(x,y) \in \overline{\Omega}$, then

$$\phi^1 \leq \phi^2 \ \text{ in } \overline{\Sigma_h^+}.$$

3.3 Monotonicity in the variable s

This subsection is devoted to the proof of the monotonicity result stated in part c) of Theorem 1.13. We shall use the maximum principles of the previous section (Lemmas 3.2 and 3.4). We also use the sliding method in infinite cylinders developped by Berestycki and Nirenberg [18].

Lemma 3.5 *Let f be a function satisfying (1.24) and (1.26). Let (c,u) be a classical solution of (1.28). Then the function ϕ defined by (3.1) is increasing in s.*

Proof. Remember that ϕ is defined by $\phi(s,x,y) = u((s-x\cdot e)/c,x,y)$. As already underlined, the function ϕ is of class $C^{1,\mu}(I\!\!R\times\overline{\Omega})$ for each $\mu \in [0,1)$. Furthermore, ϕ solves $L\phi + f(\phi) = 0$ in $\mathcal{D}'(I\!\!R\times\Omega)$, where L is defined in (3.9), together with boundary, periodicity and limiting conditions (3.4)-(3.5). For any $\tau \in I\!\!R$, one sets $\phi^\tau(s,x,y) = \phi(s+\tau,x,y)$.

Assume that one has proved that $\phi^\tau \geq \phi$ for all $\tau \geq 0$, and consider first the case $c > 0$. Then, for any $h > 0$, the function $z(t,x,y) = u(t+h,x,y) - u(t,x,y)$ is nonnegative and, since q and A depend on the variables (x,y) only, z is a classical solution of the following linear parabolic equation

$$\partial_t z - \text{div}(A\nabla z) + q\cdot\nabla z + bz = 0 \ \text{ in } I\!\!R\times\Omega$$

for some globally bounded function b, together with boundary conditions $\nu A\nabla z = 0$ on $I\!\!R\times\partial\Omega$. From the strong parabolic maximum principle and the uniqueness of the corresponding Cauchy problem, the function z is either identically 0, or positive everywhere in $I\!\!R\times\overline{\Omega}$. If $z \equiv 0$, then $\phi(s+ch,x,y) = \phi(s,x,y)$ for all $(s,x,y) \in I\!\!R\times\overline{\Omega}$. The latter is impossible because $ch \neq 0$ and because of the limiting conditions (3.5) as $s \to \pm\infty$. Therefore, $z > 0$ for any $h > 0$. Hence, the function ϕ is increasing in s. The case $c < 0$ can be treated similarly.

As a consequence, one only has to prove that $\phi^\tau \geq \phi$ for all $\tau \geq 0$. Let θ and ρ be given as in (1.26). From (3.5), there exists a real $B > 0$ such that $\phi(s,x,y) \leq \theta$ for all $s \leq -B$ and $(x,y) \in \overline{\Omega}$ and $\phi(s,x,y) \geq 1-\rho$ for all $s \geq B$ and $(x,y) \in \overline{\Omega}$. For all $\tau \geq 2B$, the functions ϕ and ϕ^τ satisfy

$$\begin{cases} \phi(s,x,y) \leq \theta & \text{for all } s \leq -B,\ (x,y) \in \overline{\Omega}, \\ \phi^\tau(s,x,y) \geq 1-\rho & \text{for all } s \geq -B,\ (x,y) \in \overline{\Omega}, \\ \phi(-B,x,y) \leq \phi^\tau(-B,x,y) & \text{for all } (x,y) \in \overline{\Omega}. \end{cases}$$

Morover, because problem (3.2)-(3.5) is invariant by translation with respect to the variable s, the function ϕ^τ solves (3.2)-(3.5). Consequently, $\phi^1 := \phi$ and $\phi^2 := \phi^\tau$ fall within the framework of Lemma 3.2 (resp. Lemma 3.4) in Σ_{-B}^- (resp. Σ_{-B}^+). Therefore, it follows that

$$\phi(s,x,y) \leq \phi^\tau(s,x,y) \ \text{ for all } (s,x,y) \in I\!\!R\times\overline{\Omega}.$$

Let us now decrease τ and define

$$\tau^* = \ \inf\ \{\tau > 0,\ \forall\tau' \geq \tau,\ \phi \leq \phi^{\tau'} \text{ in } I\!\!R\times\overline{\Omega}\}.$$

In order to complete the proof of Lemma 3.5, we only have to show that $\tau^* = 0$. Let us argue by contradiction and assume $\tau^* > 0$. By continuity, we see that $\phi \le \phi^{\tau^*}$ in $I\!R \times \overline{\Omega}$. Two cases may occur :

case 1 : suppose that

$$\sup_{[-B,B]\times\overline{\Omega}} (\phi - \phi^{\tau^*}) < 0. \tag{3.11}$$

Since the function ϕ is globally Lipschitz-continuous (as the function u is), there exists a real number $\eta \in (0, \tau^*)$ such that for all $\tau^* - \eta \le \tau \le \tau^*$, one has

$$\phi(s,x,y) < \phi^{\tau}(s,x,y) \ \text{ for all } s \in [-B,B],\ (x,y) \in \overline{\Omega}. \tag{3.12}$$

Choose any $\tau \in [\tau^* - \eta, \tau^*]$. The above formula (3.12) especially yields $\phi(\pm B,x,y) \le \phi^{\tau}(\pm B,x,y)$ for all $(x,y) \in \overline{\Omega}$. On the other hand, since $\phi \ge 1-\rho$ in $[B,+\infty) \times \overline{\Omega}$ and since $\tau \ge \tau^* - \eta > 0$, it follows that

$$\phi^{\tau}(s,x,y) \ge 1-\rho \ \text{ for all } s \ge B,\ (x,y) \in \overline{\Omega}.$$

We may now apply Lemma 3.4 in Σ_B^+, and also Lemma 3.2 in Σ_{-B}^-, for the functions ϕ and ϕ^{τ}. We then infer that $\phi(s,x,y) \le \phi^{\tau}(s,x,y)$ for all $|s| \ge B$ and $(x,y) \in \overline{\Omega}$. Together with (3.12), that yields $\phi \le \phi^{\tau}$ in $I\!R \times \overline{\Omega}$. This is in contradiction with the minimality of τ^*. Hence, (3.11) is ruled out.

case 2 : suppose that

$$\sup_{[-B,B]\times\overline{\Omega}} (\phi - \phi^{\tau^*}) = 0. \tag{3.13}$$

Then there exists a sequence $(s_n, x_n, y_n) \in [-B,B] \times \overline{\Omega}$ such that

$$\phi(s_n,x_n,y_n) - \phi^{\tau^*}(s_n,x_n,y_n) \to 0 \ \text{ as } n \to +\infty.$$

Since ϕ (and ϕ^{τ^*}) are L-periodic with respect to the variable x, one can assume that $(x_n,y_n) \in \overline{C}$. Up to extraction of some subsequence, one can then assume that $(s_n,x_n,y_n) \to (\overline{s},\overline{x},\overline{y}) \in [-B,B] \times \overline{C})$ as $n \to +\infty$. By continuity of ϕ, one gets $\phi(\overline{s},\overline{x},\overline{y}) = \phi^{\tau^*}(\overline{s},\overline{x},\overline{y})$.

Coming back to the variables (t,x,y), the function $z(t,x,y) = \phi(ct + x\cdot e, x, y) - \phi(ct + x \cdot e + \tau^*, x, y)$ is a classical solution of the following linear parabolic equation

$$\begin{cases} \partial_t z = \operatorname{div}(A\nabla z) - q(x,y)\cdot\nabla_{x,y} z + b(t,x,y) z & \text{in } I\!R \times \Omega, \\ \nu A \nabla z = 0 & \text{on } I\!R \times \partial\Omega \end{cases}$$

for some bounded function b. Furthermore, z is nonpositive and z vanishes at the point $((\overline{s}-\overline{x})/c, \overline{x}, \overline{y})$. From the strong parabolic maximum principle and the Hopf lemma, it follows that $z \equiv 0$.

In other words, $\phi(s,x,y) = \phi(s+\tau^*,x,y)$ for all $(s,x,y) \in I\!R \times \overline{\Omega}$. But because of the limiting conditions (3.5) and because $\tau^* > 0$, the function ϕ cannot be τ^*-periodic in the direction s. So case 2 is ruled out too.

Therefore, one has proved that $\tau^* = 0$ and $\phi \le \phi^{\tau}$ for all $\tau \ge 0$. As already underlined in the beginning of the proof of Lemma 3.5, it follows then that ϕ is increasing in s. □

Corollary 3.6 *Let f be a function satisfying (1.24-1.25) and (1.26). Let (c,u) be a classical solution of (1.28). Then the function cu is increasing in t. Furthermore, if q and A satisfy (1.20) and (1.23), then the function u is increasing in t.*

Proof. It immediately follows from Lemmas 3.1 and 3.5 and from the definition of ϕ. □

4 Case with ignition temperature : uniqueness of the speed and of the profile of u

This section is devoted to the proof of the uniqueness results stated in part b) of Theorem 1.13. We prove the following

Lemma 4.1 *Let f be a function satisfying (1.24) and (1.26). If (c^1,u^1) and (c^2,u^2) are two classical solutions of (1.28), then $c^1=c^2$ and there exists a real number h such that $u^1(t,x,y)=u^2(t+h,x,y)$ for all $(t,x,y)\in I\!R\times\overline{\Omega}$.*

Proof. The proof of this lemma uses the sliding method with respect to the variable s and it is similar to that of Lemma 3.5. However, it is done here for the sake of completeness.

Let (c^1,u^1) and (c^2,u^2) be two classical solutions of (1.28). Even if it means changing the superscripts 1 and 2, one can assume that $c^1\geq c^2$. By assumption, the real numbers c^1, c^2 are not zero. One can then define the functions ϕ^1 and ϕ^2 in $I\!R\times\overline{\Omega}$ by :

$$\phi^1(s,x,y)=u^1\left(\frac{s-x\cdot e}{c^1},x,y\right),\quad \phi^2(s,x,y)=u^2\left(\frac{s-x\cdot e}{c^2},x,y\right).$$

The functions ϕ^1 and ϕ^2 satisfy the boundary, periodicity and limiting conditions (3.4)-(3.5), and they are globally $C^{1,\mu}(I\!R\times\overline{\Omega})$ for each $\mu\in[0,1)$. The function ϕ^1 is a solution of

$$\begin{array}{ll}\mathrm{div}_{x,y}(A\nabla_{x,y}\phi^1)+(\tilde{e}A\tilde{e})\phi^1_{ss}+\mathrm{div}_{x,y}(A\tilde{e}\phi^1_s)+\partial_s(\tilde{e}A\nabla_{x,y}\phi^1) & \\ \quad -q\cdot\nabla_{x,y}\phi^1-(q\cdot\tilde{e}+c^1)\phi^1_s+f(x,y,\phi^1) & =0 \text{ in } \mathcal{D}'(I\!R\times\Omega).\end{array}\tag{4.1}$$

On the other hand, from Lemma 3.5, the function ϕ^2 is increasing in s and it satisfies

$$\begin{array}{ll}\mathrm{div}_{x,y}(A\nabla_{x,y}\phi^2)+(\tilde{e}A\tilde{e})\phi^2_{ss}+\mathrm{div}_{x,y}(A\tilde{e}\phi^2_s)+\partial_s(\tilde{e}A\nabla_{x,y}\phi^2) & \\ \quad -q\cdot\nabla_{x,y}\phi^2-(q\cdot\tilde{e}+c^1)\phi^2_s+f(x,y,\phi^2) & =(c^2-c^1)\ \phi^2_s \\ & \leq 0 \text{ in } \mathcal{D}'(I\!R\times\Omega).\end{array}\tag{4.2}$$

Notice that the latter holds for each function of the type $\phi^{2,\tau}(s,x,y):=\phi^2(s+\tau,x,y)$ because of the invariance of (4.2) by translation with respect to s and because the velocity field q and the diffusion matrix A depend on the variables (x,y) only.

We are now going to slide the function ϕ^2 with respect to ϕ^1. From the limiting conditions (3.5) satisfied by ϕ^1 and ϕ^2, there exists a real number $B>0$ such that

$$\left\{\begin{array}{lll}\phi^1(s,x,y)\leq\theta & \text{for all } s\leq -B, & (x,y)\in\overline{\Omega},\\ \phi^2(s,x,y)\geq 1-\rho & \text{for all } s\geq B, & (x,y)\in\overline{\Omega},\end{array}\right.$$

and

$$\phi^1(B,x,y) \geq 1-\rho \ \text{ for all } (x,y) \in \overline{\Omega}, \tag{4.3}$$

where θ and ρ are given in (1.26). As we did in the proof of Lemma 3.5, by using Lemmas 3.2 and 3.4 in Σ^-_{-B} and Σ^+_{-B}, it is found that $\phi^1 \leq \phi^{2,\tau}$ in $I\!R \times \overline{\Omega}$ for all $\tau \geq 2B$.

Let us now decrease τ and set

$$\tau^* = \ \inf\,\{\tau \in I\!R,\ \phi^1 \leq \phi^{2,\tau} \text{ in } I\!R \times \overline{\Omega}\}.$$

This real number τ^* is finite because $\phi^2(-\infty,\cdot,\cdot) = 0$ and $\phi^1(+\infty,\cdot,\cdot) = 1$. By continuity, one has $\phi^1 \leq \phi^{2,\tau^*}$. Two cases may occur :

case 1 : suppose that

$$\sup_{[-B,B]\times\overline{\Omega}} \ (\phi^1 - \phi^{2,\tau^*}) \ < 0.$$

Since the functions ϕ^1 and ϕ^2 are globally Lipschitz-continuous, there exists a positive real number η such that the above inequality holds good for all $\tau \in [\tau^*-\eta, \tau^*]$. Choose any $\tau \in [\tau^*-\eta,\tau^*]$. Lemma 3.2 applied to ϕ^1 and $\phi^{2,\tau}$ in Σ^-_{-B} yields

$$\phi^1(s,x,y) \leq \phi^{2,\tau}(s,x,y) \ \text{ for all } s \leq -B,\ (x,y) \in \overline{\Omega}.$$

On the other hand, because of (4.3) and since $\phi^1(s,x,y) \leq \phi^{2,\tau}(s,x,y)$ if $|s| \leq B$, it follows that

$$\phi^{2,\tau}(B,x,y) \geq 1-\rho \ \text{ for all } (x,y) \in \overline{\Omega}.$$

Since ϕ^2 is increasing in s, one gets that $\phi^{2,\tau} \geq 1-\rho$ in $\overline{\Sigma^+_B}$. Lemma 3.4 applied to ϕ^1 and $\phi^{2,\tau}$ in Σ^+_B yields

$$\phi^1(s,x,y) \leq \phi^{2,\tau}(s,x,y) \ \text{ for all } s \geq B,\ (x,y) \in \overline{\Omega}.$$

Eventually, $\phi^1 \leq \phi^{2,\tau}$ in $I\!R \times \overline{\Omega}$, contradicting the minimality of τ^*. Therefore, case 1 is ruled out.

case 2 : suppose that

$$\sup_{[-B,B]\times\overline{\Omega}} \ (\phi^1 - \phi^{2,\tau^*}) \ = 0.$$

There exists then a sequence of points $(s_n,x_n,y_n) \in [-B,B]\times\overline{\Omega}$ such that $\phi^1(s_n,x_n,y_n) - \phi^{2,\tau^*}(s_n,x_n,y_n) \to 0$ as $n \to +\infty$. Since both ϕ^1 and ϕ^2 are L-periodic with respect to x, one can assume that $(x_n,y_n) \in \overline{C}$. Up to extraction of some subsequence, one can also assume that $(s_n,x_n,y_n) \to (\overline{s},\overline{x},\overline{y}) \in [-B,B]\times\overline{C}$ as $n \to +\infty$. By continuity, one gets $\phi^1(\overline{s},\overline{x},\overline{y}) = \phi^{2,\tau^*}(\overline{s},\overline{x},\overline{y})$.

Coming back to the variables (t,x,y), consider the function z defined in $I\!R \times \overline{\Omega}$ by

$$\begin{aligned} z(t,x,y) \ &= \phi^1(c^1t + x\cdot e, x, y) - \phi^2(c^1t + x\cdot e + \tau^*, x, y) \\ &= u^1(t,x,y) - \phi^2(c^1t + x\cdot e + \tau^*, x, y) \end{aligned}$$

(note that the function $\phi^2(c^1t + x\cdot e + \tau^*,x,y)$ is not in general equal to the function u^2 up to translation). This function z is globally of class C^1 in $I\!R \times \overline{\Omega}$ and of class C^2 with respect to the (x,y) variables in $I\!R \times \Omega$. The function z is nonpositive and it vanishes at the point

$((\overline{s}-\overline{x}\cdot e)/c^1,\overline{x},\overline{y})$. It satisfies the Neumann boundary condition $\nu A\nabla z=0$ on $I\!R\times\partial\Omega$. Furthermore, because of (4.1) and (4.2), it follows that

$$\partial_t z-\mathrm{div}_{x,y}(A\nabla z)+q(x,y)\cdot\nabla z\le f(x,y,\phi^1(c^1t+x\cdot e,x,y))-f(x,y,\phi^2(c^1t+x\cdot e+\tau^*,x,y)).$$

Since f is globally Lipschitz-continuous in $\overline{\Omega}\times I\!R$, there exists then a bounded function $b(t,x,y)$ such that

$$\partial_t z-\mathrm{div}(A\nabla z)+q(x,y)\cdot\nabla_{x,y}z+bz\le 0,\quad \text{for all } (t,x,y)\in I\!R\times\Omega.$$

The strong parabolic maximum principle and the Hopf lemma yield $z(t,x,y)=0$ for all $t\le(\overline{s}-\overline{x}\cdot e)/c^1$ and for all $(x,y)\in\overline{\Omega}$. By definition of z and since both ϕ^1 and ϕ^2 are L-periodic with respect to x, it follows then that $z(t,x,y)=0$ for all $(t,x,y)\in I\!R\times\overline{\Omega}$, *i.e.*

$$\phi^1(s,x,y)=\phi^2(s+\tau^*,x,y)\ \text{ for all } (s,x,y)\in I\!R\times\overline{\Omega}.$$

Putting that into (4.1) and (4.2) gives $(c^2-c^1)\ \phi^2_s\equiv 0$. If $c_1\ne c_2$, then $\phi^2_s\equiv 0$, which is ruled out by Lemma 3.5. Therefore, $c^1=c^2=:c$. By definition of ϕ^1 and ϕ^2, it follows that

$$u^1(t,x,y)=u^2\left(t+\frac{\tau^*}{c},x,y\right)\ \text{ for all } (t,x,y)\in I\!R\times\overline{\Omega}.$$

That completes the proofs of Lemma 4.1 and part b) of Theorem 1.13. □

5 Case with ignition temperature : existence of a solution (c,u) of (1.28)

This section is devoted to the proof of part a) of Theorem 1.13. Throughout this section, one assumes that f satisfies (1.24-1.25) and (1.26), and that q and A are respectively globally $C^{1,\delta}(\overline{\Omega})$ and $C^3(\overline{\Omega})$ (with $\delta>0$) and that they satisfy (1.19), (1.21) and (1.22).

To do the proof of part a) of Theorem 1.13, we actually prove the existence of a weak solution (c,ϕ) of (3.2)-(3.5) such that $c>0$ and $0\le\phi(s,x,y)\le 1$ for all $(s,x,y)\in I\!R\times\overline{\Omega}$. Once this has been proved, since f vanishes at 0 and 1, parabolic regularity and the strong parabolic maximum principle applied to the function u given by (3.1) yields that u is a classical solution of (1.28) and that $0<u(t,x,y)<1$ for all $t\in I\!R$ and $(x,y)\in\overline{\Omega}$.

We divide the proof of the existence of a solution (c,ϕ) of (3.2)-(3.5) into several steps. We first work with elliptic regularizations of (3.2)-(3.5) in finite cylinders of the type $[-a,a]\times\overline{\Omega}$ (section 5.1). Next, we show the existence of exponential solutions of the corresponding linearized problem around 0 (section 5.2). Lastly, we pass to the limit $a\to+\infty$ and make the regularization parameter converge to 0 (sections 5.3 and 5.4), by proving especially that the speeds of the approximated problems are positive and bounded away from 0.

5.1 Elliptic regularization of (3.2)-(3.5) in finite cylinders

Let a and ε be two positive real numbers. Let

$$\Sigma_a=(-a,a)\times\Omega$$

and

$$\tilde{\Sigma}_a = \overline{\Sigma}_a \setminus (\{\pm a\} \times \partial\Omega)\,.$$

In order to build a solution (c, ϕ) of (3.2)-(3.5), one first works with elliptic regularizations of the type

$$\left\{\begin{array}{rcl} L_\varepsilon\phi + f(x,y,\phi) & = & 0 \quad \text{in } \Sigma_a,\\ \nu A(\tilde{e}\phi_s + \nabla_{x,y}\phi) & = & 0 \quad \text{on } (-a,a)\times\partial\Omega,\\ \phi \text{ is } L\text{-periodic w.r.t. } x, & & \\ \phi(-a,\cdot,\cdot) = 0, \quad \phi(a,\cdot,\cdot) & = & 1, \end{array}\right. \tag{5.1}$$

where L_ε is the elliptic operator defined by

$$\begin{array}{rl} L_\varepsilon\phi = & \varepsilon\phi_{ss} + \mathrm{div}_{x,y}(A\nabla_{x,y}\phi) + (\tilde{e}A\tilde{e})\phi_{ss} + \mathrm{div}_{x,y}(A\tilde{e}\ \phi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi)\\ & -q\cdot\nabla_{x,y}\phi - (q\cdot\tilde{e} + c)\phi_s. \end{array} \tag{5.2}$$

Note that this operator L_ε is elliptic because of the term $\varepsilon\phi_{ss}$, which plays the role of a regularizing term.

Following the scheme of the proof of Berestycki and Nirenberg [18] for the existence of solutions of similar problems in finite cylinders of the type $(-a,a)\times\omega$ wich Neumann boundary conditions on $(-a,a)\times\partial\omega$, this section is divided into several lemmas (Lemmas 5.1, 5.2 and 5.3 below) dealing with the properties of the solutions of (5.1). These lemmas eventually lead to an existence and uniqueness result which is stated in Proposition 5.6.

One first proves the following

Lemma 5.1 *For each $c \in I\!R$, there exists a solution $\phi^c \in C(\overline{\Sigma}_a)\cap C^2(\tilde{\Sigma}_a)$ of (5.1).*

Proof. Let ψ be the function defined by $\psi(s) = \dfrac{s+a}{2a}$. Setting $\phi = v + \psi$, problem (5.1) is equivalent to

$$\left\{\begin{array}{rcl} -L_\varepsilon v & = & f(x,y,v+\psi) - \dfrac{1}{2a}[q\cdot\tilde{e} + c - \mathrm{div}_{x,y}(A\tilde{e})] \text{ in } \Sigma_a,\\ \nu A(\tilde{e}v_s + \nabla_{x,y}v) & = & -\dfrac{1}{2a}\nu A\tilde{e} \text{ on } (-a,a)\times\partial\Omega,\\ v \text{ is } L\text{-periodic w.r.t. } x, & & \\ v(\pm a,\cdot,\cdot) & = & 0. \end{array}\right. \tag{5.3}$$

For all $\alpha > 0$, let $R_\alpha = \Sigma_a \cap \{|x| < \alpha\}$ and let L and H_0 be the Banach spaces

$$\begin{array}{rcl} L & = & \{\ v;\ \forall\alpha > 0,\ v \in L^2(R_\alpha),\ \ \forall k \in \prod_{i=1}^d L_i\mathbb{Z},\ v(s,x+k,y) - v(s,x,y) = 0 \text{ in } L^2(\Sigma_a)\ \},\\ H_0 & = & \{\ v \in L,\ \forall\alpha > 0\ v \in H^1(R_\alpha),\ \ v|_{s=\pm a} = 0 \text{ in } H^{1/2}(\{\pm a\}\times\Omega)\ \} \end{array}$$

embedded with the norms $\|v\|_L = \left(\displaystyle\int_{(-a,a)\times C} v^2\right)^{1/2}$ and $\|v\|_{H_0} = \left(\displaystyle\int_{(-a,a)\times C} |\nabla v|^2 + v^2\right)^{1/2}$.

From Poincaré's inequality, the norm $\|v\|_{H_0}$ is equivalent to $\left(\displaystyle\int_{(-a,a)\times C} |\nabla v|^2\right)^{1/2}$. Lastly, let $\mathcal{C}$ be the set

$$\mathcal{C} = \{v \in H_0,\ \|v\|_{H_0} \le C_0\} \tag{5.4}$$

where C_0 is a positive constant to be chosen later. This set $\mathcal{C}$ is convex and compactly included in L.

Since equation (5.3) is L-periodic with respect to x, solving it in the weak sense in H_0' is the same as finding a solution $v \in H_0$ of

$$B(v,z) = \int_{(-a,a)\times C} g(v)z + \int_{(-a,a)\times\partial\Omega\ \cap C} (-\frac{1}{2a}\nu A\tilde{e})z, \quad \forall z \in H_0 \tag{5.5}$$

where $g(v)$ is the function defined by

$$g(v)(s,x,y) = f(x,y,v(s,x,y)+\psi(s)) - \frac{1}{2a}[q(x,y)\cdot\tilde{e} + c - \mathrm{div}_{x,y}(A(x,y)\tilde{e})],$$

and

$$\begin{aligned} B(v,z) = \int_{(-a,a)\times C} \quad & (\tilde{e}A\tilde{e}+\varepsilon)v_s z_s + \nabla_{x,y}z\ A\ \nabla_{x,y}v + (\nabla_{x,y}z\ A\ \tilde{e})v_s + (\tilde{e}A\nabla_{x,y}v)z_s \\ & +(q\cdot\nabla_{x,y}v)z + (q\cdot\tilde{e}+c)v_s z. \end{aligned}$$

For each $v \in C$, the function $g(v)$ is in $L \subset (H_0)'$. Moreover, since f and q are globally bounded and since A is globally C^1 (it is even $C^3(\overline{\Omega})$), one has

$$\|g(v)\|_\infty \leq M_1 := \|f\|_\infty + \frac{1}{2a}(\|q\|_\infty + |c| + \|\mathrm{div}(A)\|_\infty).$$

Therefore, Cauchy-Schwarz inequality yields

$$\left|\int_{(-a,a)\times C} g(v)z\right| \leq M_1(2a|C|)^{1/2}\|z\|_{H_0}.$$

Similarly, the map $z \mapsto \int_{(-a,a)\times\partial\Omega\ \cap C}(-\frac{1}{2a}\nu A\tilde{e})z$ is continuous on H_0 and

$$\exists M_2 > 0,\ \forall z \in H_0,\ \left|\int_{(-a,a)\times\partial\Omega\ \cap C}(-\frac{1}{2a}\nu A\tilde{e})z\right| \leq M_2\|z\|_{H_0}.$$

On the other hand, the bilinear form B is clearly continuous on H_0. Let us now check that it is coercive. For any $w \in H_0$, one has

$$\begin{aligned} B(w,w) = \int_{(-a,a)\times C} \quad & (\tilde{e}A\tilde{e}+\varepsilon)w_s^2 + \nabla_{x,y}w\ A\ \nabla_{x,y}w + (\nabla_{x,y}w\ A\ \tilde{e})w_s + (\tilde{e}A\nabla_{x,y}w)w_s \\ & +(q\cdot\nabla_{x,y}w)w + (q\cdot\tilde{e}+c)w_s w. \end{aligned}$$

Since ε is positive and A is uniformly elliptic from (1.22), one has

$$\begin{aligned} \forall (x,y)\in\overline{\Omega}, \qquad & (\tilde{e}A\tilde{e}+\varepsilon)w_s^2 + \nabla_{x,y}w\ A\ \nabla_{x,y}w + (\nabla_{x,y}w\ A\ \tilde{e})w_s + (\tilde{e}A\nabla_{x,y}w)w_s \\ & = (\nabla_{x,y}w + \tilde{e}w_s)\ A\ (\nabla_{x,y}w + \tilde{e}w_s) + \varepsilon w_s^2 \\ & \geq \delta\ |\nabla_{s,x,y}w|^2 \end{aligned}$$

for some positive constant $\delta = \delta(\varepsilon)$ which does not depend on w or $(x, y) \in \overline{\Omega}$. Integrating by parts the term $\int_{(-a,a)\times C} (q \cdot \nabla_{x,y} w)\, w$ and denoting by n to outward unit normal to C leads to

$$\int_{(-a,a)\times C} (q(x,y) \cdot \nabla_{x,y} w)\, w \;\; = \int_{(-a,a)\times \partial C} (q \cdot n) w^2 \,/2 - \int_{(-a,a)\times C} (\mathrm{div}\ q)\ w^2 \,/2$$
$$= 0$$

because of (1.19) and by L-periodicity of w with respect to x. Furthermore,

$$\int_{(-a,a)\times C} (q(x,y) \cdot e + c) w_s w = \int_C [\ (q(x,y) \cdot \tilde{e} + c)(w(\cdot, x, y))^2 \,/2\ ]_{-a}^{a} \;\; = 0$$

because $w|_{s=\pm a} = 0$ in $H^{1/2}(\{\pm a\} \times \Omega)$. Therefore, because of Poincaré's inequality, it follows that $B(w,w) \geq \delta \int_{(-a,a)\times C} |\nabla_{s,x,y} w|^2 \geq \gamma \|w\|_{H_0}^2$ for some $\gamma > 0$ and for all $w \in H_0$.

From Lax-Milgram theorem, there exists then, for each $v \in C$, a unique solution $w = T(v) \in H_0$ of

$$B(w,z) = \int_{(-a,a)\times C} g(v) z + \int_{(-a,a)\times \partial\Omega\ \cap C} (-\frac{1}{2a} \nu A \tilde{e}) z \quad \forall\ z \in H_0.$$

Since f is globally Lipschitz-continuous, it easily follows that the map T is continuous. Taking $z = w$ as test function and applying the Cauchy-Schwarz inequality, one gets

$$\gamma \|w\|_{H_0}^2 \leq [M_1 (2a|C|)^{1/2} + M_2] \|w\|_L \leq [M_1 (2a|C|)^{1/2} + M_2] \|w\|_{H_0}.$$

As a consequence, by choosing $C_0 := [M_1(2a|C|)^{1/2} + M_2]\gamma^{-1}$ in the definition of the set $\mathcal{C}$ (*see* (5.4)), it follows that $w \in \mathcal{C}$.

From Schauder fixed point theorem, there exists then a fixed point $v \in \mathcal{C}$ for the map T, namely, a solution $v \in \mathcal{C}$ of (5.5). Since both v and q are L-periodic with respect to x, the first equation in (5.3) is then satisfied in the distribution sense in $\tilde{\Sigma}_a$. Furthermore, from the regularity theory up to the boundary, the function v is of class $C^2(\tilde{\Sigma}_a)$ and it satisfies (5.3) in the classical sense in $\tilde{\Sigma}_a$.

Let us now prove that the function v can be extended by continuity in the "corners" $\{\pm a\} \times \partial\Omega$ of $\overline{\Sigma}_a$. To do it, we build a supersolution $h(s)$ for (5.3). We define it as in [18] : let $b = \|q\|_\infty + |c| + \|\mathrm{div}(A)\|_\infty + 1$, and let

$$h(s) = \frac{c_1 + \varepsilon}{b^2}\ e^{\frac{ba}{c_1+\varepsilon}} \left[1 - e^{-\frac{b}{c_1+\varepsilon}(s+a)}\right] - \frac{1}{b}(s+a) \;\; \text{for all } s \in [-a, 0],$$

where c_1 is a positive number given in (1.22), and let us extend h by symmetry on $[0, a]$. This function $h(s)$ is of class C^2 on $[-a, a]$, it is nonnegative, concave, it vanishes on $\pm a$ and it satisfies

$$L_\varepsilon h \leq (\tilde{e} A \tilde{e} + \varepsilon) h_{ss} + b|h_s| \leq (c_1 + \varepsilon) h_{ss} + b|h_s| = -1 \;\; \text{in } \Sigma_a.$$

From the maximum principle, it follows then that $|v| \leq M_1 h$ in Σ_a. As a consequence, the function v can be continuously extended by 0 on the corners $\{\pm a\} \times \partial\Omega$ of the closed cylinder $\overline{\Sigma}_a$.

Eventually, the function $\phi^c = v + \psi \in C(\overline{\Sigma}_a) \cap C^2(\tilde{\Sigma}_a)$ is a classical solution of (5.1). That completes the proof of Lemma 5.1. □

Lemma 5.2 *Under the notations of Lemma 5.1, the function ϕ^c is increasing in s and it is the unique solution of (5.1) in $C(\overline{\Sigma_a})\cap C^2(\tilde{\Sigma}_a)$.*

Proof. We use the sliding method of Berestycki and Nirenberg [17].

Let us first show that any solution $\phi\in C(\overline{\Sigma_a})\cap C^2(\tilde{\Sigma}_a)$ of (5.1) is increasing in s. Since $f=0$ in $\overline{\Omega}\ \times\ (-\infty,0]\cup[1,+\infty)$, the elliptic maximum principle and the Hopf lemma yield that $0<\phi<1$ in $(-a,a)\times\overline{\Omega}$.

For any $\lambda\in(0,2a)$, let ϕ^λ be the function defined by

$$\phi^\lambda(s,x,y)=\phi(s+2a-\lambda,x,y)\ \text{ in } \overline{\Sigma_a^\lambda},\quad \Sigma_a^\lambda:=(-a,-a+\lambda)\times\Omega.$$

In order to show that ϕ is increasing in s in $\overline{\Sigma_a}$, it suffices to prove that

$$\phi<\phi^\lambda\ \text{ in } \overline{\Sigma_a^\lambda} \tag{5.6}$$

for all $\lambda\in(0,2a)$. Since ϕ is continuous and L-periodic in x and since $\phi(-a,\cdot,\cdot)=0$, $\phi(a,\cdot,\cdot)=1$, it follows that (5.6) is true for small λ.

Let us now increase λ and set

$$\lambda^*=\sup\ \{\lambda\in(0,2a),\ \phi<\phi^\mu \text{ in } \overline{\Sigma_a^\lambda} \text{ for all } \mu\in(0,\lambda)\}\ >0.$$

To complete the proof, one has to show that $\lambda^*=2a$. Assume $\lambda^*<2a$. By continuity, one has $\phi\le\phi^{\lambda^*}$ in $\overline{\Sigma_a^{\lambda^*}}$. On the other hand, there exists a sequence $\lambda_n\stackrel{>}{\to}\lambda^*$ and some points $(s_n,x_n,y_n)\in\overline{\Sigma_a^{\lambda_n}}$ such that $\phi(s_n,x_n,y_n)\ge\phi^{\lambda_n}(s_n,x_n,y_n)$. Since ϕ is L-periodic in x, one can assume that $(x_n,y_n)\in\overline{C}$ and also that $(s_n,x_n,y_n)\to(\overline{s},\overline{x},\overline{y})\in\overline{\Sigma_a^{\lambda^*}}$. Passing to the limit $n\to+\infty$, one gets $\phi(\overline{s},\overline{x},\overline{y})=\phi^{\lambda^*}(\overline{s},\overline{x},\overline{y})$.

The function $z(s,x,y)=\phi(s,x,y)-\phi^{\lambda^*}(s,x,y)$ defined in $\overline{\Sigma_a^{\lambda^*}}$ is nonpositive and it vanishes at the point $(\overline{s},\overline{x},\overline{y})$. Since the equation (5.1) is invariant by translation with respect to s and since the function f is Lipschitz-continuous, the function z satisfies

$$\left\{\begin{array}{rll}(\tilde{e}A\tilde{e}+\varepsilon)z_{ss}+\mathrm{div}(A\nabla_{x,y}z)+\mathrm{div}_{x,y}(A\tilde{e}\ z_s) & & \\ +\partial_s(\tilde{e}A\nabla_{x,y}z)-q\cdot\nabla_{x,y}z-(q\cdot\tilde{e}+c)z_s+bz & =0 & \text{in } \Sigma_a^{\lambda^*},\\ \nu A\nabla z & =0 & \text{on } (-a,-a+\lambda^*)\times\partial\Omega\end{array}\right. \tag{5.7}$$

for some bounded function $b(s,x,y)$. Furthermore,

$$z(-a,x,y)=-\phi(a-\lambda^*,x,y)<0\ \text{ for all } (x,y)\in I\!R\times\overline{\omega}$$

because $\lambda^*<2a$ and because ϕ is positive in $(-a,a)\times\overline{\Omega}$ and continuous. Similarly, one has

$$z(-a+\lambda^*,x,y)=\phi(-a+\lambda^*,x,y)-1<0\ \text{ for all } (x,y)\in\overline{\Omega}.$$

As a consequence, the point $(\overline{s},\overline{x},\overline{y})$ where z vanishes lies in $(-a,-a+\lambda^*)\times\overline{\Omega}$. But that is ruled out by (5.7) from the strong maximum principle together with Hopf lemma.

Thus, $\lambda^*=2a$ and ϕ is increasing in the variable s in $\overline{\Sigma}_a$.

Let us now turn to the proof of the uniqueness of the solution $\phi\in C(\overline{\Sigma_a})\cap C^2(\tilde{\Sigma}_a)$ of (5.1). Consider two solutions ϕ and ϕ'. By arguing as above and sliding ϕ' with respect to ϕ, it is found that $\phi(s,x,y)\le\phi'(s+2a-\lambda,x,y)$ for all $\lambda\in(0,2a)$ and for all $(s,x,y)\in\overline{\Sigma_a^\lambda}$. Passing to the limit $\lambda\to 2a$, one gets $\phi\le\phi'$ in $\overline{\Sigma}_a$. On the other hand, by sliding ϕ with respect to ϕ', it also follows that $\phi'\le\phi$ in $\overline{\Sigma}_a$. Eventually, $\phi\equiv\phi'$ and the proof is done. □

Lemma 5.3 *The functions ϕ^c are decreasing and continuous with respect to c in the following sense : if $c > c'$, then $\phi^c < \phi^{c'}$ in $(-a,a)\times\overline{\Omega}$ and if $c_n \to c \in I\!R$, then $\phi^{c_n} \to \phi^c$ in $C^{2,\alpha}_{loc}(\tilde{\Sigma}_a)$ (for all $0 \le \alpha < 1$) and in $C(\overline{\Sigma}_a)$.*

Proof. Choose $c > c'$ and set $\phi = \phi^c$ and $\phi' = \phi^{c'}$. Since ϕ' is increasing in s, it satisfies

$$\begin{array}{ll}(\tilde{e}A\tilde{e}+\varepsilon)\phi'_{ss} + \mathrm{div}_{x,y}(A\nabla_{x,y}\phi') + \mathrm{div}_{x,y}(A\tilde{e}\ \phi'_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi') & \\ \qquad -q\cdot\nabla_{x,y}\phi' - (q\cdot\tilde{e}+c)\phi'_s + f(x,y,\phi') & = (c'-c)\phi'_s \le 0 \ \text{ in } \Sigma_a.\end{array}$$

Furthermore, ϕ' verifies the same boundary and periodicity conditions as ϕ. As a consequence, ϕ' is a super-solution for the problem (5.1) which ϕ is a solution of. Using a sliding method as in Lemma 5.2 leads to

$$\phi(s,x,y) < \phi'(s+2a-\lambda,x,y) \ \text{ for all } (s,x,y) \in \overline{\Sigma^\lambda_a}$$

for all $\lambda \in (0,2a)$. The passage to the limit $\lambda \to 2a$ leads to $\phi \le \phi'$ in $\overline{\Sigma}_a$. From the strong maximum principle and the Hopf lemma, it follows that either $\phi < \phi'$ in $(-a,a)\times\overline{\Omega}$ or $\phi \equiv \phi'$. The latter would imply that $c\phi'_s \equiv c'\phi'_s$, whence $\phi'_s \equiv 0$ in Σ_a. This is clearly impossible because ϕ' is increasing in s. As a consequence, $\phi < \phi'$ in $(-a,a)\times\overline{\Omega}$.

Let us now turn to the proof of the continuity of the functions ϕ^c with respect to c. Choose a sequence $c_n \to c \in I\!R$. From standard elliptic estimates up to the boundary, the functions ϕ^{c_n} converge (up to extraction of some subsequence) in $C^{2,\alpha}_{loc}(\tilde{\Sigma}_a)$ (for all $0 \le \alpha < 1$) to a function ϕ solving (5.1).

To prove the convergence in $C(\overline{\Sigma}_a)$, choose first n_0 such that $|c_n - c| \le 1$ for $n \ge n_0$. As it was done at the end of the proof of Lemma 5.1, the maximum principle yields

$$\forall n \ge n_0, \qquad |\phi^{c_n}(s,x,y) - \psi(s)| \le M'\tilde{h}(s) \ \text{ in } \Sigma_a$$

where

$$\left\{\begin{array}{ll}\psi(s) & = \dfrac{1}{2a}(s+a), \\ M' & = \|f\|_\infty + \dfrac{1}{2a}(\|q\|_\infty + |c| + \|\mathrm{div}(A)\|_\infty + 1), \\ \tilde{h}(s) & = \dfrac{c_1+\varepsilon}{b'^2}\ e^{\frac{b'a}{c_1+\varepsilon}}\left[1 - e^{-\frac{b'}{c_1+\varepsilon}(s+a)}\right] - \dfrac{1}{b'}(s+a) \ \text{ for all } s \in [-a,0], \\ \tilde{h}(s) & = \tilde{h}(-s) \ \text{ for all } s \in [0,a], \\ b' & = \|q\|_\infty + |c| + \|\mathrm{div}(A)\|_\infty + 2.\end{array}\right.$$

Therefore, the function ϕ can be extended by continuity on the corners $\{\pm a\}\times\partial\Omega$ of Σ_a and the functions ϕ^{c_n} converge to ϕ uniformly in $\overline{\Sigma}_a$. By uniqueness of the solution of (5.1) in $C^2(\tilde{\Sigma}_a)\cap C(\overline{\Sigma}_a)$, the function ϕ is nothing else but the function ϕ^c. Lastly, by uniqueness of the limit, one concludes that the whole sequence (ϕ^{c_n}) approaches ϕ^c as $n \to +\infty$. That completes the proof of Lemma 5.3. □

In the remaining part of this subsection, for any $\varepsilon, a > 0$ and $c \in I\!R$, we call $\phi^c_{\varepsilon,a}$ the unique solution of (5.1) in $C(\overline{\Sigma}_a)\cap C^2(\tilde{\Sigma}_a)$.

Lemma 5.4 *There exist $a_1 > 0$ and K_1 such that, for all $a \geq a_1$ and $\varepsilon \in (0,1]$,*

$$(c > K_1) \implies \left(\max_{(x,y)\in\overline{\Omega}} \phi^c_{\varepsilon,a}(0,x,y) = \max_{(x,y)\in\overline{C}} \phi^c_{\varepsilon,a}(0,x,y) < \theta \right).$$

Proof. First of all, let $\psi(x,y)$ be a $C^2(\overline{\Omega})$ function such that

$$\nu A(\nabla\psi + \tilde{e}) = 0 \text{ on } \partial\Omega$$

and ψ is L-periodic with respect to x. A function ψ satisfying these properties can be found as a minimizer of the integral

$$\int_C \nabla\varphi A \nabla\varphi + \int_{\partial C\cap\{(x,y),\ x\in(0,L_1)\times\cdots\times(0,L_d)\}} (\nu A\tilde{e})\varphi$$

over all functions $\varphi \in H^1_{loc}(\Omega)$ such that $\varphi(\cdot + k,\cdot) - \varphi = 0$ in $L^2(\Omega)$ for all $k \in \prod_{i=1}^{d} L_i\mathbb{Z}$. Of course, such a minimizer satisfies the additional equation $\operatorname{div}(A\nabla\psi) = 0$ in Ω, but this property is not needed here.

Next, from (1.24-1.25) and (1.26), there exists a C^1 function g defined in $[0,2]$ such that $g(u) = 0$ for all $u \in [0,\theta/2] \cup \{2\}$, $g(u) > 0$ for all $u \in (\theta/2, 2)$, $g'(2) < 0$ and

$$f(x,y,u) \leq g(u) \text{ for all } (x,y) \in \overline{\Omega} \text{ and } u \in [0,2].$$

Let then (k,v) be the unique solution of the one-dimensional problem

$$\begin{cases} v'' - kv' + g(v) = 0 \text{ in } \mathbb{R}, \\ v(-\infty) = 0 < v(\xi) < v(+\infty) = 2 \text{ for all } \xi \in \mathbb{R} \text{ and } v(0) = \theta. \end{cases} \tag{5.8}$$

Furthermore, k is positive and the function v is increasing in $\mathbb{R}$. Since the positive and bounded function v' satisfies the linear equation

$$(v')'' - k(v')' + g'(v)v' = 0 \text{ in } \mathbb{R}$$

and since the function $g'(v)$ is itself globally bounded, it follows from standard elliptic estimates and elliptic Harnack inequality that there exists a constant C_0 such that

$$\forall \xi \in \mathbb{R}, \quad |v''(\xi)| \leq C_0 v'(\xi).$$

Therefore, (5.8) implies that

$$\forall \xi \in \mathbb{R}, \quad 0 \leq g(v(\xi)) \leq Cv'(\xi), \tag{5.9}$$

where $C = C_0 + k$.

By continuity and L-periodicity with respect to x, the function ψ defined above is globally bounded in $\overline{\Omega}$. Let us now consider the function

$$\tilde{v}(s,x,y) = v(s + \psi(x,y) - \max_{\Omega} \psi)$$

defined in $I\!R\times\overline{\Omega}$. Let us now check that for c and a large enough, this function $\tilde{v}$ is a supersolution for problem (5.1). Firstly, since $v(+\infty)=2$, there exists $a_1>0$ such that

$$\forall a\geq a_1,\quad \forall (x,y)\in\overline{\Omega},\quad \tilde{v}(a,x,y)>1.$$

Secondly, owing to the definition of the function ψ, the function $\tilde{v}$ satisfies

$$\nu A(\nabla_{x,y}\tilde{v}+\tilde{e}\tilde{v}_s)=0\ \text{ on } I\!R\times\partial\Omega.$$

Take any $\varepsilon\in(0,1]$, $a\geq a_1$ and

$$c>K_1:=\max_{(x,y)\in\overline{\Omega}}\{k[1+(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)]+\mathrm{div}(A(\tilde{e}+\nabla\psi))-q\cdot(\tilde{e}+\nabla\psi)\}+C.$$

One has

$$\begin{array}{rcl}L_\varepsilon\tilde{v}+f(x,y,\tilde{v}) & = & (\varepsilon+\tilde{e}A\tilde{e}+\nabla\psi A\nabla\psi+\tilde{e}A\nabla\psi+\nabla\psi A\tilde{e})\ v''(\xi)\\ & & +[\mathrm{div}(A(\tilde{e}+\nabla\psi))-q\cdot\nabla\psi-q\cdot\tilde{e}-c]\ v'(\xi)+f(x,y,v(\xi))\\ & & \qquad\qquad (\text{where } \xi=s+\psi(x,y)-\max_{\overline{\Omega}}\psi)\\ & = & [\varepsilon+(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)]\ v''(\xi)\\ & & +[\mathrm{div}(A(\tilde{e}+\nabla\psi))-q\cdot(\tilde{e}+\nabla\psi)-c]\ v'(\xi)+f(x,y,v(\xi))\\ & = & \{k[\varepsilon+(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)]+\mathrm{div}(A(\tilde{e}+\nabla\psi))-q\cdot(\tilde{e}+\nabla\psi)-c\}\,v'(\xi)\\ & & +f(x,y,v(\xi))-[\varepsilon+(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)]\ g(v(\xi))\end{array}$$

from (5.8). Hence,

$$\begin{array}{rcl}L_\varepsilon\tilde{v}+f(x,y,\tilde{v}) & \leq & \{k[1+(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)]+\mathrm{div}(A(\tilde{e}+\nabla\psi))-q\cdot(\tilde{e}+\nabla\psi)-c\}\,v'(\xi)\\ & & +g(v(\xi))\end{array}$$

since $\varepsilon\leq 1$, $k>0$, $v'>0$ and $0\leq f(x,y,u)\leq g(u)$ for all $(x,y,u)\in\overline{\Omega}\times[0,2]$. Eventually,

$$L_\varepsilon\tilde{v}+f(x,y,\tilde{v})<0\ \text{ in } \Sigma_a$$

from (5.9) and from the choice of c.

By sliding the function $\tilde{v}$ with respect to the variable s and comparing it to $\phi^c_{\varepsilon,a}$ (as for ϕ and ϕ' in the proof of Lemma 5.3) and by using the monotonicity of v and the fact that $\tilde{v}(-a,x,y)>0$ and $\tilde{v}(a,x,y)>1$ for all $(x,y)\in\overline{\Omega}$, it follows that

$$\phi^c_{\varepsilon,a}(s,x,y)<\tilde{v}(s,x,y)\ \text{ for all } (s,x,y)\in\overline{\Sigma}_a.$$

Therefore,

$$\max_{(x,y)\in\overline{\Omega}}\phi^c_{\varepsilon,a}(0,x,y)=\max_{(x,y)\in\overline{C}}\phi^c_{\varepsilon,a}(0,x,y)<\max_{(x,y)\in\overline{\Omega}}\tilde{v}(0,x,y)\leq v(0)$$

since v is increasing. Since $v(0)=\theta$, the proof of Lemma 5.4 is complete. □

Lemma 5.5 *There exist $a_2>0$ and K_2 such that, for all $a\geq a_2$ and $\varepsilon\in(0,1]$,*

$$(c<K_2)\ \Longrightarrow\ \left(\max_{(x,y)\in\overline{\Omega}}\phi^c_{\varepsilon,a}(0,x,y)=\max_{(x,y)\in\overline{C}}\phi^c_{\varepsilon,a}(0,x,y)>\theta\right).$$

Proof. First of all, as in Lemma 5.4, let $\psi(x,y)$ be a $C^2(\overline{\Omega})$ function such that

$$\nu A(\nabla\psi+\tilde{e})=0 \text{ on } \partial\Omega$$

and ψ is L-periodic with respect to x. Let M_0 be such that

$$(\tilde{e}+\nabla\psi(x,y))\ A(x,y)\ (\tilde{e}+\nabla\psi(x,y))\ \leq\ M_0 \ \text{ for all } (x,y)\in\overline{\Omega}. \tag{5.10}$$

Next, remembering the definition of ρ in (1.26), let χ be a smooth function defined in $\mathbb{R}$ such that $0\leq\chi\leq 1$ in $\mathbb{R}$, $\chi(u)=0$ for all $u\leq 1-\rho$ and $\chi(u)=1$ for all $u\geq 1-\rho/2$. Let $h(u)$ be the function defined in $\mathbb{R}$ by

$$\forall u\in\mathbb{R},\quad h(u)=\frac{1}{1+M_0}\chi(u)\min_{(x,y)\in\overline{\Omega}} f(x,y,u).$$

From (1.24-1.25) and (1.26), the function h is globally Lipschitz-continuous, $h(u)=0$ for all $u\in(-\infty,1-\rho]\cup[1,+\infty)$, $h(u)>0$ for all $u\in(1-\rho,1)$ and h is nonincreasing in $[1-\rho/2,1]$. Therefore, there exists a unique solution (κ,w) of the one-dimensional problem

$$\left\{\begin{array}{c} w''-\kappa w'+h(w)=0 \ \text{ in } \mathbb{R},\\ w(-\infty)=-1<w(\xi)<w(+\infty)=1 \text{ for all } \xi\in\mathbb{R} \ \text{ and } \ w(0)=\theta.\end{array}\right. \tag{5.11}$$

Furthermore, κ is positive and the function w is increasing in $\mathbb{R}$.

Let us now consider the function

$$\tilde{w}(s,x,y)=w(s+\psi(x,y)-\min_{\Omega}\psi)$$

defined in $\mathbb{R}\times\overline{\Omega}$. Let us now check that for $-c$ and a large enough, this function $\tilde{w}$ is a subsolution for problem (5.1). Firstly, since $w(-\infty)=-1$, there exists $a_2>0$ such that

$$\forall a\geq a_2,\quad \forall(x,y)\in\overline{\Omega},\quad \tilde{w}(-a,x,y)<0.$$

Secondly, owing to the definition of the function ψ, the function $\tilde{w}$ satisfies the same boundary condition as the function $\tilde{v}$ in the proof of Lemma 5.4, namely :

$$\nu A(\nabla_{x,y}\tilde{w}+\tilde{e}\tilde{w}_s)=0 \ \text{ on } \mathbb{R}\times\partial\Omega.$$

Take any $\varepsilon\in(0,1]$, $a\geq a_2$ and

$$c<K_2:=\min_{(x,y)\in\overline{\Omega}}\left[\kappa\ (\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)+\mathrm{div}(A(\tilde{e}+\nabla\psi))-q\cdot(\nabla\psi+\tilde{e})\right].$$

As in Lemma 5.4, one has

$$\begin{array}{rcl} L_\varepsilon\tilde{w}+f(x,y,\tilde{w}) & = & \{\kappa[\varepsilon+(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)]+\mathrm{div}(A(\tilde{e}+\nabla\psi))-q\cdot(\nabla\psi+\tilde{e})-c\}\,w'(\xi)\\ & & +f(x,y,w(\xi))-[\varepsilon+(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)]\ h(w(\xi))\end{array}$$

where $\xi=s+\psi(x,y)-\min_{\overline{\Omega}}\psi$. From (5.10), from the definition and the nonnegativity of h, from the positivity of κ and w' and from the choice of c, it follows then that

$$L_\varepsilon\tilde{w}+f(x,y,\tilde{w})>0 \ \text{ in } \Sigma_a.$$

By sliding the function $\tilde{w}$ with respect to the variable s and comparing it to $\phi^c_{\varepsilon,a}$ (as for ϕ and ϕ' in the proof of Lemma 5.3) and by using the monotonicity of w and the fact that $\tilde{w}(-a,x,y)<0$ and $\tilde{w}(a,x,y)<1$ for all $(x,y)\in\overline{\Omega}$, one gets that

$$\phi^c_{\varepsilon,a}(s,x,y)>\tilde{w}(s,x,y)\ \text{ for all } (s,x,y)\in\overline{\Sigma}_a.$$

Therefore,

$$\max_{(x,y)\in\overline{\Omega}}\phi^c_{\varepsilon,a}(0,x,y)=\max_{(x,y)\in\overline{C}}\phi^c_{\varepsilon,a}(0,x,y)>\max_{(x,y)\in\overline{\Omega}}\tilde{w}(0,x,y)\geq w(0)$$

since w is increasing. Since $w(0)=\theta$, the proof of Lemma 5.4 is complete. □

We complete this section by proving, for a large enough, the existence of a real number $c^{\varepsilon,a}$ and of a function $\phi^{\varepsilon,a}$ satisfying (5.1) together with an additional normalization condition, namely (5.12) below :

Proposition 5.6 *There exist $a_0>0$ and $K\geq 0$ such that, for all $a\geq a_0$ and for all $\varepsilon\in(0,1]$, there exists a unique real number $c=c^{\varepsilon,a}$ such that the solution $\phi^{\varepsilon,a}:=\phi^{c^{\varepsilon,a}}_{\varepsilon,a}\in C^2(\tilde{\Sigma}_a)\cap C(\overline{\Sigma}_a)$ of (5.1) satisfies the normalization condition*

$$\max_{\overline{\Omega}}\phi^{\varepsilon,a}(0,\cdot,\cdot)=\max_{\overline{C}}\phi^{\varepsilon,a}(0,\cdot,\cdot)=\theta. \tag{5.12}$$

Furthermore,

$$\forall 0<\varepsilon\leq 1,\ \forall a\geq a_0,\quad |c^{\varepsilon,a}|\leq K. \tag{5.13}$$

Proof. Under the notations of Lemmas 5.4 and 5.5, let us define

$$a_0=\max(a_1,a_2)\text{ and }K=\max(|K_1|,|K_2|)\geq 0.$$

Fix any $a\geq a_0$ and $\varepsilon\in(0,1]$. From Lemmas 5.4 and 5.5, it follows that

$$\begin{cases}\forall c\geq K, & \displaystyle\max_{(x,y)\in\overline{\Omega}}\phi^c_{\varepsilon,a}(0,x,y)=\max_{(x,y)\in\overline{C}}\phi^c_{\varepsilon,a}(0,x,y)<\theta,\\ \forall c\leq -K, & \displaystyle\max_{(x,y)\in\overline{\Omega}}\phi^c_{\varepsilon,a}(0,x,y)=\max_{(x,y)\in\overline{C}}\phi^c_{\varepsilon,a}(0,x,y)>\theta.\end{cases}$$

On the other hand, Lemma 5.3 yields that the function

$$\Xi(c):=\max_{\overline{\Omega}}\phi^c_{\varepsilon,a}(0,\cdot,\cdot)=\max_{\overline{C}}\phi^c_{\varepsilon,a}(0,\cdot,\cdot)$$

is decreasing and continuous with respect to c. Therefore, the conclusion of Proposition 5.6 follows. □

5.2 Exponential decay for $s \le 0$

In this section, one finds upper bounds independent of a for the functions $\phi^{\varepsilon,a}$ in the left half-cylinder $[-a,0]\times\Omega$. Since $\phi^{\varepsilon,a}$ is increasing with respect to the variable s and satisfies $\max_{\overline{\Omega}} \phi^{\varepsilon,a}(0,\cdot,\cdot)=\theta$, and since $f(x,y,u)=0$ on $\overline{\Omega}\times[0,\theta]$, it follows that $\phi^{\varepsilon,a}$ is a positive and L-periodic (with respect to x) solution of

$$\left\{\begin{array}{rcl} L_\varepsilon \phi^{\varepsilon,a} &=& 0 \ \text{ in } (-a,0)\times\Omega,\\ \nu A(\tilde{e}\phi^{\varepsilon,a}_s+\nabla_{x,y}\phi^{\varepsilon,a}) &=& 0 \ \text{ on } (-a,0)\times\partial\Omega, \end{array}\right. \tag{5.14}$$

where $L_\varepsilon v=(\tilde{e}A\tilde{e}+\varepsilon)v_{ss}+\mathrm{div}_{x,y}(A\nabla_{x,y}v)+\mathrm{div}_{x,y}(A\tilde{e}\ v_s)+\partial_s(\tilde{e}A\nabla_{x,y}v)-q(x,y)\cdot\nabla_{x,y}v-(q\cdot\tilde{e}+c^{\varepsilon,a})v_s$.

Our goal in this section is to build some positive and L-periodic (with respect to x) solutions of (5.14) of the exponential type $w_\varepsilon(s,x,y)=e^{\lambda_\varepsilon s}\psi_\varepsilon(x,y)$ for some positive real numbers λ_ε. In other words, we look for a positive real number λ_ε and for a positive function $\psi_\varepsilon(x,y)$, defined in $\overline{\Omega}$, such that

$$\left\{\begin{array}{rcll} -\mathrm{div}(A\nabla\psi_\varepsilon)-\lambda_\varepsilon[\mathrm{div}(A\tilde{e}\ \psi_\varepsilon)+\tilde{e}A\nabla\psi_\varepsilon]+q(x,y)\cdot\nabla\psi_\varepsilon && & \\ +\lambda_\varepsilon(q\cdot\tilde{e}+c^{\varepsilon,a})\psi_\varepsilon-\lambda_\varepsilon^2(\tilde{e}A\tilde{e})\psi_\varepsilon &=& \varepsilon\lambda_\varepsilon^2\psi_\varepsilon & \text{in } \Omega,\\ \nu A(\tilde{e}\lambda_\varepsilon\psi_\varepsilon+\nabla\psi_\varepsilon) &=& 0 & \text{on } \partial\Omega,\\ \psi_\varepsilon \text{ is } L\text{-periodic w.r.t. } x. && & \end{array}\right.$$

The existence of such exponential solutions $w_\varepsilon(s,x,y)=e^{\lambda_\varepsilon s}\psi_\varepsilon(x,y)$ is a consequence of the following proposition :

Proposition 5.7 *1) Let $\zeta(x,y)$ be a $C^{0,\delta'}(\overline{\Omega})$ (for some $\delta'>0$) and assume that ζ is L-periodic with respect to x. For all γ and $\lambda\in I\!R$, there exists a unique $\mu=\mu_{\gamma,\zeta}(\lambda)\in I\!R$ (principal eigenvalue) and a positive function $\psi=\psi_{\gamma,\zeta}(\lambda)\in C^2(\overline{\Omega})$, unique up to multiplication, such that*

$$\left\{\begin{array}{rcll} -L_{\gamma,\lambda,\zeta}\psi &=& \mu\psi & \text{in } \Omega,\\ \nu A(\tilde{e}\lambda\psi+\nabla\psi) &=& 0 & \text{on } \partial\Omega,\\ \psi \text{ is } L\text{-periodic w.r.t. } x, && & \end{array}\right. \tag{5.15}$$

where

$$L_{\gamma,\lambda,\zeta}\psi=\mathrm{div}(A\nabla\psi)+\lambda[\mathrm{div}(A\tilde{e}\ \psi)+\tilde{e}A\nabla\psi]-q\cdot\nabla\psi-\lambda(q\cdot\tilde{e}+\gamma)\psi+\lambda^2(\tilde{e}A\tilde{e})\psi+\zeta\psi.$$

2) For all γ, $\lambda\in I\!R$ and ζ as in 1), the principal eigenvalue $\mu_{\gamma,\zeta}(\lambda)$ is equal to

$$\mu_{\gamma,\zeta}(\lambda)=\max_{\varphi\in E_\lambda}\ \inf_{\overline{\Omega}}\ \frac{-L_{\gamma,\lambda,\zeta}\varphi}{\varphi} \tag{5.16}$$

where

$$E_\lambda=\{\varphi\in C^2(\overline{\Omega}),\ \varphi>0 \text{ in } \overline{\Omega},\ \varphi \text{ is } L\text{-periodic w.r.t. } x,\ \nu A(\tilde{e}\lambda\varphi+\nabla\varphi)=0 \text{ on } \partial\Omega\}.$$

Furthermore, $\mu_{\gamma,\zeta}(\lambda)$ is nonincreasing with respect to ζ in the sense that, if $\zeta_1(x,y)\le\zeta_2(x,y)$ for all $(x,y)\in\overline{\Omega}$, then $\mu_{\gamma,\zeta_1}(\lambda)\ge\mu_{\gamma,\zeta_2}(\lambda)$ for all γ and $\lambda\in I\!R$.

3) Let $\gamma \in I\!R$ and $\zeta = \zeta_0$ be constant in $\overline{\Omega}$. There exists a concave function $h : I\!R \to I\!R$ such that $h(0) = h'(0) = 0$ and $\mu_{\gamma,\zeta_0}(\lambda) = -\zeta_0 + \gamma\lambda + h(\lambda)$ for all $\lambda \in I\!R$.

4) Assume that $\zeta(x, y) = 0$ for all $(x, y) \in \overline{\Omega}$. For each $\gamma > 0$ and $\alpha > 0$, there exists a unique positive $\lambda = \lambda^{\alpha,\gamma}$ such that $\mu_{\gamma,0}(\lambda) = \alpha\lambda^2$. Lastly, $\lambda^{\alpha,\gamma}$ is decreasing with respect to $\alpha > 0$ and increasing with respect to $\gamma > 0$.

Proof. It is divided into four steps.

Step 1 : Solving the eigenvalue problem (5.15). This cell-problem is not completely standard because of the periodicity and boundary conditions. We do its proof here for the sake of completeness. Let γ and λ be two given real numbers and let ζ be a continuous function in $\overline{\Omega}$, which is L-periodic with respect to x. For all $\eta > 0$, set $\Omega_\eta = \Omega \cap \{|x| < \eta\}$. Let $\mathcal{H}$ and E be the Banach spaces

$$\begin{aligned} \mathcal{H} &= \{v;\ \forall \eta > 0,\ v \in H^1(\Omega_\eta) \text{ and } \forall k \in \prod_{i=1}^{d} L_i\mathbb{Z},\ v(x+k, y) - v(x, y) = 0 \text{ in } L^2(\Omega)\}, \\ F &= \{v \in C^{0,\delta'}(\overline{\Omega}), v \text{ is } L\text{-periodic w.r.t. } x\} \end{aligned}$$

(with the same δ' as for ζ), embedded with the norms

$$\|v\|_{\mathcal{H}} = \left(\int_C |\nabla v|^2 + v^2\right)^{1/2} \text{ and } \|v\|_F = \|v\|_{C^{0,\delta'}(\overline{C})}.$$

Set $M = M_{\gamma,\zeta}(\lambda) := |\lambda|(\|q\|_\infty + |\gamma|) + \lambda^2 c_2 + \|\zeta\|_\infty + 1$, where c_2 is given in (1.21). Let g be any function in F ($\subset \mathcal{H}'$). Since q and A are L-periodic with respect to x, solving

$$-\text{div}(A\nabla v) - \lambda[\text{div}(A\tilde{e}\ v) + \tilde{e}A\nabla v] + q \cdot \nabla v + [\lambda(q \cdot \tilde{e} + \gamma) - \lambda^2\tilde{e}A\tilde{e} - \zeta + M]v = g$$

in the weak sense in $\mathcal{H}'$, together with the boundary conditions $\nu A(\tilde{e}\lambda v + \nabla v) = 0$ on $\partial\Omega$, is the same as finding a solution $v \in \mathcal{H}$ of

$$\forall z \in \mathcal{H}, \quad \mathcal{B}(v, z) = \int_C gz,$$

where

$$\begin{aligned} \mathcal{B}(v, z) &= \int_C \nabla v\ A\ \nabla z + \lambda(\nabla z A\tilde{e})v - \lambda(\tilde{e}A\nabla v)z \\ &\quad +(q \cdot \nabla v)z + [\lambda(q \cdot \tilde{e} + \gamma) - \lambda^2\tilde{e}A\tilde{e} - \zeta + M]vz. \end{aligned}$$

The bilinear form $\mathcal{B}$ is clearly continuous on $\mathcal{H}$. Let us now check that it is coercive. Choose any $v \in \mathcal{H}$. One has

$$\begin{aligned} \mathcal{B}(v, v) &= \int_C \nabla v\ A\ \nabla v + \lambda(\nabla v A\tilde{e} - \tilde{e}A\nabla v)v \\ &\quad +(q \cdot \nabla v)v + [\lambda(q \cdot \tilde{e} + \gamma) - \lambda^2\tilde{e}A\tilde{e} - \zeta + M]v^2. \end{aligned}$$

As in section 5.1, by integrating by parts the term $\int_C (q \cdot \nabla v)v$ and using (1.19), it is found that this term is zero. Since A is symmetric, it follows then that

$$\mathcal{B}(v, v) = \int_C \nabla v\ A\ \nabla v + [\lambda(q \cdot \tilde{e} + \gamma) - \lambda^2\tilde{e}A\tilde{e} - \zeta + M]v^2.$$

Thus, our choice of M implies

$$\forall v \in \mathcal{H}, \quad \mathcal{B}(v,v) \geq \min(c_1, 1)\|v\|_{\mathcal{H}}^2,$$

where c_1 is given in (1.21). Lax-Milgram theorem yields then the existence of a unique $v \in \mathcal{H}$ such that $\mathcal{B}(v,z) = \int_C gz$ for all $z \in \mathcal{H}$.

Elliptic regularity theory up to the boundary implies that v is a classical $C^{2,\delta'}(\overline{\Omega})$ solution of

$$\left\{\begin{array}{rcll} -\mathrm{div}(A\nabla v) - \lambda[\mathrm{div}(A\tilde{e}\ v) + \tilde{e}A\nabla v] & & & \\ +q\cdot\nabla v + [\lambda(q\cdot\tilde{e}+\gamma) - \lambda^2\tilde{e}A\tilde{e} - \zeta + M]v & = & g & \text{in } \overline{\Omega}, \\ \nu A(\tilde{e}\lambda v + \nabla v) & = & 0 & \text{on } \partial\Omega, \\ v \text{ is } L\text{-periodic w.r.t. } x. & & & \end{array}\right. \tag{5.17}$$

The map $T\ :\ g \in F \mapsto v = Tg \in F$ is linear, and, from standard elliptic estimates, it is compact.

Let now K be the cone $K = \{v \in F,\ v \geq 0\}$. Its interior K° is equal to $K^\circ = \{v \in F,\ v > 0 \text{ in } \overline{\Omega}\} \neq \emptyset$ and $K \cap (-K) = \{0\}$. For each $g \in K^\circ$, the solution $v = Tg$ of (5.17) is globally bounded in $\overline{\Omega}$ and L-periodic with respect to the variable x. We claim that this function v is positive in $\overline{\Omega}$. Indeed, multiply equation (5.17) by $v^- = \max(-v, 0)$ and integrate by parts over C. It follows that

$$-\int_C \nabla v^- A\nabla v^- + [\lambda(q\cdot\tilde{e}+\gamma) - \lambda^2\tilde{e}A\tilde{e} - \zeta + M](v^-)^2 = \int_C gv^-.$$

Our choice of M yields $v^- \equiv 0$, that is to say that v is nonnegative. From Hopf lemma and the strong maximum principle, one concludes that v is positive in $\overline{\Omega}$. Therefore, $T(K^\circ) \subset K^\circ$.

From Krein-Rutman theory, there exists a unique positive real number $\tilde{\mu} = \tilde{\mu}_{\gamma,\zeta}(\lambda)$ and a unique (up to multiplication by positive constants) function $\psi = \psi_{\gamma,\zeta}(\lambda) \in K^\circ$ such that $\tilde{\mu}T\psi = \psi$, *i.e.*,

$$\left\{\begin{array}{rcll} -L_{\gamma,\lambda,\zeta}\psi + M_{\gamma,\zeta}(\lambda)\psi & = & \tilde{\mu}\psi & \text{in } \overline{\Omega}, \\ \nu A(\tilde{e}\lambda\psi + \nabla\psi) & = & 0 & \text{on } \partial\Omega, \\ \psi \text{ is } L\text{-periodic w.r.t. } x. & & & \end{array}\right.$$

The principal eigenvalue $\tilde{\mu}$ depends both on γ and λ. Set $\mu = \mu_{\gamma,\zeta}(\lambda) := \tilde{\mu}_{\gamma,\zeta}(\lambda) - M_{\gamma,\zeta}(\lambda)$. The function $\lambda \mapsto \mu_{\gamma,\zeta}(\lambda)$ is defined on $\mathbb{R}$ and, for each $\lambda \in \mathbb{R}$, the function $\psi_{\gamma,\zeta}(\lambda) := \psi$ is the unique (up to multiplication by positive constants) positive solution of

$$\left\{\begin{array}{rcll} -L_{\gamma,\lambda,\zeta}\psi = -\mathrm{div}(A\nabla\psi) - \lambda[\mathrm{div}(A\tilde{e}\ \psi) + \tilde{e}A\nabla\psi] & & & \\ +q\cdot\nabla\psi + [\lambda(q\cdot\tilde{e}+\gamma) - \lambda^2\tilde{e}A\tilde{e} - \zeta]\psi & = & \mu\psi & \text{in } \overline{\Omega}, \\ \nu A(\tilde{e}\lambda\psi + \nabla\psi) & = & 0 & \text{on } \partial\Omega, \\ \psi \text{ is } L\text{-periodic w.r.t. } x. & & & \end{array}\right. \tag{5.18}$$

Step 2 : Proof of the formula (5.16). Since the function $\psi = \psi_{\gamma,\zeta}(\lambda)$ is in E_λ, it follows that

$$\mu_{\gamma,\zeta}(\lambda) \leq \max_{\{\varphi \in E_\lambda\}} \inf_{\overline{\Omega}} \frac{-L_{\gamma,\lambda,\zeta}\varphi}{\varphi}.$$

Let us now assume that there exists $\varphi \in E_\lambda$ such that $\mu_{\gamma,\zeta}(\lambda) < \inf_{\overline{\Omega}} \frac{-L_{\gamma,\lambda,\zeta}\varphi}{\varphi}$. In other words, one has $-L_{\gamma,\lambda,\zeta}\varphi - \mu_{\gamma,\zeta}(\lambda)\varphi \geq \eta\varphi$ in Ω for some $\eta > 0$. Since both functions φ and ψ are continuous, positive and L-periodic with respect to x in $\overline{\Omega}$, there exists $\tau > 0$ such that $\varphi \geq \tau\psi$ in $\overline{\Omega}$ with equality somewhere. Let $w = \varphi/\psi$. A straightforward calculation leads to

$$\left\{\begin{array}{lll} -\text{div}(A\nabla w) - \dfrac{\nabla\psi}{\psi}\ A\ \nabla w - \nabla w\ A\ \dfrac{\nabla\psi}{\psi} & & \\ \qquad -\lambda(\nabla w\ A\ \tilde{e} + \tilde{e}A\nabla w) + q\cdot\nabla w & \geq \eta w > 0 & \text{in } \Omega, \\ \nu A\nabla w & = 0 & \text{on } \partial\Omega. \end{array}\right.$$

Since $w \geq \tau$ in $\overline{\Omega}$ with equality somewhere, the strong maximum principle together with Hopf lemma yields that $w \equiv \tau$ in $\overline{\Omega}$, *i.e.* $\varphi \equiv \tau\psi$. Putting that into the inequation $-L_{\gamma,\lambda,\zeta}\varphi - \mu_{\gamma,\zeta}(\lambda)\varphi \geq \eta\varphi$ leads to $0 \geq \eta\varphi$ in Ω. The latter is impossible because both η and φ are positive. Therefore, formula (5.16) is proved and the maximum in (5.16) is reached by the function $\psi_{\gamma,\zeta}(\lambda)$.

Because of the definition of the operators $L_{\gamma,\lambda,\zeta}$, formula (5.16) immediately implies that, if $\zeta_1 \leq \zeta_2$ in $\overline{\Omega}$, then $\mu_{\gamma,\zeta_1}(\lambda) \geq \mu_{\gamma,\zeta_2}(\lambda)$ for all γ and $\lambda \in \mathbb{R}$.

Step 3 : Properties of the function $\lambda \mapsto \mu_{\gamma,\zeta}(\lambda)$. Let $\zeta = \zeta_0$ be constant in $\overline{\Omega}$. Let us make the change of functions $\varphi = e^{-\lambda x\cdot e}\tilde{\varphi}$ in formula (5.16). After a straightforward calculation, it is found that

$$-\frac{L_{\gamma,\lambda,\zeta_0}\varphi}{\varphi} = \frac{-\text{div}(A\nabla\tilde{\varphi}) + q\cdot\nabla\tilde{\varphi}}{\tilde{\varphi}} + \gamma\lambda - \zeta_0.$$

Therefore, $\mu_{\gamma,\zeta_0}(\lambda) = -\zeta_0 + \gamma\lambda + h(\lambda)$ where

$$h(\lambda) = \max_{\tilde{\varphi}\in E'_\lambda}\ \inf_{\overline{\Omega}}\ \frac{-\text{div}(A\nabla\tilde{\varphi}) + q\cdot\nabla\tilde{\varphi}}{\tilde{\varphi}}$$

and

$$E'_\lambda = \{\tilde{\varphi} \in C^2(\overline{\Omega}),\ \tilde{\varphi} > 0 \text{ in } \overline{\Omega},\ e^{-\lambda x\cdot e}\tilde{\varphi} \text{ is } L\text{-periodic with respect to } x,\ \nu A\nabla\tilde{\varphi} = 0 \text{ on } \partial\Omega\}.$$

By definition, the function h only depends on λ (and not on γ or ζ_0). Let us now prove that it is concave. Let $\lambda_1,\ \lambda_2 \in \mathbb{R}$, $t \in [0,1]$ and set $\lambda = t\lambda_1 + (1-t)\lambda_2$. One has to check that $h(\lambda) \geq th(\lambda_1) + (1-t)h(\lambda_2)$. Let $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ be arbitrarily chosen in E'_{λ_1} and E'_{λ_2} respectively. Define $z_1 = \ln(\tilde{\varphi}_1)$, $z_2 = \ln(\tilde{\varphi}_2)$, $z = tz_1 + (1-t)z_2$ and $\tilde{\varphi} = e^z$. It is easy to see that $\tilde{\varphi} \in E'_\lambda$. Therefore, it follows from the definition of h that

$$h(\lambda) \geq \inf_{\overline{\Omega}}\ \frac{-\text{div}(A\nabla\tilde{\varphi}) + q\cdot\nabla\tilde{\varphi}}{\tilde{\varphi}}.$$

On the other hand, it is found that

$$\frac{-\text{div}(A\nabla\tilde{\varphi}) + q\cdot\nabla\tilde{\varphi}}{\tilde{\varphi}} = -\text{div}(A\nabla z) - \nabla z\ A\ \nabla z + q\cdot\nabla z$$

and

$$\begin{array}{rcl} \nabla z\ A\ \nabla z & = & t\ \nabla z_1\ A\ \nabla z_1 + (1-t)\ \nabla z_2 A\ \nabla z_2 - t(1-t)(\nabla z_1 - \nabla z_2)A(\nabla z_1 - \nabla z_2) \\ & \leq & t\ \nabla z_1\ A\ \nabla z_1 + (1-t)\ \nabla z_2 A\ \nabla z_2 \end{array}$$

since $t(1-t) \geq 0$. Hence,

$$\begin{array}{rcl} \dfrac{-\mathrm{div}(A\nabla\tilde{\varphi}) + q\cdot\nabla\tilde{\varphi}}{\tilde{\varphi}} & \geq & t[-\mathrm{div}(A\nabla z_1) - \nabla z_1\ A\ \nabla z_1 + q\cdot\nabla z_1] \\ & & +(1-t)[-\mathrm{div}(A\nabla z_2) - \nabla z_2\ A\ \nabla z_2 + q\cdot\nabla z_2] \\ & \geq & t\left(\dfrac{-\mathrm{div}(A\nabla\tilde{\varphi}_1) + q\cdot\nabla\tilde{\varphi}_1}{\tilde{\varphi}_1}\right) + (1-t)\left(\dfrac{-\mathrm{div}(A\nabla\tilde{\varphi}_2) + q\cdot\nabla\tilde{\varphi}_2}{\tilde{\varphi}_2}\right). \end{array}$$

Eventually,

$$\begin{array}{rcl} h(\lambda) & \geq & \inf\limits_{\overline{\Omega}} \dfrac{-\mathrm{div}(A\nabla\tilde{\varphi}) + q\cdot\nabla\tilde{\varphi}}{\tilde{\varphi}} \\ & \geq & t\inf\limits_{\overline{\Omega}} \dfrac{-\mathrm{div}(A\nabla\tilde{\varphi}_1) + q\cdot\nabla\tilde{\varphi}_1}{\tilde{\varphi}_1} + (1-t)\inf\limits_{\overline{\Omega}} \dfrac{-\mathrm{div}(A\nabla\tilde{\varphi}_2) + q\cdot\nabla\tilde{\varphi}_2}{\tilde{\varphi}_2}. \end{array}$$

Since $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ were arbitrary in E'_{λ_1} and E'_{λ_2} respectively, one concludes, by definition of h, that $h(\lambda) \geq th(\lambda_1) + (1-t)h(\lambda_2)$. That shows that h is concave.

An immediate consequence of the concavity of the function h is its continuity. Hence, for any $\gamma \in I\!R$, the function μ_γ is continuous.

Let us now turn to the proof of the other properties of the function $\mu_{\gamma,\zeta_0}(\lambda)$. First, by uniqueness of the solutions of (5.18), one has $\mu_{\gamma,\zeta_0}(0) = -\zeta_0$ and $\psi_{\gamma,\zeta_0}(0) = 1$ (up to multiplication by positive constants). As a consequence, $h(0) = 0$.

Let us now check that $\mu'_{\gamma,\zeta_0}(0) = \gamma$ (which implies that $h'(0) = 0$). Take an arbitrary sequence $\lambda_n \to 0$ as $n \to +\infty$. Let $\psi_n = \psi_{\gamma,\zeta_0}(\lambda_n)$ be the unique positive solution of (5.18) with $\lambda = \lambda_n$, $\zeta = \zeta_0$ and $\mu = \mu_{\gamma,\zeta_0}(\lambda_n)$. Up to normalization, one can assume that $\max_{\overline{\Omega}}\ \psi_n = \max_{\overline{C}}\ \psi_n = 1$. Since, the sequence $(\mu_{\gamma,\zeta_0}(\lambda_n))$ is bounded (actually, $\mu_{\gamma,\zeta_0}(\lambda_n) \to \mu_{\gamma,\zeta_0}(0) = -\zeta_0$), standard elliptic estimates and Sobolev injections imply that the functions ψ_n converge locally (and then uniformly in $\overline{\Omega}$ by periodicity) in $C^{2,\mu}$ (for all $0 \leq \mu < 1$) to a nonnegative function ψ such that $\max_{\overline{C}}\ \psi = 1$ and solving

$$\left\{ \begin{array}{rcll} -\mathrm{div}(A\nabla\psi) + q\cdot\nabla\psi & = & 0 & \text{in } \overline{\Omega} \\ \nu A\nabla\psi & = & 0 & \text{on } \partial\Omega \\ \psi \text{ is } L\text{-periodic w.r.t. } x. & & & \end{array} \right.$$

The strong maximum principle and Hopf lemma imply that ψ is positive in $\overline{\Omega}$ and by uniqueness (up to multiplication) of the positive solutions of (5.18), one gets that $\psi = \psi_{\gamma,\zeta_0}(0) \equiv 1$. On the other hand, integrating by parts over C the equation (5.18) satisfied by ψ_n with λ_n and $\mu_n = \mu_{\gamma,\zeta_0}(\lambda_n)$ leads to

$$-\lambda_n \int_C \tilde{e}A\nabla\psi_n + \int_C [\lambda_n(q\cdot\tilde{e} + \gamma) - \lambda_n^2 \tilde{e}A\tilde{e} - \zeta_0]\psi_n = \mu_n \int_C \psi_n.$$

In other words,

$$\begin{aligned}\frac{\mu_n+\zeta_0}{\lambda_n} &= \frac{-\displaystyle\int_C \tilde{e}A\nabla\psi_n + \int_C (q\cdot\tilde{e}+\gamma-\lambda_n\tilde{e}A\tilde{e})\psi_n}{\displaystyle\int_C \psi_n} \\ &\to |C|^{-1}\int_C (q\cdot\tilde{e}+\gamma) \ \text{ as } n\to+\infty \ \text{ (since } \psi_n\to 1 \text{ in } C^2(\overline{\Omega})) \\ &= \gamma \ \text{ (from (1.19)).}\end{aligned}$$

Thus, the function μ_{γ,ζ_0} is differentiable at 0 and $\mu'_{\gamma,\zeta_0}(0)=\gamma$, which implies that $h'(0)=0$.

Step 4 : Solving $\mu_{\gamma,0}(\lambda)=\alpha\lambda^2$. Assume $\zeta=0$ and let γ and α be two given positive numbers. From Step 3, it immediately follows that the equation $\mu_{\gamma,0}(\lambda)=\alpha\lambda^2$ has two and only two solutions : 0 and a positive real number denoted by $\lambda^{\alpha,\gamma}$. Furthermore, $\mu_{\gamma,0}(\lambda)>\alpha\lambda^2$ for all $\lambda\in(0,\lambda^{\alpha,\gamma})$ and $\mu_{\gamma,0}(\lambda)<\alpha\lambda^2$ for all $\lambda\in(-\infty,0)\cup(\lambda^{\alpha,\gamma},+\infty)$.

Let now $\gamma>0$ and $0<\alpha_1<\alpha_2$ be given. One has $\mu_{\gamma,0}(\lambda^{\alpha_2,\gamma})=\alpha_2(\lambda^{\alpha_2,\gamma})^2>\alpha_1(\lambda^{\alpha_2,\gamma})^2$. Therefore, $\lambda^{\alpha_2,\gamma}<\lambda^{\alpha_1,\gamma}$. In other words, for each $\gamma>0$, $\lambda^{\alpha,\gamma}$ is decreasing with respect to $\alpha>0$.

Let now $0<\gamma_1<\gamma_2$ and $\alpha>0$ be given. From the formula $\mu_{\gamma,0}(\lambda)=\gamma\lambda+h(\lambda)$, it follows that $\mu_{\gamma_2,0}(\lambda)=\mu_{\gamma_1,0}(\lambda)+(\gamma_2-\gamma_1)\lambda$. In particular, $\mu_{\gamma_2,0}(\lambda^{\alpha,\gamma_1})>\mu_{\gamma_1,0}(\lambda^{\alpha,\gamma_1})=\alpha(\lambda^{\alpha,\gamma_1})^2$, whence $\lambda^{\alpha,\gamma_1}<\lambda^{\alpha,\gamma_2}$. That means that, for each $\alpha>0$, $\lambda^{\alpha,\gamma}$ is increasing with respect to $\gamma>0$.

That completes the proof of Proposition 5.7. □

5.3 Passage to the limit in the infinite cylinder

In this section, making use of the results of Propositions 5.6 and 5.7 in the previous two sections, we pass to the limit $a\to+\infty$ in the infinite cylinder $\mathbb{R}\times\overline{\Omega}$ for the solutions $\phi^{\varepsilon,a}$ of (5.1), (5.12).

Proposition 5.8 *Under the notations of Proposition 5.6, one has*

$$\forall\varepsilon>0,\quad 0<c^\varepsilon:=\liminf_{a\to+\infty,\ a\geq a_0} c^{\varepsilon,a}\leq K.$$

Proof. From Proposition 5.6, one knows that $|c^\varepsilon|\leq K$. Take a sequence $a^n\to+\infty$ such that $c^{\varepsilon,a^n}\to c^\varepsilon$ as $n\to+\infty$.

Let $\phi^n:=\phi^{\varepsilon,a^n}$ be the solution of (5.1) satisfying the normalization conditon (5.12). From Lemma 5.2, each function ϕ^n is continuous in $\overline{\Sigma}_{a^n}$ and increasing with respect to s. Therefore, there exists a unique real number $s^n\in(0,a^n)$ such that

$$\max_{\overline{C}}\phi^n(s^n,\cdot,\cdot)=\max_{\overline{\Omega}}\phi^n(s^n,\cdot,\cdot)=\frac{1+\theta}{2}. \tag{5.19}$$

Let now V^n be the function defined by

$$V^n(s,x,y)=\phi^n(s+s^n,x,y)\ \text{ for all } s\in[-a^n-s^n,a^n-s^n],\ (x,y)\in\overline{\Omega}.$$

Let us pass to the limit $n \to +\infty$. Up to extraction of some subsequence, two cases may occur :

case 1 : $a^n - s^n \to +\infty$ as $n \to +\infty$. From standard elliptic estimates and Sobolev's injections, the functions V^n converge (up to extraction of some subsequence) in $C^{2,\alpha}_{loc}(I\!\!R \times \overline{\Omega})$, for $0 < \alpha < 1$, to a function V satisfying

$$\left\{\begin{array}{rcl} (\tilde{e}A\tilde{e}+\varepsilon)V_{ss}+\text{div}_{x,y}(A\nabla_{x,y}V)+\text{div}_{x,y}(A\tilde{e}\ V_s) & & \\ +\partial_s(\tilde{e}A\nabla_{x,y}V)-q\cdot\nabla_{x,y}V-(q\cdot\tilde{e}+c^\varepsilon)V_s+f(x,y,V) & = & 0 \text{ in } I\!\!R\times\overline{\Omega}, \\ \nu A(\tilde{e}V_s+\nabla_{x,y}V) & = & 0 \text{ on } I\!\!R\times\partial\Omega, \\ V \text{ is } L\text{-periodic w.r.t. } x, & & \\ \max_{\overline{C}} V(0,\cdot,\cdot)\ =\ \max_{\overline{\Omega}} V(0,\cdot,\cdot) & = & \dfrac{1+\theta}{2}, \\ V \text{ is nondecreasing w.r.t. } s. & & \end{array}\right. \tag{5.20}$$

From standard elliptic estimates and from the monotonicity of V, it follows that

$$V(s,x,y) \to \psi_\pm(x,y) \text{ in } C^{2,\alpha}(\overline{\Omega}) \text{ as } s \to \pm\infty,$$

where the functions $\psi_\pm(x,y)$ satisfy

$$\left\{\begin{array}{rcl} \text{div}(A\nabla_{x,y}\psi_\pm)-q\cdot\nabla\psi_\pm+f(x,y,\psi_\pm) & = & 0 \quad \text{in } \overline{\Omega}, \\ \nu A\nabla\psi_\pm & = & 0 \quad \text{on } \partial\Omega, \\ \psi_\pm \text{ is } L\text{-periodic w.r.t. } x. & & \end{array}\right. \tag{5.21}$$

Integrating by parts these equations over the period cell C leads to $\displaystyle\int_C f(x,y,\psi_\pm(x,y))\,dxdy = 0$. Since $f \geq 0$, it follows that $f(x,y,\psi_\pm(x,y)) \equiv 0$. By multiplying the equations (5.21) by $\psi_\pm$ and by integrating by parts over C, it is found that $\displaystyle\int_C \nabla\psi_\pm A\nabla\psi_\pm = 0$. As a consequence, both $\psi_\pm$ are constant and satisfy $f(x,y,\psi_\pm) = 0$ for all $(x,y) \in \overline{\Omega}$. Because of (1.26) and because of the choice of the normalization for V on $\{0\} \times \overline{\Omega}$, one gets $\psi_- \in [0,\theta]$ and $\psi_+ = 1$.

As it was done in the course of the proof of Lemma 3.1, one can now integrate by parts, over $[-B,B] \times C$, the equation (5.20) satisfied by V. One obtains

$$\int_C [\ (\tilde{e}A\tilde{e}+\varepsilon)V_s+\tilde{e}A\nabla_{x,y}V-(q\cdot\tilde{e}+c^\varepsilon)V\ ]_{-B}^{B}+\int_{[-B,B]\times C} f(x,y,V)=0. \tag{5.22}$$

Since V converges to two constants $\psi_\pm$ as $s \to \pm\infty$ in $C^2_{loc}(I\!\!R \times \overline{\Omega})$, it follows that $V_s \to 0$ and $\nabla_{x,y}V \to 0$ as $s \to \pm\infty$, uniformly in (x,y). The passage to the limit $B \to +\infty$ in equality (5.22) yields that the function $(s,x,y) \mapsto f(x,y,V(s,x,y))$ is integrable over the whole cylinder $I\!\!R \times C$ and that

$$-c^\varepsilon|C|(1-\psi_-)+\int_{I\!\!R\times C} f(x,y,V(s,x,y))\ ds\ dx\ dy = 0.$$

On the other hand, the continuous function $(s,x,y) \mapsto f(x,y,V(s,x,y))$ is nonnegative, L-periodic with respect to x and it is not identically equal to 0 because of the normalization for V. Therefore, $\displaystyle\int_{I\!\!R\times C} f(x,y,V(s,x,y))\ dsdxdy > 0$ and $c^\varepsilon > 0$.

case 2 : $a^n - s^n \to b \in [0,+\infty)$. One just has to slightly modify the above arguments of case 1. Up to extraction of some subsequence, the functions V^n converge in $C^{2,\alpha}_{loc}((-\infty,b)\times\overline{\Omega})$ to a function V solving (5.20) in the set $(-\infty,b)\times\overline{\Omega}$. Furthermore, the functions ϕ^n are equi-Lipschitz-continuous in any set of the type $[a^n-1,a^n]\times\mathcal{K}$ for any compact subset $\mathcal{K}\subset\Omega$. Therefore, for all $\eta>0$, there exists $\kappa>0$ such that

$$\forall n,\quad \forall s\in[a^n-\kappa,a^n],\quad 1-\eta\le\max_{\overline{C}}\phi^n(s,\cdot,\cdot)\le 1.$$

Because of (5.19), it follows then that $s^n\le a^n-\delta$ for some $\delta>0$, whence $b\ge\delta$ and $\max_{\overline{C}}V(0,\cdot,\cdot)=\max_{\overline{\Omega}}V(0,\cdot,\cdot)=(1+\theta)/2$. Furthermore, the same arguments as above imply that V can be extended by continuity on $\{b\}\times\overline{\Omega}$ with $V(b,\cdot,\cdot)=1$ and from standard elliptic estimates up to the boundary, the function V is actually $C^1((-\infty,b]\times\Omega)$.

As it was done in case 1, one can easily prove that $V(-\infty,\cdot,\cdot)$ is equal to a constant $\psi_-\in[0,\theta]$. Integrating the equation satisfied by V over $[-B,b]\times C$ and passing to the limit $B\to+\infty$ leads to

$$\int_C(\tilde{e}A\tilde{e}+\varepsilon)V_s(b,\cdot,\cdot)\ -\ c^\varepsilon|C|(1-\psi_-)+\int_{(-\infty,b)\times C}f(x,y,V)=0.$$

Since $V_s\ge0$ in $(-\infty,b)\times\Omega$ and the continuous function $(s,x,y)\mapsto f(x,y,V(s,x,y))$ is nonnegative and not identically equal to 0 on $(-\infty,b)\times\overline{C}$, one concludes as in case 1 that $c^\varepsilon>0$.□

As above, consider now a sequence $a^n\to+\infty$ such that $c^{\varepsilon,a^n}\to c^\varepsilon(>0)$ and let $\phi^n:=\phi^{\varepsilon,a^n}$. The following proposition deals with the passage to the limit $n\to+\infty$ for the functions ϕ^n.

Proposition 5.9 *Up to extraction of some subsequence, the functions ϕ^n converge in $C^{2,\alpha}_{loc}(\mathbb{R}\times\overline{\Omega})$ (for $0<\alpha<1$) to a function ϕ^ε such that*

$$\left\{\begin{array}{rcll}(\tilde{e}A\tilde{e}+\varepsilon)\phi^\varepsilon_{ss}+\mathrm{div}_{x,y}(A\nabla_{x,y}\phi^\varepsilon)+\mathrm{div}_{x,y}(A\tilde{e}\ \phi^\varepsilon_s) & & & \\ +\partial_s(\tilde{e}A\nabla_{x,y}\phi^\varepsilon)-q\cdot\nabla_{x,y}\phi^\varepsilon-(q\cdot\tilde{e}+c^\varepsilon)\phi^\varepsilon_s+f(x,y,\phi^\varepsilon) & = & 0 & \text{in } \mathbb{R}\times\overline{\Omega},\\ \nu A(\tilde{e}\phi^\varepsilon_s+\nabla_{x,y}\phi^\varepsilon) & = & 0 & \text{on } \mathbb{R}\times\partial\Omega,\\ \phi^\varepsilon \text{ is } L\text{-periodic w.r.t. } x, & & & \\ \max_{\overline{C}}\phi^\varepsilon(0,\cdot,\cdot)\ =\ \max_{\overline{\Omega}}\phi^\varepsilon(0,\cdot,\cdot) & = & \theta, & \\ \phi^\varepsilon \text{ is increasing with respect to } s. & & & \end{array}\right. \tag{5.23}$$

Furthermore, $\phi^\varepsilon(-\infty,\cdot,\cdot)=0$ and $\phi^\varepsilon(+\infty,\cdot,\cdot)=1$.

Proof. The convergence of the functions ϕ^n to a function ϕ^ε in $C^{2,\alpha}_{loc}(\mathbb{R}\times\overline{\Omega})$ follows from standard elliptic estimates. Furthermore, the function ϕ^ε is nondecreasing with respect to the variable s since each function ϕ^n is increasing in s.

The only thing that remains to be proved is that $\phi^\varepsilon(-\infty,\cdot,\cdot)=0$ and $\phi^\varepsilon(+\infty,\cdot,\cdot)=1$. Assume temporarily that has been proved. For any $h>0$, the function $\phi^\varepsilon(s+h,x,y)-\phi^\varepsilon(s,x,y)$ is a nonnegative solution of a linear elliptic equation with bounded coefficients. It follows then from the strong maximum principle and Hopf lemma that $\phi^\varepsilon(s+h,x,y)-\phi^\varepsilon(s,x,y)>0$ for all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$, which proves that the function ϕ^ε is increasing in s.

Let us now prove that $\phi^\varepsilon(-\infty,\cdot,\cdot)=0$. Observe first that since each ϕ^n is increasing with respect to s and $\max_{\overline{\Omega}}\phi^n(0,\cdot,\cdot)=\theta$, one has $\phi^n(s,x,y)\le\theta$ for all $s\in[-a^n,0]$ and $(x,y)\in\overline{\Omega}$. In $(-a^n,0)\times\overline{\Omega}$, the function ϕ^n satisfies

$$\left\{\begin{array}{rcl}(\tilde{e}A\tilde{e}+\varepsilon)\phi^n_{ss}+\mathrm{div}_{x,y}(A\nabla_{x,y}\phi^n)+\mathrm{div}(A\tilde{e}\ \phi^n_s) & & \\ +\partial_s(\tilde{e}A\nabla_{x,y}\phi_n)-q\cdot\nabla_{x,y}\phi^n-(q\cdot\tilde{e}+c^{\varepsilon,a^n})\phi^n_s & = & 0\ \text{ in } (-a^n,0)\times\overline{\Omega}\\ \nu A(\tilde{e}\phi^n_s+\nabla_{x,y}\phi^n) & = & 0\ \text{ on } (-a^n,0)\times\partial\Omega.\end{array}\right. \tag{5.24}$$

On the other hand, from Proposition 5.8 and since $c^{\varepsilon,a^n}\to c^\varepsilon>0$ as $a^n\to+\infty$, one has $\frac{3}{2}c_\varepsilon\ge c^{\varepsilon,a^n}\ge\frac{1}{2}c^\varepsilon>0$ for n large enough. In the sequel, choose n large enough such that the latter holds. Proposition 5.7, part 4), applied to $\alpha=\varepsilon>0$ and $\gamma=c^{\varepsilon,a^n}>0$ yields the existence of a positive real number $\lambda_n=\lambda^{\varepsilon,\ c^{\varepsilon,a^n}}$ and a positive function $\psi_n=\psi^{\varepsilon,\ c^{\varepsilon,a^n}}$ solving (5.18) with $(\gamma,\lambda,\zeta,\mu)=(c^{\varepsilon,a^n},\lambda_n,0,\varepsilon\lambda_n^2)$. In other words, the function $e^{\lambda_n s}\psi_n(x,y)$ solves (5.24) in $I\!R\times\Omega$. Since $\lambda^{\alpha,\gamma}$ is increasing with respect to $\gamma>0$, it follows that

$$0<\lambda^{\varepsilon,\ c^\varepsilon/2}\le\lambda_n\le\lambda^{\varepsilon,\ 3c^\varepsilon/2}.$$

Up to extraction of some subsequence, one can then assume that $\lambda_n\to\lambda_\varepsilon\in[\lambda^{\varepsilon,\ c^\varepsilon/2},\lambda^{\varepsilon,\ 3c^\varepsilon/2}]$.

On the other hand, one can assume, up to multiplication, that

$$\max_{\overline{\Omega}}\psi_n=\max_{\overline{C}}\psi_n=1.$$

One now claims that there exists a constant $\alpha_\varepsilon>0$ such that

$$\forall n,\ \forall(x,y)\in\overline{\Omega},\quad 0<\alpha_\varepsilon\le\psi_n(x,y)\le 1. \tag{5.25}$$

Indeed, if that were not true, then, since the functions ψ_n are L-periodic with respect to x, there would exist, at least for some subsequence, some points $(x_n,y_n)\in\overline{C}$ such that $\psi_n(x_n,y_n)\to 0$ as $n\to+\infty$. On the other hand, the functions ψ_n are bounded between 0 and 1 and solve the elliptic equations (5.18) with $\gamma=c^{\varepsilon,a^n}\in[-K,K]$, $\lambda=\lambda_n\in[\lambda^{\varepsilon,\ c^\varepsilon/2},\lambda^{\varepsilon,\ 3c^\varepsilon/2}]$, $\zeta=0$ and $\mu=\varepsilon\lambda_n^2$. Therefore, up to extraction of some subsequence, the positive functions ψ_n converge in $C^{2,\alpha}_{loc}(\overline{\Omega})$ to a nonnegative function ψ solving (5.18) with $(\gamma,\lambda,\zeta,\mu)=(c^\varepsilon,\lambda_\varepsilon,0,\varepsilon\lambda_\varepsilon^2)$. Furthermore, $0\le\psi\le 1$ in $\overline{\Omega}$ and one can also assume that

$$\exists\ (\underline{x},\underline{y}),\ (\overline{x},\overline{y})\in\overline{\Omega},\quad \psi(\underline{x},\underline{y})=0 \text{ and } \psi(\overline{x},\overline{y})=1.$$

The strong maximum principle implies that $(\underline{x},\underline{y})\in\Omega$ and the Hopf lemma then yields that $\psi\equiv 0$, which is impossible. Therefore, the proof of the claim (5.25) is complete.

Let now $\tilde{\psi}_n$ be the function $\tilde{\psi}_n=\beta_n\psi_n$ such that

$$\min_{\overline{\Omega}}\tilde{\psi}_n=\min_{\overline{C}}\tilde{\psi}_n=\theta.$$

It follows from (5.25) that $\theta\le\beta_n\le\theta/\alpha_\varepsilon$, whence $\tilde{\psi}_n(x,y)\le\theta/\alpha_\varepsilon$ for all $(x,y)\in\overline{\Omega}$ and for all n. Furthermore, as done above, the functions $\tilde{\psi}_n$ converge, up to extraction of some subsequence, to a positive function $\tilde{\psi}^\varepsilon$ solving (5.18) with $(\gamma,\lambda,\zeta,\mu)=(c^\varepsilon,\lambda_\varepsilon,0,\varepsilon\lambda_\varepsilon^2)$. One can also say that $\min_{\overline{\Omega}}\tilde{\psi}^\varepsilon=\min_{\overline{C}}\tilde{\psi}^\varepsilon=\theta$ and $\max_{\overline{\Omega}}\tilde{\psi}^\varepsilon\le\theta/\alpha_\varepsilon$.

Since ϕ^n and $w_n(s,x,y) = e^{\lambda_n s}\tilde{\psi}_n(x,y)$ are both classical solutions of the same elliptic equation (5.24), with no zero-order term, in $(-a^n, 0) \times \Omega$ with the same L-periodicity with respect to x and the same boundary condition $\nu A(\nabla_{x,y}\phi^n + \tilde{e}\phi^n_s) = \nu A[\nabla_{x,y}w_n + \tilde{e}(w_n)_s]$ on $(-a^n, 0) \times \partial\Omega$, and since $\phi^n \le \theta \le w_n$ for $s = 0$ and $\phi^n = 0 \le w_n$ for $s = -a^n$, one concludes from the maximum principle and Hopf lemma that

$$\phi^n(s,x,y) \le e^{\lambda_n s}\tilde{\psi}_n(x,y) \text{ for all } s \in [-a^n, 0],\ (x,y) \in \overline{\Omega}.$$

The passage to the limit $n \to +\infty$ leads to

$$\forall s \le 0,\ \forall (x,y) \in \overline{\Omega}, \quad \phi^\varepsilon(s,x,y) \le e^{\lambda_\varepsilon s}\tilde{\psi}^\varepsilon(x,y), \tag{5.26}$$

where $\lambda_\varepsilon \in [\lambda^{\varepsilon,\, c^\varepsilon/2}, \lambda^{\varepsilon,\, 3c^\varepsilon/2}]$ and $\min_{\overline{C}} \tilde{\psi}^\varepsilon = \theta$, $\max_{\overline{C}} \tilde{\psi}^\varepsilon \le \theta/\alpha_\varepsilon$. Hence, $\phi^\varepsilon(-\infty,\cdot,\cdot) = 0$.

Lastly, as done in Proposition 5.8, there exists a function ψ_+ such that $\phi^\varepsilon(s,x,y) \to \psi^+(x,y)$ as $s \to +\infty$. Moreover, ψ_+ is constant and $f(x,y,\psi_+) = 0$ for all $(x,y) \in \overline{\Omega}$. Since $\max_{\overline{\Omega}} \phi^\varepsilon(0,\cdot,\cdot) = \theta$ and ϕ^ε is nondecreasing with respect to s, it follows from (1.26) that $\psi_+ \equiv \theta$ or $\psi_+ \equiv 1$. If $\psi_+ \equiv \theta$, then the strong mamimum principle implies that $\phi^\varepsilon \equiv \theta$ for all (s,x,y), which is impossible because $\phi^\varepsilon(-\infty,\cdot,\cdot) = 0$. As a consequence, $\psi_+ \equiv 1$ and the proof of Proposition 5.9 is complete. □

5.4 Passage to the limit $\varepsilon \to 0$

The last step in the proof of Theorem 1.13 consists in passing to the limit $\varepsilon \to 0$ for the functions ϕ^ε. The two key points are to prove that the real numbers c^ε are bounded from below by a positive constant and that the solution ϕ obtained at the limit is not trivial. The latter follows from the comparison with exponentially decaying functions, from the monotonicity with respect to s and from uniform (independent of ε) L^∞ *a priori* estimates of the (x,y)-gradient of ϕ^ε.

The positiveness of the limit of c^ε is the purpose of the following

Proposition 5.10 *Under the notations of Proposition 5.8, one has*

$$0 < \liminf_{\varepsilon \to 0} c^\varepsilon \le K.$$

Before doing the proof of this proposition, let us first state an auxiliary lemma :

Lemma 5.11 *Let $u^\varepsilon(t,x,y)$ be the function defined for all $t \in \mathbb{R}$ and $(x,y) \in \overline{\Omega}$ by $u^\varepsilon(t,x,y) = \phi^\varepsilon(x \cdot e + c^\varepsilon t, x, y)$, where ϕ^ε is given in Proposition 5.9. For any compact subset $\mathcal{K}$ of $\overline{\Omega}$, there exists a constant $C(\mathcal{K})$, only depending on $\mathcal{K}$, such that*

$$\forall \varepsilon > 0,\ \int_{\mathbb{R}\times\mathcal{K}} \left[(u^\varepsilon_t)^2 + |\nabla_{x,y}u^\varepsilon|^2\right] \, dtdxdy \le C(\mathcal{K}) \left(\frac{1 + N\|q\|^2_\infty}{2c_1} + 2 \max_{(x,y)\in\overline{\Omega}} F(x,y,1)\right) \tag{5.27}$$

where $F(x,y,t) = \int_0^t f(x,y,\tau)d\tau$ and c_1 is given in (1.21).

Proof. For each ε, under the notation of Proposition 5.9, the function $\phi^\varepsilon(s,x,y)$ is a classical solution of (5.23). First, as it was done in Lemma 3.1, the equality (3.7) holds, that is to say that the nonnegative function $(s,x,y)\mapsto f(x,y,\phi^\varepsilon(s,x,y))$ is integrable over $I\!R\times C$ and that

$$c^\varepsilon|C| = \int_{I\!R\times C} f(x,y,\phi^\varepsilon(s,x,y))\;ds\;dx\;dy. \tag{5.28}$$

Next, multiply the equation (5.23) by ϕ^ε and integrate by parts over $(-B,B)\times C$, where B is an arbitrary positive number. By using the periodicity and boundary conditions in (1.19), (1.22) and (5.23), it follows that

$$\begin{aligned}-\int_{(-B,B)\times C}&(\tilde e A\tilde e+\varepsilon)(\phi^\varepsilon_s)^2+\nabla_{x,y}\phi^\varepsilon A\nabla_{x,y}\phi^\varepsilon+(\nabla_{x,y}\phi^\varepsilon A\tilde e+\tilde eA\nabla_{x,y}\phi^\varepsilon)\phi^\varepsilon_s\\ &+\int_C\Big[(\tilde eA\tilde e+\varepsilon)\phi^\varepsilon_s\phi^\varepsilon+(\tilde eA\nabla_{x,y}\phi^\varepsilon)\phi^\varepsilon-\frac12(q\cdot\tilde e+c^\varepsilon)(\phi^\varepsilon)^2\Big]_{-B}^{B}\\ &\qquad\qquad+\int_{(-B,B)\times C}f(x,y,\phi^\varepsilon)\phi^\varepsilon \;=\;0.\end{aligned}$$

From standard elliptic estimates, one knows that $\nabla_{s,x,y}\phi^\varepsilon\to0$ as $s\to\pm\infty$ uniformly in $(x,y)\in\overline{\Omega}$. Moreover, the term $\int_C\Big[\frac12(q\cdot\tilde e+c^\varepsilon)(\phi^\varepsilon)^2\Big]_{-B}^{B}$ converges to $(1/2)c^\varepsilon|C|$ as $B\to+\infty$ since the first d components of q have zero mean value on C. On the other hand,

$$\begin{aligned}(\tilde eA\tilde e+\varepsilon)(\phi^\varepsilon_s)^2+\nabla_{x,y}\phi^\varepsilon A\nabla_{x,y}\phi^\varepsilon+(\nabla_{x,y}\phi^\varepsilon A\tilde e+\tilde eA\nabla_{x,y}\phi^\varepsilon)\phi^\varepsilon_s&\\ =(\nabla_{x,y}\phi^\varepsilon+\tilde e\phi^\varepsilon_s)A(\nabla_{x,y}\phi^\varepsilon+\tilde e\phi^\varepsilon_s)+\varepsilon(\phi^\varepsilon_s)^2\ge0&\end{aligned}$$

and the integral $\int_{I\!R\times C}f(x,y,\phi^\varepsilon)\phi^\varepsilon$ converges since $0\le f(x,y,\phi^\varepsilon)\phi^\varepsilon\le f(x,y,\phi^\varepsilon)$ and $\int_{I\!R\times C}f(x,y,\phi^\varepsilon)$ converges. Therefore, one concludes that $\int_{I\!R\times C}(\nabla_{x,y}\phi^\varepsilon+\tilde e\phi^\varepsilon_s)A(\nabla_{x,y}\phi^\varepsilon+\tilde e\phi^\varepsilon_s)+\varepsilon(\phi^\varepsilon_s)^2$ converges and that

$$\begin{aligned}\frac12c^\varepsilon|C|+\int_{I\!R\times C}&(\nabla_{x,y}\phi^\varepsilon+\tilde e\phi^\varepsilon_s)A(\nabla_{x,y}\phi^\varepsilon+\tilde e\phi^\varepsilon_s)+\varepsilon(\phi^\varepsilon_s)^2\\ &=\int_{I\!R\times C}f(x,y,\phi^\varepsilon)\phi^\varepsilon\le\int_{I\!R\times C}f(x,y,\phi^\varepsilon)=c^\varepsilon|C|.\end{aligned}$$

Because of (1.21), it follows in particular that

$$\int_{I\!R\times C}|\nabla_{x,y}\phi^\varepsilon+\tilde e\phi^\varepsilon_s|^2\;ds\;dx\;dy\le\frac{1}{2c_1}c^\varepsilon|C|. \tag{5.29}$$

Let us now multiply the equation (5.23) by ϕ^ε_s and integrate by parts over $(-B,B)\times C$. One obtains

$$\begin{aligned}\int_C[(\tilde eA\tilde e+\varepsilon)(\phi^\varepsilon_s)^2\;/2]_{-B}^{B}&-\int_{(-B,B)\times C}\nabla_{x,y}\phi^\varepsilon_s\;A\;\nabla_{x,y}\phi^\varepsilon\\ +\int_{(-B,B)\times C}(-\nabla_{x,y}\phi^\varepsilon_sA\tilde e+\tilde eA\nabla_{x,y}\phi^\varepsilon_s)\phi^\varepsilon_s&-\int_{(-B,B)\times C}(q\cdot\nabla_{x,y}\phi^\varepsilon)\;\phi^\varepsilon_s\\ -\int_{(-B,B)\times C}(q\cdot\tilde e+c^\varepsilon)(\phi^\varepsilon_s)^2&+\int_C[F(x,y,\phi^\varepsilon)]_{-B}^{B}\;=\;0.\end{aligned}\tag{5.30}$$

As already underlined, since $\phi_s^\varepsilon \to 0$ as $s \to \pm\infty$ uniformly in (x,y), the first integral in (5.30) approaches 0 as $B \to +\infty$. On the other hand, since A is symmetric, the third integral in (5.30) is zero; moreover, the second integral can be rewritten as

$$\begin{aligned}\int_{(-B,B)\times C} \nabla_{x,y}\phi_s^\varepsilon \; A \; \nabla_{x,y}\phi^\varepsilon &= \int_{(-B,B)\times C} \partial_s \left(\frac{1}{2}\nabla_{x,y}\phi^\varepsilon \; A \; \nabla_{x,y}\phi^\varepsilon\right) \\ &= \int_C \left[\frac{1}{2}\nabla_{x,y}\phi^\varepsilon \; A \; \nabla_{x,y}\phi^\varepsilon\right]_{-B}^{B} \to 0 \;\text{ as } B \to +\infty.\end{aligned}$$

Therefore, it follows from (5.30) that

$$\begin{aligned}c^\varepsilon \int_{(-B,B)\times C} (\phi_s^\varepsilon)^2 &= \int_C [F(x,y,\phi^\varepsilon)]_{-B}^{B} - \int_{(-B,B)\times C} q\cdot(\nabla_{x,y}\phi^\varepsilon + \tilde{e}\phi_s^\varepsilon)\; \phi_s^\varepsilon \; + \eta(B) \\ &\le \int_C F(x,y,1) \; + \; \int_{(-B,B)\times C} \frac{\|q\|_\infty}{2}\left(\alpha|\nabla_{x,y}\phi^\varepsilon + \tilde{e}\phi_s^\varepsilon|^2 + \frac{N}{\alpha}(\phi_s^\varepsilon)^2\right) \; + \eta(B)\end{aligned}$$

where $\eta(B) \to 0$ as $B \to +\infty$ and α denotes any positive real number. If $\|q\|_\infty = 0$, it follows that $c^\varepsilon \int_{I\!R\times C} (\phi_s^\varepsilon)^2 \le \int_C F(x,y,1)$ (actually, the equality holds in this case). In the general case where $\|q\|_\infty > 0$, then the choice $\alpha = N\|q\|_\infty / c^\varepsilon > 0$ leads to

$$\frac{c^\varepsilon}{2}\int_{I\!R\times C} (\phi_s^\varepsilon)^2 \le \int_C F(x,y,1) + \int_{I\!R\times C} \frac{N\|q\|_\infty^2}{2c^\varepsilon}|\nabla_{x,y}\phi^\varepsilon + \tilde{e}\phi_s^\varepsilon|^2$$

by passing to the limit $B \to +\infty$. Combining that together with (5.29) gives :

$$\frac{c^\varepsilon}{2}\int_{I\!R\times C} (\phi_s^\varepsilon)^2 \le \int_C F(x,y,1) \; + \; \frac{N}{4c_1}\|q\|_\infty^2 |C|. \tag{5.31}$$

Note that the latter also holds in the case $\|q\|_\infty = 0$.

Now, multiply (5.31) by $2c^\varepsilon > 0$ and add (5.29). By using the fact that each function ϕ^ε is L-periodic with respect to x, it follows that, for any compact subset $\mathcal{K}$ of $\overline{\Omega}$, there exists a constant $C(\mathcal{K})$ (depending only on $\mathcal{K}$) such that, for all $\varepsilon > 0$,

$$\int_{I\!R\times\mathcal{K}} \left[(c^\varepsilon\phi_s^\varepsilon)^2 + |\nabla_{x,y}\phi^\varepsilon + \tilde{e}\phi_s^\varepsilon|^2\right] \, ds\, dx\, dy \le c^\varepsilon C(\mathcal{K}) \left(\frac{1+N\|q\|_\infty^2}{2c_1} + 2\max_{(x,y)\in\overline{\Omega}} F(x,y,1)\right).$$

By making the change of variables $(s,x,y) = (c^\varepsilon t + x\cdot e, x, y)$ and coming back to the functions $u^\varepsilon(t,x,y) = \phi^\varepsilon(c^\varepsilon t + x\cdot e, x, y)$, one eventually gets the estimates (5.27) and the proof of Lemma 5.11 is complete. □

Let us now turn to

Proof of Proposition 5.10. For each $\varepsilon > 0$, one has $0 < c^\varepsilon \le K$ from Proposition 5.8. Assume the conclusion of Proposition 5.10 does not hold. There exists then a sequence $\varepsilon_n \to 0^+$ such that $c^{\varepsilon_n} \to 0^+$ as $n \to +\infty$. In the sequel, for the sake of simplicity, we drop the index n.

For each ε, under the notation of Proposition 5.9, the function $\phi^\varepsilon(s,x,y)$ is a solution of (5.23). The function $u^\varepsilon(t,x,y)=\phi^\varepsilon(x\cdot e+c^\varepsilon t,x,y)$ is a classical solution of

$$\begin{cases} \dfrac{\varepsilon}{(c^\varepsilon)^2}\partial_{tt}u^\varepsilon+\mathrm{div}(A\nabla u^\varepsilon)-\partial_t u^\varepsilon-q\cdot\nabla u^\varepsilon+f(x,y,u^\varepsilon) &= 0 \text{ in } I\!R\times\overline{\Omega}, \\ \nu A\nabla u^\varepsilon &= 0 \text{ on } I\!R\times\partial\Omega, \\ \forall k\in\prod_{i=1}^d L_i Z\!\!Z,\ \ \forall (t,x,y)\in I\!R\times\overline{\Omega},\quad u^\varepsilon\left(t+\dfrac{k\cdot e}{c^\varepsilon},x,y\right) &= u^\varepsilon(t,x+k,y), \\ u^\varepsilon(t,x,y)\underset{x\cdot e\to-\infty}{\longrightarrow}0,\ u^\varepsilon(t,x,y) &\underset{x\cdot e\to+\infty}{\longrightarrow} 1. \end{cases} \tag{5.32}$$

Moreover, $0<u^\varepsilon<1$ and u^ε is increasing with respect to t in $I\!R\times\overline{\Omega}$. Note that the convergences of $u^\varepsilon(t,x,y)$ to 1 and 0 as $x\cdot e\to\pm\infty$ are local in t and uniform in y and in the directions of $I\!R^d$ which orthogonal to e. Furthermore, it follows from the definition of u^ε that $u^\varepsilon(t,x,y)\to 0$ (resp. 1) as $t\to-\infty$ (resp. $t\to+\infty$) locally in (x,y).

Up to extraction of some subsequence, three cases may occur : $\varepsilon/(c^\varepsilon)^2\to\kappa\in(0,+\infty)$, $\varepsilon/(c^\varepsilon)^2\to+\infty$ or $\varepsilon/(c^\varepsilon)^2\to 0$.

Let us first begin with :

Case 1 : Assume that $\varepsilon/(c^\varepsilon)^2\to\kappa\in(0,+\infty)$. From assumption (1.18), there exists a point $(k_0,y_0)\in\overline{\Omega}$ such that $k_0\cdot e>0$ and $k_0\in\prod_{i=1}^d L_i Z\!\!Z$. Since $u^\varepsilon(t,k_0,y_0)\to 0$ (resp. 1) as $t\to-\infty$ (resp. $t\to+\infty$), one can assume, up to translation with respect ot t, that $u^\varepsilon(0,k_0,y_0)=(1+\theta)/2$.

Since $\varepsilon/(c^\varepsilon)^2\to\kappa\in(0,+\infty)$, standard elliptic estimates imply that, up to extraction of some subsequence, the functions u^ε converge, in $C^{2,\alpha}_{loc}(I\!R\times\overline{\Omega})$ (for $0<\alpha<1$), to a function u satisfying

$$\begin{cases} \kappa\partial_{tt}u+\mathrm{div}_{x,y}(A\nabla_{x,y}u)-\partial_t u-q\cdot\nabla_{x,y}u+f(x,y,u) &= 0 \ \text{ in } I\!R\times\overline{\Omega}, \\ \nu A\nabla_{x,y}u &= 0 \ \text{ on } I\!R\times\partial\Omega, \\ 0\le u\le 1,\quad \partial_t u &\ge 0 \ \text{ in } I\!R\times\overline{\Omega} \end{cases}$$

and $u(0,k_0,y_0)=(1+\theta)/2$. Furthermore, for any $B\in I\!R$, one has $u^\varepsilon(B,0,y_0)\le u^\varepsilon((k_0\cdot e)/c^\varepsilon,0,y_0)$ for ε small enough since $c^\varepsilon\to 0^+$, $k_0\cdot e>0$ and u^ε is increasing in t. But $u^\varepsilon((k_0\cdot e)/c^\varepsilon,0,y_0)=u^\varepsilon(0,k_0,y_0)=(1+\theta)/2$. Passing to the limit $\varepsilon\to 0$ gives

$$u(B,0,y_0)\le\frac{1+\theta}{2},\quad \text{for all } B\in I\!R. \tag{5.33}$$

On the other hand, it follows from Lemma 5.11 and Fatou's lemma that

$$\int_{I\!R\times\mathcal{K}}(u_t^2+|\nabla_{x,y}u|^2)\,dt\,dx\,dy\le C(\mathcal{K})\left(\frac{1+N\|q\|_\infty^2}{2c_1}+2\max_{(x,y)\in\overline{\Omega}}F(x,y,1)\right) \tag{5.34}$$

for all compact subset $\mathcal{K}$ of $\overline{\Omega}$, where the constant $C(\mathcal{K})$ only depends on $\mathcal{K}$. Let $u^\pm(x,y)$ be the functions defined in $\overline{\Omega}$ by

$$u^\pm(x,y)=\lim_{t\to\pm\infty}u(t,x,y).$$

These functions can be defined since u is bounded and nondecreasing in t. From standard elliptic estimates, the convergence $u(t,x,y) \to u^{\pm}(x,y)$ as $t \to \pm\infty$ holds in $C^{2,\alpha}_{loc}(\overline{\Omega})$. From (5.34), it follows then that $u^{\pm}$ are constant. But since $u^{\pm}$ solve

$$\mathrm{div}(A\nabla u^{\pm}) - q\cdot\nabla u^{\pm} + f(x,y,u^{\pm}) = 0 \ \text{ in } \overline{\Omega},$$

one concludes that $f(x,y,u^{\pm}) \equiv 0$ in $\overline{\Omega}$.

From our choice of (k_0,y_0) and since u is nondecreasing in t, one has $u^+ \geq (1+\theta)/2$ while (5.33) yields $u^+ \leq (1+\theta)/2$. Finally, $u^+ = (1+\theta)/2$ and $f(x,y,(1+\theta)/2) = 0$ for all $(x,y)\in\overline{\Omega}$. The latter contradicts (1.26). Case 1 is then ruled out.

Case 2 : Assume that $\varepsilon/(c^{\varepsilon})^2 \to +\infty$. Make the change of variables $\tau = (c^{\varepsilon}/\sqrt{\varepsilon})\, t$. The function $v^{\varepsilon}(\tau,x,y) = u^{\varepsilon}(\frac{\sqrt{\varepsilon}}{c^{\varepsilon}}\tau,x,y)$ satisfies

$$\begin{aligned} \partial_{\tau\tau}v^{\varepsilon} + \mathrm{div}_{x,y}(A\nabla_{x,y}v^{\varepsilon}) - \frac{c^{\varepsilon}}{\sqrt{\varepsilon}}\partial_{\tau}v^{\varepsilon} - q\cdot\nabla_{x,y}v^{\varepsilon} + f(x,y,v^{\varepsilon}) &= 0 \ \text{ in } I\!R\times\overline{\Omega}, \\ \nu A\nabla_{x,y}v^{\varepsilon} &= 0 \ \text{ on } I\!R\times\partial\Omega \end{aligned}$$

and it is globally bounded and nondecreasing with respect to τ. Since $c^{\varepsilon}/\sqrt{\varepsilon} \to 0^+$ by assumption, the functions v^{ε} converge in $C^{2,\alpha}_{loc}(I\!R\times\overline{\Omega})$, up to extraction of some subsequence, to a globally bounded and nondecreasing in τ function v solving

$$\begin{aligned} \partial_{\tau\tau}v + \mathrm{div}_{x,y}(A\nabla_{x,y}v) - q\cdot\nabla_{x,y}v + f(x,y,v) &= 0 \ \text{ in } I\!R\times\overline{\Omega}, \\ \nu A\nabla_{x,y}v &= 0 \ \text{ on } I\!R\times\partial\Omega. \end{aligned} \tag{5.35}$$

Furthermore, it follows from Lemma 5.11 that

$$\int_{I\!R\times\mathcal{K}} \left[\frac{(c^{\varepsilon})^2}{\varepsilon}(v^{\varepsilon}_{\tau})^2 + |\nabla_{x,y}v^{\varepsilon}|^2\right]\frac{\sqrt{\varepsilon}}{c^{\varepsilon}}\, d\tau\, dx\, dy \ \leq C(\mathcal{K})\left(\frac{1+N\|q\|^2_{\infty}}{2c_1} + 2\max_{(x,y)\in\overline{\Omega}} F(x,y,1)\right)$$

for all compact subset $\mathcal{K}$ of $\overline{\Omega}$ and for all $\varepsilon > 0$. The passage to the limit $\varepsilon \to 0$ together with Fatou's lemma leads to $\int_{I\!R\times\mathcal{K}} |\nabla_{x,y}v|^2 = 0$ for all compact subset $\mathcal{K}$ of $\overline{\Omega}$. Hence, the function v only depends on τ. By defining $v^{\pm} := \lim_{\tau\to\pm\infty} v(\tau)$ and passing to the limit $\tau \to +\infty$ in (5.35), one gets that $f(x,y,v^+) \equiv 0$ for all $(x,y)\in\overline{\Omega}$.

On the other hand, as in case 1, up to translation in τ, one can have assumed that $v^{\varepsilon}(0,k_0,y_0) = (1+\theta)/2$ for some $(k_0,y_0)\in\overline{\Omega}$ such that $k_0 \in \prod_{i=1}^{d} L_i\mathbb{Z}$ and $k_0\cdot e > 0$. Since $k_0\cdot e/\sqrt{\varepsilon} \to +\infty$, it easily follows that, for all $B\in I\!R$,

$$v^{\varepsilon}(B,0,y_0) \leq v^{\varepsilon}\left(\frac{k_0\cdot e}{\sqrt{\varepsilon}},0,y_0\right) = v^{\varepsilon}(0,k_0,y_0) = \frac{1+\theta}{2},$$

for ε small enough. Hence, $v(B,0,y_0) \leq (1+\theta)/2$ for all $B\in I\!R$. Therefore, $v^+ = \lim_{\tau\to+\infty} v(\tau,0,y_0) \leq (1+\theta)/2$ while $v^+ = \lim_{\tau\to+\infty} v(\tau,k_0,y_0) \geq (1+\theta)/2$. Finally, $v^+ = (1+\theta)/2$ and one is led to a contradiction as in case 1. Case 2 is then ruled out too.

Case 3 : Suppose that $\varepsilon/(c^\varepsilon)^2 \to 0$. The elliptic operators $(\varepsilon/(c^\varepsilon)^2)\partial_{tt} + \mathrm{div}_{x,y}(A\nabla_{x,y})$ in (5.32) become degenerate at the limit and the arguments used in cases 1 and 2 do not work anymore as such. Nevertheless, one can still reach a contradiction by slightly modifying the proof in case 1.

Since $0 \le u^\varepsilon \le 1$ and since (5.27) holds for all $\varepsilon > 0$, there exists a function $u \in H^1_{loc}(I\!R\times\Omega)$ such that, up to extraction of some subsequence, $u^\varepsilon \to u$ almost everywhere in $I\!R \times \Omega$ and

$$(u^\varepsilon, u^\varepsilon_t, \nabla_{x,y}u^\varepsilon) \overset{\text{weak}}{\rightharpoonup} (u, u_t, \nabla_{x,y}u) \text{ in } L^2(I\!R \times \mathcal{K})$$

for all compact subset $\mathcal{K} \subset \overline{\Omega}$. Moreover, the function u is such that $0 \le u \le 1$, $u_t \ge 0$ and

$$\int_{I\!R\times\mathcal{K}} (u_t^2 + |\nabla_{x,y}u|^2)\ dt\ dx\ dy \le C(\mathcal{K}) \left(\frac{1 + N\|q\|^2_\infty}{2c_1} + 2 \max_{(x,y)\in\overline{\Omega}} F(x, y, 1) \right) \tag{5.36}$$

for all compact set $\mathcal{K} \subset \overline{\Omega}$.

Take now any compactly supported function $\varphi \in C^2(I\!R \times \overline{\Omega})$. By multiplying the first equation in (5.32) by φ, integrating by parts and passing to the limit $\varepsilon \to 0$, it follows that

$$\int_{I\!R\times\Omega} -\nabla_{x,y}\varphi\ A\ \nabla_{x,y}u - u_t\varphi - (q \cdot \nabla_{x,y}u)\varphi + f(x, y, u)\varphi = 0.$$

From parabolic regularity, the function u is then a classical solution of

$$\left\{ \begin{array}{rcl} u_t - \mathrm{div}_{x,y}(A\nabla_{x,y}u) + q \cdot \nabla_{x,y}u - f(x, y, u) &=& 0 \text{ in } I\!R \times \Omega, \\ \nu A\nabla_{x,y}u &=& 0 \text{ on } I\!R \times \partial\Omega, \\ 0 \le u \le 1, \quad \partial_t u &\ge& 0 \text{ in } I\!R \times \overline{\Omega}. \end{array} \right. \tag{5.37}$$

On the other hand, one can assume, up to translation in t, that

$$\forall \varepsilon > 0, \quad \int_{(0,1)\times\{C+(k_0,0)\}} u^\varepsilon(t, x, y)\ dt\ dx\ dy\ = |C|\frac{1+\theta}{2} \tag{5.38}$$

for some $k_0 \in \prod_{i=1}^d L_i\mathbb{Z}$ such that $k_0 \cdot e > 0$. Here, $C + (k_0, 0) = \{(x + k_0, y),\ (x, y) \in C\}$. Since $c^\varepsilon \to 0^+$ and u^ε is increasing in t, it follows that, for all $B \in I\!R$, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$,

$$\begin{array}{rcl} \forall (t, x, y) \in (0, 1) \times C, \quad u^\varepsilon(B + t, x, y) &\le& u^\varepsilon\left(t + \dfrac{k_0 \cdot e}{c^\varepsilon}, x, y\right) \\ &=& u^\varepsilon(t, x + k_0, y) \quad \text{(from (5.32))}. \end{array}$$

Integrate over $(0, 1) \times C$ and pass to the limit $\varepsilon \to 0^+$. By using (5.38), it follows that

$$\forall B \in I\!R, \quad \int_{(0,1)\times C} u(B + t, x, y)\ dt\ dx\ dy \le |C|\frac{1+\theta}{2}. \tag{5.39}$$

From (5.36) and (5.37), it follows as in case 1 that $u(t, x, y) \to u^+$ as $t \to +\infty$ where u^+ is a constant function such that $f(x, y, u^+) = 0$ for all $(x, y) \in \overline{\Omega}$. Formula (5.39) implies that $u^+ \le (1+\theta)/2$ while formula (5.38), after passing to the limit $\varepsilon \to 0$ and to the limit $t \to +\infty$, leads to $u^+ \ge (1 + \theta)/2$.

Eventually, $u^+ = (1 + \theta)/2$, which leads to a contradiciton as in case 1. Case 3 is then ruled out too and the proof of Proposition 5.10 is complete. □

Remark 5.12 A result similar to Proposition 5.10 was proved by Heinze [51] for pulsating travelling fronts in straight infinite cylinders in the homogenization limit. Heinze did a short proof based on gradients estimates similar to (5.27). Unfortunately, his technique does not work in general in our framework because of the (x, y) dependance of the nonlinear source term f.

Completion of the proof of part a) of Theorem 1.13. Choose a subsequence $\varepsilon \to 0$ such that $c^\varepsilon \to c := \liminf_{\varepsilon \to 0} c^\varepsilon > 0$ and remember that each function u^ε is defined in $\mathbb{R} \times \overline{\Omega}$ by $u^\varepsilon(t, x, y) = \phi^\varepsilon(c^\varepsilon t + x \cdot e, x, y)$.

As in case 3) in the proof of Proposition 5.10 (one has $\varepsilon/(c^\varepsilon)^2 \to 0$ as $\varepsilon \to 0$ since $c^\varepsilon \to c > 0$), the functions u^ε converge, up to extraction of some subsequence, in $H^1_{loc}(\mathbb{R} \times \Omega)$ and almost everywhere, to a classical solution u of (5.37) satisfying the gradient estimates (5.36), namely

$$\int_{\mathbb{R}\times\mathcal{K}} (u_t^2 + |\nabla_{x,y} u|^2)\, dt\, dx\, dy \leq C(\mathcal{K}) \left(\frac{1 + N\|q\|_\infty^2}{2c_1} + 2 \max_{(x,y)\in\overline{\Omega}} F(x, y, 1) \right) \tag{5.40}$$

for any compact subset $\mathcal{K} \subset \overline{\Omega}$.

The functions u^ε are such that $u^\varepsilon(t + (k \cdot e)/c^\varepsilon, x, y) = u^\varepsilon(t, x + k, y)$ for all $k \in \prod_{i=1}^d L_i\mathbb{Z}$ and $(x, y) \in \overline{\Omega}$. Let us now prove that $u(t+(k\cdot e)/c, x, y) = u(t, x+k, y)$ for all $(t, x, y) \in \mathbb{R}\times\overline{\Omega}$ and for all $k \in \prod_{i=1}^d L_i\mathbb{Z}$. Take $k \in \prod_{i=1}^d L_i\mathbb{Z}$. For all $B > 0$ and all compact set $\mathcal{K} \subset \overline{\Omega}$, one has

$$\begin{aligned}
&\int_{(-B,B)\times\mathcal{K}} \left[u^\varepsilon \left(t + \frac{k\cdot e}{c}, x, y \right) - u^\varepsilon(t, x + k, y) \right]^2 \\
&\quad = \int_{(-B,B)\times\mathcal{K}} \left[u^\varepsilon \left(t + \frac{k\cdot e}{c}, x, y \right) - u^\varepsilon \left(t + \frac{k\cdot e}{c^\varepsilon}, x, y \right) \right]^2 \\
&\quad \leq \left(\frac{k\cdot e}{c} - \frac{k\cdot e}{c^\varepsilon} \right)^2 \int_{\mathbb{R}\times\mathcal{K}} (u_t^\varepsilon)^2 \\
&\quad \leq \left(\frac{k\cdot e}{c} - \frac{k\cdot e}{c^\varepsilon} \right)^2 C(\mathcal{K}) \left(\frac{1 + N\|q\|_\infty^2}{2c_1} + 2 \max_{(x,y)\in\overline{\Omega}} F(x, y, 1) \right)
\end{aligned}$$

from (5.27). Therefore, one concludes by passage to the limit $\varepsilon \to 0$ that $u(t+(k\cdot e)/c, x, y) = u(t, x+k, y)$ almost everywhere in $\mathbb{R} \times \Omega$, and then this equality holds for all $(t, x, y) \in \mathbb{R} \times \overline{\Omega}$ since u is continuous.

At this stage, in order to complete the proof of part a) of Theorem 1.13, we only have to prove that u satisfies the limits $u(t, x, y) \to 0$ as $x \cdot e \to -\infty$ and $u(t, x, y) \to 1$ as $x \cdot e \to +\infty$, locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e. Since u is periodic in (x, t), namely u satisfies $u(t + (k \cdot e)/c, x, y) = u(t, x + k, y)$ for all $k \in \prod_{i=1}^d L_i\mathbb{Z}$ and for all $(t, x, y) \in \mathbb{R} \times \overline{\Omega}$, and since c is positive, it is sufficient to prove that $u(t, x, y) \to 0$ as $t \to -\infty$ and $u(t, x, y) \to 1$ as $t \to +\infty$ locally in (x, y).

The functions $\phi^\varepsilon(s, x, y)$ are L-periodic with respect to x and satisfy $\max_{\overline{\Omega}} \phi^\varepsilon(0, x, y) = \max_{\overline{C}} \phi^\varepsilon(0, x, y) = \theta$. Therefore,

$$\max_{(x,y)\in\overline{\Omega}} u^\varepsilon \left(-\frac{x\cdot e}{c^\varepsilon}, x, y \right) = \max_{(x,y)\in\overline{C}} u^\varepsilon \left(-\frac{x\cdot e}{c^\varepsilon}, x, y \right) = \theta. \tag{5.41}$$

One would like to pass to the limit in this last equality. One especially would like to be sure that there is at least a point $(\overline{x},\overline{y}) \in \overline{C}$ such that $u(-\overline{x}\cdot e/c,\overline{x},\overline{y}) = \theta$. The uniform L^2 estimates for the gradient of u^ε with respect to the variables (t,x,y) (*see* Lemma 5.11) are not sufficient here to pass to the limit.

Nevertheless, we will use uniform estimates for the gradient of u^ε with respect to the space variables (x,y). These estimates, which are independent of ε are stated in the following

Lemma 5.13 *Assume that the restriction of the function $f\ :\ (x,y,u)\mapsto f(x,y,u)$ to $\overline{\Omega}\times[0,1]$ is globally $C^{1,\delta'}$ with respect to u, for some $\delta'>0$. Assume also that $\|f\|_{C^1(\overline{\Omega}\times[0,1])}$, $\|q\|_{C^1(\overline{\Omega})}$, $\|A\|_{C^3(\overline{\Omega})}\le b$. Then there exists a constant C, which only depends on Ω and b, such that the functions u^ε solving (5.32) satisfy*

$$\|\nabla_{x,y}u^\varepsilon\|_{L^\infty(I\!\!R\times\overline{\Omega})}\le C$$

for ε small enough.

This lemma is stated in a more general framework in Theorem 7.1 of Section 7.

Let us now turn to the completion of the proof of part a) of Theorem 1.13.

Step 1 : let us now assume temporarily that the restriction of f to $\overline{\Omega}\times[0,1]$ is globally $C^{1,\delta'}$ with respect to u, for some $\delta'>0$.

First of all, it follows from (5.41) and the monotonicity of u^ε with respect to t that $u^\varepsilon(t,x,y)\le\theta$ whenever $c^\varepsilon t+x\cdot e\le 0$. Therefore, for every $(t_0,x_0,y_0)\in I\!\!R\times\overline{\Omega}$ such that $ct_0+x_0\cdot e<0$, there exists $r>0$ such that for ε small enough and for all $(t,x,y)\in I\!\!R\times\overline{\Omega}$ satisfying $|t-t_0|+|(x,y)-(x_0,y_0)|\le r$, one has $u^\varepsilon(t,x,y)\le\theta$. Since u^ε converges to u almost everywhere and since u is continuous, one concludes that $u(t_0,x_0,y_0)\le\theta$. By continuity, it also follows that

$$\forall (t,x,y)\in I\!\!R\times\overline{\Omega},\qquad ct+x\cdot e\le 0\implies u(t,x,y)\le\theta. \tag{5.42}$$

On the other hand, (5.41) yields the existence of a point $(x^\varepsilon,y^\varepsilon)\in\overline{C}$ such that $u^\varepsilon(-x^\varepsilon\cdot e/c^\varepsilon,x^\varepsilon,y^\varepsilon)=\theta$. Up to extraction of some subsequence, one can assume that $(x^\varepsilon,y^\varepsilon)\to(\overline{x},\overline{y})\in\overline{C}$. Fix any positive real number η. It follows from Lemma 5.13 that there exist $r>0$ and $\varepsilon_0>0$ such that

$$\forall\varepsilon\in(0,\varepsilon_0),\quad \forall(x,y)\in B_r(\overline{x},\overline{y})\cap\overline{\Omega},\quad u^\varepsilon\left(-\frac{x^\varepsilon\cdot e}{c^\varepsilon},x,y\right)\ge\theta-\eta,$$

where $B_r(\overline{x},\overline{y})\cap\overline{\Omega}=\{(x,y)\in\overline{\Omega},\ |(x,y)-(\overline{x},\overline{y})|\le r\}$ and the Lebesgue-measure of $B_r(\overline{x},\overline{y})\cap\overline{\Omega}$ is positive. Since u^ε is increasing in t, $u^\varepsilon(t,x,y)\ge\theta-\eta$ for all $(t,x,y)\in[-(x^\varepsilon\cdot e)/c^\varepsilon,+\infty)\times(B_r(\overline{x},\overline{y})\cap\overline{\Omega})$ and for all $\varepsilon\in(0,\varepsilon_0)$. Since u^ε converges almost everywhere in $I\!\!R\times\Omega$ to the continuous function u, and since $(x^\varepsilon\cdot e)/c^\varepsilon\to(\overline{x}\cdot e)/c$, one then gets that $u(t,x,y)\ge\theta-\eta$ for all $t\ge-\overline{x}\cdot e/c$ and for all $(x,y)\in B_r(\overline{x},\overline{y})\cap\overline{\Omega}$. Since η was an arbitrary positive real number, it follows that $u(-\overline{x}\cdot e/c,\overline{x},\overline{y})\ge\theta$.

Together with (5.42), that yields $u(-\overline{x}\cdot e/c,\overline{x},\overline{y})=\theta$, whence

$$\max_{(x,y)\in\overline{\Omega}} u\left(-\frac{x\cdot e}{c},x,y\right)=\max_{(x,y)\in\overline{C}} u\left(-\frac{x\cdot e}{c},x,y\right)=\theta. \tag{5.43}$$

Let us now prove that $u(t,x,y)\to 0$ as $t\to-\infty$, locally in $(x,y)\in\overline{\Omega}$. From (5.26), one has

$$\forall (t,x,y)\in I\!\!R\times\overline{\Omega},\quad c^\varepsilon t+x\cdot e\le 0 \implies u^\varepsilon(t,x,y)\le e^{\lambda_\varepsilon(c^\varepsilon t+x\cdot e)}\tilde{\psi}^\varepsilon(x,y), \tag{5.44}$$

where, under the notations of Proposition 5.7-4), $\lambda_\varepsilon\in[\lambda^{\varepsilon,c^\varepsilon/2},\lambda^{\varepsilon,3c^\varepsilon/2}]$ and $\tilde{\psi}^\varepsilon$ is L-periodic with respect to x and satisfies $\min_{\overline{\Omega}}\tilde{\psi}^\varepsilon=\min_{\overline{C}}\tilde{\psi}^\varepsilon=\theta$. Furthermore, for each $\varepsilon>0$, the function $\tilde{\psi}^\varepsilon$ solves (5.18) with $(\gamma,\lambda,\zeta,\mu)=(c^\varepsilon,\lambda_\varepsilon,0,\varepsilon\lambda_\varepsilon^2)$. Since the positive real numbers $\lambda^{\alpha,\gamma}$ are decreasing with respect to $\alpha>0$ and increasing with respect to $\gamma>0$ (from Proposition 5.7), one has $\lambda^{\varepsilon,c^\varepsilon/2}\ge\lambda^{1,c/4}>0$ for ε small enough.

Assume now by contradiction that, up to extraction of some subsequence, $\lambda_\varepsilon\to+\infty$. Take a sequence of points $(x_\varepsilon,y_\varepsilon)\in\overline{C}$ such that $\tilde{\psi}^\varepsilon(x_\varepsilon,y_\varepsilon)=\theta$. Up to extraction of some subsequence, one can assume that $(x_\varepsilon,y_\varepsilon)\to(\underline{x},\underline{y})\in\overline{C}$ as $\varepsilon\to 0$. Define $\underline{t}=(-1-\underline{x}\cdot e)/c$. From (5.44), one has

$$u^\varepsilon(\underline{t},x_\varepsilon,y_\varepsilon)\le\theta e^{-\lambda_\varepsilon\left(\frac{c^\varepsilon}{c}(-1-\underline{x}\cdot e)+x_\varepsilon\cdot e\right)}.$$

Since $c^\varepsilon\to c>0$ and $x_\varepsilon\to\underline{x}$, one gets $u^\varepsilon(\underline{t},x_\varepsilon,y_\varepsilon)\to 0$ as $\varepsilon\to 0$. Fix now any positive number η. From Lemma 5.13, there exist $r>0$ and $\varepsilon_0>0$ such that

$$\forall\varepsilon\in(0,\varepsilon_0),\quad\forall(x,y)\in B_r(\underline{x},\underline{y})\cap\overline{\Omega},\quad u^\varepsilon(\underline{t},x,y)\le\eta.$$

Since u^ε is increasing in t, this last estimate holds for all $t\le\underline{t}$. Since u^ε converges to u almost everywhere in $I\!\!R\times\Omega$ and since u is continuous, it follows then that $u(t,x,y)\le\eta$ for all $(t,x,y)\in(-\infty,\underline{t}]\times(B_r(\underline{x},\underline{y})\cap\overline{\Omega})$. Since $\eta>0$ was arbitrary, one concludes that $u(t,\underline{x},\underline{y})=0$ for all $t\le\underline{t}$. The strong maximum principle implies that $u(t,x,y)=0$ for all $t\le\underline{t}$ and $(x,y)\in\overline{\Omega}$. Eventually, u is identically equal to 0 in $I\!\!R\times\overline{\Omega}$. The latter is in contradiction with (5.43).

Therefore, the numbers λ_ε are bounded from above, and from below by $\lambda^{1,c/4}$ for ε small enough. Up to extraction of some subsequence, there exists then a real number $\overline{\lambda}$ such that

$$\lambda_\varepsilon\to\overline{\lambda}\ge\lambda^{1,c/4}>0\text{ as }\varepsilon\to 0. \tag{5.45}$$

Remember now that the functions $\tilde{\psi}^\varepsilon$ satisfy $\min_{\overline{\Omega}}\tilde{\psi}^\varepsilon=\min_{\overline{C}}\tilde{\psi}^\varepsilon=\theta$. One claims that

$$\exists\tilde{C}>0,\ \forall\varepsilon>0,\quad\max_{\overline{\Omega}}\tilde{\psi}^\varepsilon\le\tilde{C}. \tag{5.46}$$

Indeed, if that were not true, then $\max_{\overline{\Omega}}\tilde{\psi}^\varepsilon\to+\infty$ up to extraction of some subsequence. The L-periodic with respect to x functions

$$\overline{\psi}^\varepsilon:=\frac{\tilde{\psi}^\varepsilon}{\max_{\overline{\Omega}}\tilde{\psi}^\varepsilon}$$

solve (5.18) with $(\gamma,\lambda,\zeta,\mu)=(c^\varepsilon,\lambda_\varepsilon,0,\varepsilon\lambda_\varepsilon^2)$, and are such that $\max_{\overline{\Omega}}\overline{\psi}^\varepsilon=\max_{\overline{C}}\overline{\psi}^\varepsilon=1$ and $\min_{\overline{\Omega}}\overline{\psi}^\varepsilon=\min_{\overline{C}}\overline{\psi}^\varepsilon\to 0$ as $\varepsilon\to 0$. Therefore, as done in the proof of Proposition 5.9, the positive functions $\overline{\psi}^\varepsilon$ converge in $C^{2,\alpha}(\overline{\Omega})$ (for all $0\le\alpha<1$) to a nonnegative solution $\overline{\psi}$ of

(5.18) with $(\gamma,\lambda,\zeta,\mu)=(c,\overline{\lambda},0,0)$. Furthermore, there are some points $(\overline{x},\overline{y})$ and $(\underline{x},\underline{y})\in\overline{C}$ such that $\overline{\psi}(\underline{x},\underline{y})=0$ and $\overline{\psi}(\overline{x},\overline{y})=1$. The strong maximum principle and Hopf lemma imply that $\overline{\psi}\equiv 0$, which is impossible.

Therefore, (5.46) holds. As a consequence, the bounded sequence of functions $\tilde{\psi}^{\varepsilon}$ converges, up to extraction of some subsequence, to a solution $\tilde{\psi}$ of (5.18) with $(\gamma,\lambda,\zeta,\mu)=(c,\overline{\lambda},0,0)$. Furthermore, $\min_{\overline{\Omega}}\tilde{\psi}=\min_{\overline{C}}\tilde{\psi}=\theta$ and $\tilde{\psi}(x,y)\le\tilde{C}$ for all $(x,y)\in\overline{\Omega}$. The passage to the limit $\varepsilon\to 0$ in (5.44) provides

$$\forall(t,x,y)\in I\!R\times\overline{\Omega},\quad ct+x\cdot e\le 0\implies u(t,x,y)\le e^{\overline{\lambda}(ct+x\cdot e)}\tilde{\psi}(x,y). \tag{5.47}$$

As a consequence, $u(t,x,y)$ converges locally to 0 as $x\cdot e\to-\infty$ or $t\to-\infty$.

Furthermore, as already underlined, the function u satisfies the gradient estimates (5.40). Since u is nondecreasing with respect to the variable t, one finally gets, by using the same arguments as in cases 1 or 3 of Proposition 5.10, that $u(t,x,y)$ approaches a constant u^+ as $t\to+\infty$, and that $f(x,y,u^+)=0$ for all $(x,y)\in\overline{\Omega}$. From the normalization condition (5.43), it follows then that $u^+\ge\theta$, whence, due to the profile of f, $u^+=\theta$ or $u^+=1$. The first case would imply, thanks to the maximum pinciple, that $u\equiv\theta$. That is impossible because $u(-\infty,x,y)=0$. Eventually, $u^+=1$ and $u(t,x,y)\to 1$ as $t\to+\infty$.

Lastly, from the strong maximum principle, one has $0<u(t,x,y)<1$ for all $(t,x,y)\in I\!R\times\overline{\Omega}$. That completes the proof of part a) of Theorem 1.13 in the case where f is of class $C^{1,\delta'}$ with respect to u in $\overline{\Omega}\times[0,1]$.

Step 2 : consider now the case where the function f satisfies (1.24-1.26) and is just globally Lipschitz-continuous in $\overline{\Omega}\times I\!R$, instead of being $C^{1,\delta'}$ with respect to u in $\overline{\Omega}\times[0,1]$.

Remember that the fields q and A are respectively of class $C^{1,\delta}(\overline{\Omega})$ and $C^3(\overline{\Omega})$, where $\delta>0$. Since f satisfies (1.24-1.26), there exists a sequence of functions f_n satisfying (1.24-1.26) and such that $\sup_n\ Lip(f_n)<+\infty$, $f_n\to f$ uniformly in $\overline{\Omega}\times I\!R$ and, for each n, the restriction $\tilde{f}_n$ of f_n to $\overline{\Omega}\times[0,1]$ is of class $C^{1,\delta}$ with respect to u.

From step 1, for each n, there exists a classical solution (c_n,u_n) of (1.28) with the nonlinear source term f_n. Furthermore, $c_n>0$, $0<u_n<1$ in $I\!R\times\overline{\Omega}$, u_n is increasing with respect to t and one can assume that $\max_{\overline{\Omega}}u_n(-x\cdot e/c_n,x,y)=\max_{\overline{C}}u_n(-x\cdot e/c_n,x,y)=\theta$. Lastly, the functions u_n satisfy the gradient estimate (5.40) with the nonlinearity $F_n(x,y,t)=\int_0^t f_n(x,y,\tau)d\tau$. Namely, for any compact subset $\mathcal{K}\subset\overline{\Omega}$, there exists a constant $C(\mathcal{K})$ only depending on $\mathcal{K}$ such that

$$\forall n,\quad\int_{I\!R\times\mathcal{K}}(\partial_t u_n^2+|\nabla_{x,y}u_n|^2)\ dt\ dx\ dy\le C(\mathcal{K})\left(\frac{1+N\|q\|_\infty^2}{2c_1}+2\max_{(x,y)\in\overline{\Omega}}F_n(x,y,1)\right).$$

From standard parabolic estimates, there exists a constant C such that $\|u_n\|_{C^1(I\!R\times\overline{\Omega})}\le C$ for all n. Therefore, up to extraction of some subsequence, the functions u_n converge locally uniformly to a function u which, from parabolic regularity, is a classical solution of

$$\left\{\begin{array}{rcl}\dfrac{\partial u}{\partial t}-\mathrm{div}(A\nabla u)+q\cdot\nabla u&=&f(x,y,u)\ \text{ in } I\!R\times\Omega,\\ \nu A\nabla u&=&0\ \text{ on } I\!R\times\partial\Omega,\\ 0\le u\le 1,\quad\dfrac{\partial u}{\partial t}&\ge&0\text{ in } I\!R\times\overline{\Omega}.\end{array}\right.$$

Furthermore, the function u satisfies the gradient estimates (5.40).

In order to pass to the limit $n \to +\infty$ for the speeds c_n, one proves the following

Lemma 5.14 *There exist* $0 < \underline{c} \le \overline{c} < +\infty$ *such that*

$$\forall n, \quad 0 < \underline{c} \le c_n \le \overline{c} < +\infty. \tag{5.48}$$

Proof. Since the functions f_n satisfy (1.24-1.26) and are uniformly Lipschitz-continuous, there exists a $C^{1,\delta}([0,1])$ function $\overline{f}$ such that $\overline{f}(s) = 0$ for all $s \in [0,\theta/2] \cup \{1\}$, $\overline{f}(s) > 0$ for all $s \in (\theta/2, 1)$, $\overline{f}'(1) < 0$, and

$$\forall n, \quad \forall (x,y,s) \in \overline{\Omega} \times [0,1], \quad f_n(x,y,s) \le \overline{f}(s).$$

From the results of step 1, there exists a classical solution $(\overline{c}, \overline{u})$ of problem (1.28) with the nonlinear source term $\overline{f}$. Furthermore, $\overline{c} > 0$ and $\overline{u}_t > 0$.

By using a sliding method as in the proof of Lemma 4.1, one is going to prove that $c_n \le \overline{c}$ for all n. Choose any arbitrary n and let ϕ_n and $\overline{\phi}$ be the functions defined by $\phi_n(s,x,y) = u_n((s - x\cdot e)/c_n, x, y)$ and $\overline{\phi}(s,x,y) = \overline{u}((s - x\cdot e)/\overline{c}, x, y)$. The functions ϕ_n and $\overline{\phi}$ are of class $C^{1,\mu}(I\!R \times \overline{\Omega})$ (for each $\mu \in [0,1)$). Assume now by contradiction that $c_n > \overline{c}$. Since $f_n \le \overline{f}$ and $\partial_s \overline{\phi} \ge 0$, it follows that the function $\overline{\phi}$ is a supersolution for the equation satisfied (3.2) satisfied by ϕ_n (with $c = c_n$ and $f = f_n$), in the sense that

$$\begin{array}{rcl} \mathrm{div}_{x,y}(A\nabla_{x,y}\overline{\phi}) + (\tilde{e}A\tilde{e})\overline{\phi}_{ss} + \mathrm{div}_{x,y}(A\tilde{e}\overline{\phi}_s) & & \\ +\partial_s(\tilde{e}A\nabla_{x,y}\overline{\phi}) - q\cdot\nabla_{x,y}\overline{\phi} & & \\ -(q\cdot\tilde{e} + c_n)\overline{\phi}_s + f_n(x,y,\overline{\phi}) & = & (\overline{c} - c_n)\overline{\phi}_s + f_n(x,y,\overline{\phi}) - \overline{f}(\overline{\phi}) \\ & \le & 0 \quad \text{in } \mathcal{D}'(I\!R \times \overline{\Omega}). \end{array} \tag{5.49}$$

As in the proof of Lemma 4.1, by sliding $\overline{\phi}$ with respect to ϕ_n and by using the fact that the function $f_n(x,y,u)$ is nonincreasing with respect to the variable u in a right neighborhood of 0 as well as in a left neighborhood of 1, uniformly in (x,y), one gets then the existence of a real number τ^* such that $\overline{\phi}(s+\tau^*, x, y) \ge \phi_n(s,x,y)$ for all $(s,x,y) \in I\!R \times \overline{\Omega}$, with equality at a point $(\overline{s}, \overline{x}, \overline{y})$. Therefore, the function $z(t,x,y) := \phi_n(c_n t + x\cdot e, x, y) - \overline{\phi}(c_n t + x\cdot e + \tau^*, x, y) = u_n(t,x,y) - \overline{u}((c_n/\overline{c})t, x, y)$ is nonpositive, it vanishes at $(\overline{t}, \overline{x}, \overline{y}) = ((\overline{s} - \overline{x}\cdot e)/c_n, \overline{x}, \overline{y})$, and it is a classical subsolution of

$$\begin{array}{rcl} \partial_t z - \mathrm{div}_{x,y}(A\nabla_{x,y}z) + q\cdot\nabla_{x,y}z & & \\ +f_n(x,y,\overline{\phi}(c_n t + x\cdot e + \tau^*, x, y)) - f_n(x,y,\phi_n(c_n t + x\cdot e, x, y)) & \le & 0 \quad \text{in } I\!R \times \Omega. \end{array}$$

Since the function f_n is globally Lipschitz-continuous, there exists a bounded function $C(t,x,y)$ such that

$$\partial_t z - \mathrm{div}_{x,y}(A\nabla_{x,y}z) + q\cdot\nabla_{x,y}z + C(t,x,y)z \le 0 \quad \text{in } I\!R \times \Omega.$$

On the other hand, one has $\nu A\nabla_{x,y}z = 0$ on $I\!R \times \partial\Omega$. The strong parabolic maximum principle implies that $z(t,x,y) = 0$ for all $t \le \overline{t}$ and for all $(x,y) \in \overline{\Omega}$. By definition of z, and since both $\overline{\phi}$ and ϕ_n are L-periodic with respect to x, one concludes that $z \equiv 0$ in $I\!R \times \overline{\Omega}$, that is $\phi_n(s,x,y) \equiv \overline{\phi}(s+\tau^*, x, y)$. Coming back to (5.49), that means that $\overline{\phi}_s \equiv 0$ which is impossible. As a conclusion, the assumption $\overline{c} < c_n$ cannot hold, which shows the right-hand side of (5.48).

Let us now prove the existence of a positive real number $\underline{c}$ such that $\underline{c} \le c_n$ for all n. Firstly, under the notation in (1.26), fix two real numbers a, b such that $1 - \rho < a < b < 1$. From (1.24-1.25) and (1.26), the function $f(x, y, u)$ is uniformly bounded from below by a positive constant in $\overline{\Omega} \times [a, b]$. Therefore, there exists a function $\underline{f}$ of class $C^{1,\delta}([0, b])$ such that $\underline{f}(s) = 0$ for all $s \in [0, a] \cup \{b\}$, $\underline{f}(s) > 0$ for all $s \in (a, b)$, $\underline{f}'(b) < 0$ and

$$\forall n, \quad \forall (x, y, s) \in \overline{\Omega} \times [0, b], \quad \underline{f}(s) \le f_n(x, y, s).$$

One also extends $\underline{f}$ by 0 outside the interval $[0, b]$. From step 1, there exists then a classical solution $(\underline{c}, \underline{u})$ of (1.28) with the nonlinear source term $\underline{f}$, and where the limit of $\underline{u}(t, x, y)$ as $x \cdot e \to +\infty$ is equal to b (instead of 1 in (1.28)). Furthermore, $\underline{c} > 0$ and $\underline{u}_t > 0$. As above, by using a sliding method, it easily follows then that $\underline{c} \le c_n$ for all n.

That completes the proof of Lemma 5.14. □

Lemma 5.14 yields immediately that, up to extraction of some subsequence, one can assume that $c_n \to c \in [\underline{c}, \overline{c}]$ as $n \to +\infty$. Furthermore, since the functions u_n are globally and uniformly $C^1(\mathbb{R} \times \overline{\Omega})$ and locally converge to u, the function u satisfies the periodicity condition

$$\forall k \in \prod_{i=1}^{d} L_i \mathbb{Z}, \quad u\left(t + \frac{k \cdot e}{c}, x, y\right) = u(t, x + k, y) \text{ for all } (t, x, y) \in \mathbb{R} \times \overline{\Omega} \tag{5.50}$$

and the normalization condition

$$\max_{\overline{\Omega}} u\left(-\frac{x \cdot e}{c}, x, y\right) = \max_{\overline{C}} u\left(-\frac{x \cdot e}{c}, x, y\right) = \theta. \tag{5.51}$$

Since the eigenvalue problem (5.18), with $\zeta = 0$, does not depend on the nonlinear source term f_n, it follows then from (5.45) and (5.47) that, for each n, there exists a real number $\overline{\lambda}_n \ge \lambda^{1, c_n/4} > 0$ and there exists a $C^2(\overline{\Omega})$ function $\tilde{\psi}_n(x, y)$ solving (5.18) with $(\gamma, \lambda, \zeta, \mu) = (c_n, \overline{\lambda}_n, 0, 0)$, and such that $\min_{\overline{\Omega}} \tilde{\psi}_n = \min_{\overline{C}} \tilde{\psi}_n = \theta$ and

$$\forall (t, x, y) \in \mathbb{R} \times \overline{\Omega}, \quad c_n t + x \cdot e \le 0 \implies u_n(t, x, y) \le e^{\overline{\lambda}_n (c_n t + x \cdot e)} \tilde{\psi}_n(x, y).$$

Furthermore, it follows from Lemma 5.14 and from the fact that the numbers $\lambda^{\alpha,\gamma}$ are increasing with respect to $\gamma > 0$, that $\overline{\lambda}_n \ge \lambda^{1, \underline{c}/4} > 0$ for all n.

Let now $(x_n, y_n) \in \overline{C}$ be such that $\tilde{\psi}_n(x_n, y_n) = \theta$. Up to extraction of some subsequence, one can assume that $(x_n, y_n) \to (x_\infty, y_\infty) \in \overline{C}$. For all n and for all $t \le -x_n \cdot e / c_n$, one has

$$u_n(t, x_n, y_n) \le \theta e^{\lambda^{1, \underline{c}/4} (c_n t + x_n \cdot e)}.$$

Since the functions u_n are globally and uniformly C^1 and converge locally to the continuous function u, one finally gets that

$$\forall t \le -x_\infty \cdot e / c, \quad u(t, x_\infty, y_\infty) \le \theta e^{\lambda^{1, \underline{c}/4} (ct + x_\infty \cdot e)}.$$

In particular, $u(t, x_\infty, y_\infty) \to 0$ as $t \to -\infty$.

On the other hand, as already underlined, the function u is nondecreasing with respect to t, and it satisfies the gradient estimates (5.40). It follows then that $u(t,x,y)$ converges to two constants $u^{\pm}$ locally in (x,y) as $t \to \pm\infty$. Moreover, $u^{\pm}$ are such that $f(x,y,u^{\pm}) = 0$ for all $(x,y) \in \overline{\Omega}$. From the previous paragraph, one has $u^{-} = 0$. The normalization condition (5.51), the monotonicity of u in t and the profile of the nonlinearity f imply that either $u^{+} = \theta$ or $u^{+} = 1$. The case $u^{+} = \theta$ would mean that $u(t,x,y) \le \theta$ for all $(t,x,y) \in I\!R \times \overline{\Omega}$. Because of (5.51), the strong maximum principle would imply that $u(t,x,y) \equiv \theta$. That is impossible because $u^{-} = 0$. Therefore, $u^{+} = 1$.

Eventually, $u(t,x,y) \to 0$ (resp. $\to 1$) as $t \to -\infty$ (resp. $t \to +\infty$) locally in $(x,y) \in \overline{\Omega}$. As already observed, the periodicity condition (5.50) and the positivity of c imply then that $u(t,x,y) \to 0$ (resp. $\to 1$) as $x \cdot e \to -\infty$ (resp. $x \cdot e \to +\infty$) locally in t and uniformly in y and in the directions of $I\!R^d$ which are orthogonal to e. Lastly, the strong maximum principle implies that $0 < u < 1$ in $I\!R \times \overline{\Omega}$ and the proof of part a) of Theorem 1.13 is complete. □

6 Nonlinearity f without ignition temperature

This section is devoted to the proof Theorem 1.14. Throughout this section, f denotes a function satisfying (1.24-1.25) and (1.27). One assumes that q and A are respectively globally $C^{1,\delta}(\overline{\Omega})$ and $C^{3}(\overline{\Omega})$ (with $\delta > 0$) and that they satisfy (1.19) and (1.21-1.22).

This section is divided into four main subsections : firstly, we prove of the existence of a solution (c^*, u^*) of (1.28) for a "minimal" speed c^*; secondly, we prove the existence of a solution (c,u) for each $c > c^*$; thirdly we show that there is no solution (c,u) as soon as $c < c^*$; lastly, under the additional assumption $f_u^{+}(x,y,0) > 0$ for all $(x,y) \in \overline{\Omega}$, we prove that any solution u of (1.28) is increasing with respect to time t.

6.1 Existence of a solution (c^*, u^*) of (1.28)

Following the notations of Berestycki and Nirenberg [18], let χ be a $C^1(I\!R)$ function such that $0 \le \chi \le 1$ in $I\!R$, $\chi(u) = 0$ for all $u \le 1$, $0 < \chi(u) < 1$ for all $u \in (1,2)$ and $\chi(u) = 1$ for all $u \ge 2$. Assume moreover that χ is nondecreasing in $I\!R$. For all $\theta \in (0, 1/2)$, let χ_θ be the function defined by

$$\forall u \in I\!R, \quad \chi_\theta(u) = \chi(u/\theta).$$

This function χ_θ is such that $0 \le \chi_\theta \le 1$, $\chi_\theta = 0$ in $(-\infty, \theta]$, $0 < \chi_\theta < 1$ in $(\theta, 2\theta)$ and $\chi_\theta = 1$ in $[2\theta, +\infty)$. Furthermore, the functions χ_θ are nonincreasing with respect to θ, namely, $\chi_{\theta_1} \ge \chi_{\theta_2}$ if $0 < \theta_1 \le \theta_2 < 1/2$.

Lastly, set

$$f_\theta(x,y,u) = f(x,y,u)\chi_\theta(u) \ \text{ for all } (x,y,u) \in \overline{\Omega} \times I\!R.$$

In other words, one cuts off the source term f near $u = 0$.

For each $\theta \in (0, 1/2)$, the function f_θ satisfies (1.24-1.25) and (1.26) with the ignition temperature θ. Therefore, Theorem 1.13 yields the existence of a classical solution (c_θ, u_θ) of (1.28) with the nonlinearity f_θ. Furthermore, the function u_θ is increasing in t and unique up to translation in t and the speed c_θ is unique and positive.

One has then the following lemma :

Lemma 6.1 *The speeds c_θ are nonincreasing with respect to θ.*

Proof. The proof is omitted since it is identical to the proof of the estimates $\underline{c} \le c_n \le \overline{c}$ in Lemma 5.14. □

Lemma 6.2 *There exists a constant K^* such that $c_\theta \le K^*$ for all $\theta \in (0, 1/2)$.*

Proof. Let g be a $C^1([0,1])$ function such that $g(0) = g(1) = 0$, $g > 0$ in $(0,1)$, $g'(1) < 0$ and

$$\forall (x, y, u) \in \overline{\Omega} \times [0, 1], \quad f(x, y, u) \le g(u).$$

Let us define $g_\theta(u) = \chi_\theta(u)g(u)$ for all $\theta \in (0, 1/2)$ and for all $u \in [0, 1]$. Each function g_θ satisfies (1.6). For each $\theta \in (0, 1/2)$, there exists then a solution (k_θ, v_θ) of the one-dimensional problem

$$\begin{cases} v_\theta'' - k_\theta v_\theta' + g_\theta(v_\theta) = 0 \ \text{ in } \mathbb{R}, \\ v_\theta(-\infty) = 0 < v_\theta(\xi) < v_\theta(+\infty) = 1 \ \text{ for all } \xi \in \mathbb{R}. \end{cases} \tag{6.1}$$

The speed k_θ is unique and positive and the function v_θ is increasing and unique up to translation. Furthermore, it follows from a result in §8 in [18] that there exists a real number κ such that $0 \le k_\theta \le \kappa$ for all $0 < \theta < 1/2$.

On the other hand, for each $\theta \in (0, 1/2)$, the positive function v_θ' satisfies the linear elliptic equation

$$(v_\theta')'' - k_\theta(v_\theta')' + g_\theta'(v_\theta)v_\theta' = 0 \ \text{ in } \mathbb{R}.$$

The function g_θ' is defined in $[0, 1]$ by

$$\forall u \in [0, 1], \quad g_\theta'(u) = \frac{1}{\theta}\chi'\left(\frac{u}{\theta}\right) g(u) + \chi_\theta(u)g'(u).$$

The term $\chi'(u/\theta)$ vanishes outisde the interval $[\theta, 2\theta]$ and,

$$\forall u \in [\theta, 2\theta], \quad \left|\frac{1}{\theta}\chi'\left(\frac{u}{\theta}\right) g(u)\right| \le \frac{1}{\theta}\|\chi'\|_\infty \, 2\theta Lip(g) \le 2\|\chi'\|_\infty Lip(g).$$

Since $|\chi_\theta(u)g'(u)| \le \|g'\|_\infty$ for all $\theta \in (0, 1/2)$ and $u \in [0, 1]$, it follows then that the functions g_θ' are globally bounded in $[0, 1]$, uniformly with respect to $\theta \in (0, 1/2)$. Since the coefficients k_θ are also bounded independently of θ, one concludes from standard elliptic estimates and from the elliptic Harnack inequality that there exists a constant C such that

$$\forall \theta \in (0, 1/2), \quad \forall \xi \in \mathbb{R}, \quad |v_\theta''(\xi)| \le C v_\theta'(\xi).$$

From (6.1) and the boundedness of the speeds k_θ, one gets then the existence of a constant $\hat{C}$ such that

$$\forall \theta \in (0, 1/2), \quad \forall \xi \in \mathbb{R}, \quad 0 \le g_\theta(v_\theta(\xi)) \le \hat{C} v_\theta'(\xi). \tag{6.2}$$

Next, as in Lemmas 5.4 and 5.5, let $\psi \in C^2(\overline{\Omega})$ be an L-periodic with respect to x function such that $\nu A(\nabla\psi + \tilde{e}) = 0$ on $\partial\Omega$.

Take any $\theta \in (0, 1/2)$ and consider the function $\tilde{v}_\theta$ defined by

$$\forall (s, x, y) \in \mathbb{R} \times \overline{\Omega}, \quad \tilde{v}_\theta(s, x, y) = v_\theta(s + \psi(x, y)).$$

As in the proof of Lemma 5.4, it is straighforward to check that the function $\tilde{v}_\theta$ satisfies

$$\nu A(\nabla_{x,y}\tilde{v}_\theta + \tilde{e}\partial_s\tilde{v}_\theta) = 0 \ \text{ on } \mathbb{R}\times\partial\Omega$$

and

$$\begin{array}{rcl} L\tilde{v}_\theta + f_\theta(x,y,\tilde{v}_\theta) & := & \mathrm{div}_{x,y}(A\nabla_{x,y}\tilde{v}_\theta) + (\tilde{e}A\tilde{e})\partial_{ss}\tilde{v}_\theta + \mathrm{div}(A\tilde{e}\partial_s\tilde{v}_\theta) + \partial_s(\tilde{e}A\nabla_{x,y}\tilde{v}_\theta) \\ & & -q\cdot\nabla_{x,y}\tilde{v}_\theta - (q\cdot\tilde{e}+c_\theta)\partial_s\tilde{v}_\theta + f_\theta(x,y,\tilde{v}_\theta) \\ & = & [k_\theta(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi) + \mathrm{div}_{x,y}(A(\tilde{e}+\nabla\psi)) - q\cdot(\tilde{e}+\nabla\psi) - c_\theta]v_\theta'(\xi) \\ & & +f_\theta(x,y,v_\theta(\xi)) - (\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi)g_\theta(v_\theta(\xi)) \ \text{ in } \mathbb{R}\times\Omega, \end{array}$$

where $\xi = s+\psi(x,y)$. Remember that $f_\theta(x,y,u)\le g_\theta(u)$ for all $(x,y)\in\overline{\Omega}$ and $u\in[0,1]$. From (6.2) and the nonnegativity of g_θ, one gets

$$L\tilde{v}_\theta + f_\theta(x,y,\tilde{v}_\theta) \le [k_\theta(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi) + \mathrm{div}_{x,y}(A(\tilde{e}+\nabla\psi)) - q\cdot(\tilde{e}+\nabla\psi) - c_\theta + \hat{C}]v_\theta'(\xi).$$

Assume now by contradiction that $c_\theta > \max_{\overline{\Omega}}[k_\theta(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi) + \mathrm{div}_{x,y}(A(\tilde{e}+\nabla\psi)) - q\cdot(\tilde{e}+\nabla\psi)+\hat{C}]$. Since function v_θ' is positive, the function $\tilde{v}_\theta$ satisfies

$$L\tilde{v}_\theta + f_\theta(x,y,\tilde{v}_\theta) < 0 \ \text{ in } \mathbb{R}\times\Omega.$$

In other words, $\tilde{v}_\theta$ is a supersolution for the equation (3.2) satisfied by the function $\phi_\theta(s,x,y) = u_\theta((s-x\cdot e)/c_\theta,x,y)$. By sliding $\tilde{v}_\theta$ with respect to ϕ_θ -as in Lemmas 4.1 or 5.14- one is then led to a contradiction.

Therefore,

$$c_\theta \le \max_{\overline{\Omega}}\,[k_\theta(\tilde{e}+\nabla\psi)A(\tilde{e}+\nabla\psi) + \mathrm{div}_{x,y}(A(\tilde{e}+\nabla\psi)) - q\cdot(\tilde{e}+\nabla\psi)+\hat{C}].$$

Since the real numbers k_θ are bounded independently of θ (and since $\hat{C}$ in (6.2) does not depend on θ), the conclusion of Lemma 6.2 follows. □

Lemmas 6.1 and 6.2 yield the existence of a positive real number c^* such that

$$c_\theta \nearrow c^* > 0 \ \text{ as } \theta\searrow 0.$$

Consider a sequence $\theta_n \searrow 0$. Up to translation in time t, one can assume that $u_{\theta_n}(0,x_0,y_0) = 1/2$, where (x_0,y_0) is an arbitrarily chosen point in $\overline{\Omega}$. From standard parabolic regularity theory, the functions u_{θ_n} converge locally uniformly, up to extraction of some subsequence, to a function u^*, which is a classical solution of

$$\left\{\begin{array}{rcl} \partial_t u^* - \mathrm{div}(A\nabla u^*) + q\cdot\nabla u^* - f(x,y,u^*) & = & 0 \ \text{ in } \mathbb{R}\times\Omega, \\ \nu A\nabla u^* & = & 0 \ \text{ on } \mathbb{R}\times\partial\Omega, \\ \forall k\in\prod_{i=1}^d L_i\mathbb{Z}, \quad \forall(t,x,y)\in\mathbb{R}\times\overline{\Omega}, \quad u^*\left(t+\dfrac{k\cdot e}{c^*},x,y\right) & = & u^*(t,x+k,y). \end{array}\right. \tag{6.3}$$

Furthermore, by passage to the limit, u^* is such that $0\le u^*\le 1$, $u^*(0,x_0,y_0)=1/2$, and u^* is nondecreasing with respect to the variable t. Lastly, since each function u_{θ_n} satisfies the inequality (5.40) with

$$\max_{(x,y)\in\overline{\Omega}} F_{\theta_n}(x,y,1) = \max_{(x,y)\in\overline{\Omega}}\int_0^1 f_{\theta_n}(x,y,\tau)d\tau \ \left(\le \max_{(x,y)\in\overline{\Omega}} F(x,y,1)\right)$$

instead of $\max_{(x,y)\in\overline{\Omega}} F(x,y,1)$, it easily follows from Fatou's lemma that the function u^* itself satisfies (5.40). One concludes then as in the proof of part a) of Theorem 1.13 that the function $u^*(t,x,y)$ converges locally in (x,y) as $t\to\pm\infty$ to two numbers $u^\pm\in[0,1]$ such that $f(x,y,u^\pm)=0$ for all $(x,y)\in\overline{\Omega}$. Since $u^*(0,x_0,y_0)=1/2$ and $\partial_t u^*\ge 0$, and since f is positive in $\overline{\Omega}\times(0,1)$, one concludes that $u^*(t,x,y)\to 0$ (resp. 1) as $t\to-\infty$ (resp. $t\to+\infty$) locally in (x,y). From the (t,x)-periodicity of u^* (*see* the first assertion in (6.3)) and from the positivity of c^*, it follows that $u^*(t,x,y)\to 0$ (resp. 1) as $x\cdot e\to-\infty$ (resp. $x\cdot e\to+\infty$), locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e.

Eventually, the couple (c^*,u^*) is a classical solution of (1.28) with the nonlinearity f. Furthermore, the strong maximum principle applied to any function of the type $u^*(t+h,x,y)-u^*(t,x,y)$ for any $h>0$ implies that u^* is increasing with respect to t in $\mathbb{R}\times\overline{\Omega}$.

6.2 Existence of a solution (c,u) for all $c\ge c^*$

This subsection is devoted to the proof of the following

Proposition 6.3 *For each $c\ge c^*$, there exists a solution u of (1.28), with the speed c, and such that u is increasing with respect to t.*

Proof. The case $c=c^*$ has been done above in section 6.1. Let us now fix a real number $c>c^*$. One shall construct a solution u of (1.28) for the speed c. The way of doing that consists, as in section 5, in solving a regularized problem in finite cylinders with respect to the variables (s,x,y), and then in passing to the limit in the whole cylinder and in making the regularization parameter converge to 0.

Since the restriction of the function f to $\overline{\Omega}\times[0,1]$ is of class $C^{1,\delta}(\overline{\Omega}\times[0,1])$ (where $\delta>0$) with respect to the variable u, the function $v=\partial_t u^*$ is bounded in $C^1(\mathbb{R}\times\overline{\Omega})$ and of class C^2 with respect to the variables (x,y) in $\mathbb{R}\times\Omega$. Furthermore, it is a nonnegative solution of the linear parabolic equation

$$\partial_t v-\mathrm{div}(A\nabla v)+q\cdot\nabla v-f_u(x,y,u^*)v=0 \text{ in } \mathbb{R}\times\Omega$$

with Neumann boundary conditions $\nu A\nabla v=0$ on $\mathbb{R}\times\partial\Omega$. From the strong maximum principle and Hopf lemma, it follows that v is positive everywhere in $\mathbb{R}\times\overline{\Omega}$. From Schauder interior estimates [69], one has

$$\forall(t_0,x_0,y_0)\in\mathbb{R}\times\overline{\Omega},\quad |v_t(t_0,x_0,y_0)|\le C_1\max_{\{t_0-1\le t\le t_0,\ (x,y)\in\overline{\Omega},\ |(x,y)-(x_0,y_0)|\le 1\}} v(t,x,y)$$

for some constant C_1 independent of (t_0,x_0,y_0). Choose now a given vector $k_0\in\prod_{i=1}^d L_i\mathbb{Z}$ such that $k_0\cdot e>0$. It follows then from Krylov-Safonov-Harnack-type inequalities (*see e.g.* [43], [56], [68]) that

$$\forall(t_0,x_0,y_0)\in\mathbb{R}\times\overline{\Omega},\quad \max_{\{t_0-1\le t\le t_0,\ (x,y)\in\overline{\Omega},\ |(x,y)-(x_0,y_0)|\le 1\}} v(t,x,y)\le C_2\, v\left(t+\frac{k_0\cdot e}{c^*},x_0-k_0,y_0\right)$$

for some constant C_2 independent of (t_0,x_0,y_0). Therefore, $|v_t(t_0,x_0,y_0)| \le C_1C_2 v(t+(k_0\cdot e)/c^*, x_0-k_0,y_0)$ for all $(t_0,x_0,y_0)\in I\!R\times\overline{\Omega}$. But the function $v=\partial_t u^*$ satisfies the same periodicity condition in the variables (t,x,y) as u^*. As a consequence,

$$\forall (t_0,x_0,y_0)\in I\!R\times\overline{\Omega},\quad |v_t(t_0,x_0,y_0)|\le C_1C_2 v(t_0,x_0,y_0).$$

In other words, the function $\phi^*(s,x,y)=u^*((s-x\cdot e)/c^*,x,y)$ satisfies $|\partial_{ss}\phi^*(s,x,y)|\le (C_1C_2/c^*)\ \partial_s\phi^*(s,x,y)$ for all $(s,x,y)\in I\!R\times\overline{\Omega}$.

For any $\varepsilon>0$, let L_ε be the elliptic operator defined as in (5.2) by $L_\varepsilon\phi=(\tilde e A\tilde e+\varepsilon)\phi_{ss}+\mathrm{div}_{x,y}(A\nabla_{x,y}\phi)+\mathrm{div}_{x,y}(A\tilde e\phi_s)+\partial_s(\tilde eA\nabla_{x,y}\phi)-q\cdot\nabla_{x,y}\phi-(q\cdot\tilde e+c)\phi_s$. From the definition of ϕ^*, one has $L_\varepsilon\phi^*+f(x,y,\phi^*)=\varepsilon\phi^*_{ss}+(c^*-c)\phi^*_s$. Hence,

$$L_\varepsilon\phi^*+f(x,y,\phi^*)<\left(\varepsilon\frac{C_1C_2}{c^*}+c^*-c\right)\phi^*_s<0\ \text{ in } I\!R\times\Omega \tag{6.4}$$

for ε small enough.

Choose ε small enough so that (6.4) holds. Let a be any positive real number. For any $\tau\in I\!R$, let $h_\tau:=\min_{\overline{\Omega}}\phi^*(-a+\tau,\cdot,\cdot)=\min_{\overline{C}}\phi^*(-a+\tau,\cdot,\cdot)$ (remember that the function ϕ^* is L-periodic with respect to x). Under the notations of section 5.1 and by using Schauder fixed point theorem as in Lemma 5.1, it follows that there exists a solution $\phi_\tau(s,x,y)\in C(\overline{\Sigma}_a)\cap C^2(\tilde\Sigma_a)$ of

$$\left\{\begin{array}{rcl} L_\varepsilon\phi_\tau+f(x,y,\phi_\tau) &=& 0\ \text{ in } \Sigma_a,\\ \nu A(\nabla_{x,y}\phi_\tau+\tilde e\partial_s\phi_\tau) &=& 0\ \text{ on } (-a,a)\times\partial\Omega,\\ \phi_\tau \text{ is } L\text{-periodic w.r.t. } x, && \\ \phi_\tau(-a,x,y)=h_\tau,\quad \phi_\tau(a,x,y) &=& \phi^*(a+\tau,x,y)\ \text{ for all } (x,y)\in\overline{\Omega}.\end{array}\right. \tag{6.5}$$

This can be done by writting $\phi_\tau=v+[h_\tau+\frac{s+a}{2a}(\phi^*(a+\tau,x,y)-h_\tau)]$ and solving a problem with unknown v.

The function f being nonnegative, one has $L_\varepsilon\phi_\tau\le 0$. Since ϕ_τ is not constant in $\overline{\Sigma}_a$ because $\phi_\tau(-a,\cdot,\cdot)=h_\tau\le\phi^*(-a+\tau,\cdot,\cdot)<\phi^*(a+\tau,\cdot,\cdot)=\phi_\tau(a,\cdot,\cdot)$, the maximum principle and the Hopf lemma yield that

$$\forall (s,x,y)\in(-a,a]\times\overline{\Omega},\quad h_\tau<\phi_\tau(s,x,y). \tag{6.6}$$

Similarly, since $f(x,y,u)=0$ for all $(x,y)\in\overline{\Omega}$ and for all $u\ge 1$, one gets $\phi_\tau<1$ in $\overline{\Sigma}_a$. Therefore, the limit $\phi^*(+\infty,\cdot,\cdot)=1$ implies that $\phi^*(s+\tau+k,x,y)\ge\phi_\tau(s,x,y)$ in $\overline{\Sigma}_a$ for k large enough. Let $\overline{k}$ be the smallest nonnegative k such that the latter holds and assume that $\overline{k}>0$. Since $\overline{\Sigma}_a$ is compact, it necessarily follows that $\phi^*(s+\tau+\overline{k},x,y)\ge\phi_\tau(s,x,y)$ in $\overline{\Sigma}_a$ with equality somewhere at a point $(\overline{s},\overline{x},\overline{y})$. Since ϕ^* is increasing in s, it is found that $\phi^*(-a+\tau+\overline{k},\cdot,\cdot)>\phi^*(-a+\tau,\cdot,\cdot)\ge h_\tau=\phi_\tau(-a,\cdot,\cdot)$ and $\phi^*(a+\tau+\overline{k},\cdot,\cdot)>\phi^*(a+\tau,\cdot,\cdot)=\phi_\tau(a,\cdot,\cdot)$. Therefore, $(\overline{s},\overline{x},\overline{y})\in(-a,a)\times\overline{\Omega}$. But, from (6.4), the function $\phi^*(s+\tau+\overline{k},x,y)$ is a supersolution for the elliptic equation which ϕ_τ is a solution of, and both $\phi^*(s+\tau+\overline{k},x,y)$ and ϕ_τ satisfy the same boundary conditions on $(-a,a)\times\partial\Omega$. Therefore, the strong maximum principle and Hopf lemma imply that $\phi^*(s+\tau+\overline{k},x,y)=\phi_\tau(s,x,y)$ for all $(s,x,y)\in\overline{\Sigma}_a$, which is impossible because of the different boundary conditions at $s = \pm a$. As a conclusion, $\overline{k} = 0$ and

$$\forall (s,x,y) \in \overline{\Sigma}_a, \quad \phi_\tau(s,x,y) \le \phi^*(s+\tau,x,y). \tag{6.7}$$

Note in particular that, since ϕ^* is increasing in s, one has $\phi_\tau(s,x,y) < \phi^*(a+\tau,x,y)$ if $s < a$. Putting that together with (6.6) leads to

$$\forall (s,x,y) \in (-a,a) \times \overline{\Omega}, \quad h_\tau < \phi_\tau(s,x,y) < \phi^*(a+\tau,x,y).$$

Then, by using the same sliding method as in Lemma 5.2, it follows that ϕ_τ is increasing with respect to the variable s and that ϕ_τ is the unique solution of (6.5) in $C(\overline{\Sigma}_a) \cap C^2(\tilde{\Sigma}_a)$. Furthermore, since the boundary conditions for ϕ_τ at $s = \pm a$ are continuous and increasing with respect to τ, one can prove, by using similar arguments as in Lemma 5.3, that the functions ϕ_τ depend continuously on τ in $C(\overline{\Sigma}_a) \cap C^2(\tilde{\Sigma}_a)$, and are also increasing with respect to τ, in the sense that if $\tau_1 < \tau_2$, then $\phi_{\tau_1} < \phi_{\tau_2}$ in $\overline{\Sigma}_a$. On the other hand, since $\phi^*(-\infty,\cdot,\cdot) = 0$ and $\phi^*(+\infty,\cdot,\cdot) = 1$, it follows from (6.6) and (6.7) that $\phi_\tau \to 0$ (resp. 1) as $\tau \to -\infty$ (resp. $\tau \to +\infty$), uniformly in $\overline{\Sigma}_a$. Therefore, there exists a unique $\tau(a) \in I\!R$ such that $\phi^{\varepsilon,a} := \phi_{\tau(a)}$ solves (6.5) and satisfies, say for $a > 1$,

$$\int_{(0,1)\times C} \phi^{\varepsilon,a}(s,x,y)\, ds\, dx\, dy \;=\; \frac{1}{2}|C|.$$

Choose a sequence $a_n \to +\infty$. From standard elliptic estimates, the functions ϕ^{ε,a_n} converge in $C^{2,\alpha}_{loc}(I\!R \times \overline{\Omega})$ (for $0 < \alpha < 1$), up to extraction of some subsequence, to a function ϕ^ε solving

$$\left\{ \begin{array}{rcl} L_\varepsilon \phi^\varepsilon + f(x,y,\phi^\varepsilon) &=& 0 \text{ in } I\!R \times \overline{\Omega}, \\ \nu A(\nabla_{x,y}\phi^\varepsilon + \tilde{e}\phi^\varepsilon_s) &=& 0 \text{ on } I\!R \times \partial\Omega, \\ \phi^\varepsilon \text{ is } L\text{-periodic w.r.t. } x. && \end{array} \right. \tag{6.8}$$

Furthermore, the function ϕ^ε is nonincreasing with respect to s and it satisfies $0 \le \phi^\varepsilon \le 1$ in $I\!R \times \overline{\Omega}$ and

$$\int_{(0,1)\times C} \phi^\varepsilon(s,x,y)\, ds\, dx\, dy \;=\; \frac{1}{2}|C|. \tag{6.9}$$

Standard elliptic estimates together with the monotonicity of ϕ^ε with respect to the variable s imply that $\phi^\varepsilon(s,x,y) \to \phi^\varepsilon_\pm(x,y)$ as $s \to \pm\infty$ in $C^{2,\alpha}_{loc}(\overline{\Omega})$, where the functions $\phi_\pm$ range in $[0,1]$, are L-periodic with respect to x and satisfy

$$\left\{ \begin{array}{rcl} \text{div}(A\nabla\phi^\varepsilon_\pm) - q\cdot\nabla\phi^\varepsilon_\pm + f(x,y,\phi^\varepsilon_\pm) &=& 0 \;\text{ in } \overline{\Omega}, \\ \nu A \nabla \phi^\varepsilon_\pm &=& 0 \;\text{ on } \partial\Omega. \end{array} \right. \tag{6.10}$$

By integrating (6.10) on a cell C, one gets $\displaystyle\int_C f(x,y,\phi^\varepsilon_\pm) = 0$, whence $f(x,y,\phi^\varepsilon_\pm) \equiv 0$ since the function f is nonnegative. By multiplying (6.10) by $\phi^\varepsilon_\pm$ are integrating over C, one concludes that $\displaystyle\int_C \nabla\phi^\varepsilon_\pm A \nabla\phi^\varepsilon_\pm = 0$. Eventually, both functions $\phi^\varepsilon_\pm$ are constant and such that $f(x,y,\phi^\varepsilon_\pm) \equiv 0$ in $\overline{\Omega}$. Since f satisfies (1.27), the normalization condition (6.9) implies then that $\phi^\varepsilon_- = 0$ and $\phi^\varepsilon_+ = 1$.

Let us now come back to the variables (t,x,y). As it was done in the proof of Lemma 5.11, it follows from (6.8) and from the limiting behaviour of $\phi^\varepsilon(s,x,y)$ as $s\to\pm\infty$, that the functions $u^\varepsilon(t,x,y) := \phi^\varepsilon(x\cdot e+ct,x,y)$ satisfy the gradient estimates (5.27), independently of ε. As done in section 5.4, there exists then a function $u\in H^1_{loc}(I\!R\times\Omega)$ such that, up to extraction of some subsequence, $u^\varepsilon \rightharpoonup u$ weakly in $H^1_{loc}(I\!R\times\Omega)$. From parabolic regularity, the function u is then a classical solution of

$$\begin{cases} u_t-\mathrm{div}(A\nabla_{x,y}u)+q\cdot\nabla_{x,y}u-f(x,y,u) &= 0 \text{ in } I\!R\times\Omega,\\ \nu A\nabla_{x,y}u &= 0 \text{ on } I\!R\times\partial\Omega,\\ 0\ \le\ u\ \le\ 1,\quad u_t &\ge 0 \text{ in } I\!R\times\overline{\Omega}. \end{cases}$$

Furthermore, for all $k\in\prod_{i=1}^d L_i\mathbb{Z}$, the equality $u^\varepsilon(t+(k\cdot e)/c,x,y)=u^\varepsilon(t,x+k,y)$ in $I\!R\times\overline{\Omega}$ carries over for u at the limit $\varepsilon\to 0$, almost everywhere and then everywhere in $I\!R\times\overline{\Omega}$ by continuity of u. Lastly, the function u satisfies

$$\int_{0<x\cdot e+ct<1,\ (x,y)\in C} u(t,x,y)\ dt\ dx\ dy\ =\ \frac{1}{2c}|C| \tag{6.11}$$

and the gradients estimates (5.40).

Next, let us prove that the function u satisfies the limiting conditions $u(t,x,y)\to 0$ (resp. 1) as $t\to-\infty$ (resp. $t\to+\infty$). As done in section 5.4, there exist two reals numbers $u^\pm\in[0,1]$ such that $u(t,x,y)\to u^\pm$ locally in (x,y) as $t\to\pm\infty$, and $f(x,y,u^\pm)=0$ for all $(x,y)\in\overline{\Omega}$. From (1.27) and (6.11) and from the monotonicity of u with respect to t, one concludes that $u^-=0$ and $u^+=1$. Eventually, the (x,t)-periodicity of u and the positivity of c imply that $u(t,x,y)\to 0$ as $x\cdot e\to-\infty$ and $u(t,x,y)\to 1$ as $x\cdot e\to+\infty$, locally in t and uniformly in y and in the directions of $I\!R^d$ which are orthogonal to e.

Hence, the couple (c,u) is a classical solution of (1.28). Furthermore, for any $h>0$, the nonnegative function $u(t+h,x,y)-u(t,x,y)$ is actually positive everywhere in $I\!R\times\overline{\Omega}$ from the strong parabolic maximum principle. That means that u is increasing with respect to time t. That completes the proof of Proposition 6.3. □

6.3 Nonexistence of solutions (c,u) if $c<c^*$

Under the notation of the beginning of section 6.1, for each $\theta\in(0,1/2)$, let (c_θ,u_θ) be the unique (up to translation in t for u_θ) solution of (1.28) with the nonlinearity f_θ. One knows that u_θ is increasing with respect to the variable t. Remember that each function f_θ is extended by 0 outside the interval $[0,1]$.

In order to complete the proof of Theorem 1.14, part a), let us prove the following

Lemma 6.4 *Let $c<c^*$. Then there is no solution (c,u) of (1.28).*

Proof. Assume by contradiction that there exists a solution (c,u) of (1.28) for a speed $c<c^*$. From Lemma 6.1, there exists a positive real number θ small enough so that $c<c_\theta$.

Lemma 3.1 implies that $c>0$. The function $\phi_\theta(s,x,y) := u_\theta((s-x\cdot e)/c_\theta,x,y)$ is of class $C^{1,\mu}(I\!R\times\overline{\Omega})$ (for each $\mu\in[0,1)$) and it is a subsolution for the equation (3.2) satisfied by

$\phi(s,x,y) := u((s-x\cdot e)/c,x,y)$, in the sense that

$$\begin{array}{rcl}(\tilde{e}A\tilde{e})\partial_{ss}\phi_\theta+\mathrm{div}_{x,y}(A\nabla_{x,y}\phi_\theta) & & \\ +\mathrm{div}_{x,y}(A\tilde{e}\ \partial_s\phi_\theta)+\partial_s(\tilde{e}A\nabla_{x,y}\phi_\theta) & & \\ -q\cdot\nabla_{x,y}\phi_\theta-(q\cdot\tilde{e}+c)\partial_s\phi_\theta+f(x,y,\phi_\theta) & = & (c_\theta-c)\ \partial_s\phi_\theta \\ & & +f(x,y,\phi_\theta)-f_\theta(x,y,\phi_\theta) \\ & \geq & 0 \quad \text{in } \mathcal{D}'(I\!R\times\overline{\Omega})\end{array} \tag{6.12}$$

since $\partial_s\phi_\theta\geq 0$ in $I\!R\times\overline{\Omega}$, $c<c_\theta$ and $f_\theta\leq f$. On the other hand, $\phi_\theta(-\infty,\cdot,\cdot)=\phi(-\infty,\cdot,\cdot)=0$, $\phi_\theta(+\infty,\cdot,\cdot)=\phi(+\infty,\cdot,\cdot)=1$ and the function f_θ is nonincreasing in a left neighborhood of 0 (in fact, $f_\theta(x,y,u)=0$ in $\overline{\Omega}\times[0,\theta]$) as well as in a right neighborhood of 1. Furthermore, both functions ϕ_θ and ϕ are L-periodic with respect to x and satisfy the same Neumann-type boundary conditions on $I\!R\times\partial\Omega$. Therefore, with the same sliding method as in Lemmas 4.1 and 6.1, it is found that there exists a real number τ^* such that $\phi_\theta(s+\tau^*,x,y)=\phi(s,x,y)$ for all $(s,x,y)\in I\!R\times\overline{\Omega}$. Putting that into (6.12) implies that $(c_\theta-c)\ \partial_s\phi_\theta+f(x,y,\phi_\theta)-f_\theta(x,y,\phi_\theta)\equiv 0$, whence $\partial_s\phi_\theta\equiv 0$. One has then reached a contradiction.

That completes the proof of Lemma 6.4 as well as that of Theorem 1.14, part a). □

6.4 Monotonicity of u with respect to t in the case $f_u^+(x,y,0)>0$

In this subsection, one proves that every solution u of (1.28) is increasing in t provided that f satisfies the additional assumption that $f_u^+(x,y,0):=\lim_{u\to 0^+} f(x,y,u)/u$ is positive in $\overline{\Omega}$. Let us first state the following

Lemma 6.5 *Let f be a function satisfying (1.24-1.25) and (1.27) and assume that the function $(x,y)\mapsto\zeta(x,y):=f_u^+(x,y,0)$ is positive for all $(x,y)\in\overline{\Omega}$. Let (c,u) be a classical solution of (1.28). Then $c>0$,*

$$0<\Lambda:=\liminf_{t\to-\infty,\ (x,y)\in\overline{C}}\frac{u_t(t,x,y)}{u(t,x,y)}<+\infty,$$

and, under the notations of Proposition 5.7, one has $\mu_{c,\zeta}(\Lambda/c)=0$.

Proof. The positivity of c follows from part a) of Theorem 1.14 (that can also be obtained directly by integrating the equation satisfied by $\phi(s,x,y)=u((s-x\cdot e)/c,x,y)$ over $I\!R\times C$ as in Lemma 3.1).

Next, as it was done for the function u_t^* in the course of Proposition 6.3, it follows from standard interior estimates, from Harnack type inequalities and from the (t,x)-periodicity of u, that both fields u_t/u and $\nabla_{x,y}u/u$ are globally bounded. Let now Λ be defined as in Lemma 6.5. One has $\Lambda\in I\!R$.

Take a sequence (t_n,x_n,y_n) such that $(x_n,y_n)\in\overline{C}$, $t_n\to-\infty$ and

$$u_t(t_n,x_n,y_n)/u(t_n,x_n,y_n)\to\Lambda \text{ as } n\to+\infty.$$

Up to extraction of some subsequence, one can assume that $(x_n,y_n)\to(x_\infty,y_\infty)\in\overline{C}$ as $n\to+\infty$. From the (t,x)-periodicity of u and from the limiting behavior of u as $x\cdot e\to-\infty$, one has $u(t,x,y)\to 0$ as $t\to-\infty$ locally in (x,y). Consider now the positive functions

$$w_n(t,x,y)=\frac{u(t+t_n,x,y)}{u(t_n,x_n,y_n)}.$$

Since u_t/u and $\nabla_{x,y}u/u$ are globally bounded, the functions w_n are locally bounded. They satisfy the equations

$$\partial_t w_n - \mathrm{div}(A\nabla w_n) + q\cdot\nabla w_n - \frac{f(x,y,u(t+t_n,x,y))}{u(t+t_n,x,y)}\, w_n = 0 \ \text{ in } I\!\!R\times\overline{\Omega}$$

together with boundary conditions $\nu A\nabla w_n = 0$ on $I\!\!R\times\partial\Omega$. From standard parabolic estimates, the positive functions w_n converge in $C^1_{loc}(I\!\!R\times\overline{\Omega})$, up to extraction of some subsequence, to a nonnegative $C^1(I\!\!R\times\overline{\Omega})$ solution $w_\infty(t,x,y)$ of

$$\left\{\begin{array}{rcl}\partial_t w_\infty - \mathrm{div}(A\nabla w_\infty) + q\cdot\nabla w_\infty - \zeta(x,y)w_\infty & = & 0 \text{ in } I\!\!R\times\Omega,\\ \nu A\nabla w_\infty & = & 0 \text{ on } I\!\!R\times\partial\Omega.\end{array}\right.$$

Furthermore, $w_\infty(0,x_\infty,y_\infty) = 1$, whence w_∞ is positive from the strong maximum principle and Hopf lemma. The function w_∞ is also such that $w_\infty(t+(k\cdot e)/c,x,y) = w_\infty(t,x+k,y)$ for all $k\in\prod_{i=1}^d L_i Z\!\!\!Z$ and for all $(t,x,y)\in I\!\!R\times\overline{\Omega}$.

On the other hand, one has

$$\partial_t w_n(t,x,y) = \frac{u_t(t+t_n,x,y)}{u(t+t_n,x,y)}\, w_n(t,x,y) \ \text{ for all } (t,x,y)\in I\!\!R\times\overline{\Omega}.$$

It follows then from the definition of Λ and from the choice of (t_n,x_n,y_n) that $\partial_t w_\infty(0,x_\infty,y_\infty) = \Lambda = \Lambda w_\infty(0,x_\infty,y_\infty)$ and that $\partial_t w_\infty(t,x,y) \ge \Lambda w_\infty$ for all $t\in I\!\!R$ and for all $(x,y)\in\overline{C}$. Together with the (t,x)-periodicity of w_∞, one gets $\partial_t w_\infty(t,x,y)\ge\Lambda w_\infty(t,x,y)$ for all $(t,x,y)\in I\!\!R\times\overline{\Omega}$. Since $\partial_t w_n/w_n = u_t(t+t_n,x,y)/u(t_n,x_n,y_n)$ and $\nabla_{x,y}w_n/w_n = \nabla_{x,y}u(t+t_n,x,y)/u(t_n,x_n,y_n)$, and since u_t/u and $\nabla_{x,y}u/u$ are globally bounded, it follows that $\partial_t w_\infty/w_\infty$ and $\nabla_{x,y}w_\infty/w_\infty$ are globally bounded. Consider now the function $z = \partial_t w_\infty/w_\infty$. It is actually a claissical solution of the following linear parabolic equation

$$\left\{\begin{array}{rcl} z_t - \mathrm{div}(A\nabla z) - 2\dfrac{\nabla w_\infty}{w_\infty}\cdot\nabla z + q\cdot\nabla z & = & 0 \text{ in } I\!\!R\times\Omega,\\ \nu A\nabla z & = & 0 \text{ on } I\!\!R\times\partial\Omega,\end{array}\right.$$

and $z\ge\Lambda$ with equality somewhere. From the strong maximum principle and Hopf lemma together with uniqueness of the corresponding Cauchy problem, it follows that $z(t,x,y)\equiv\Lambda$ for all $(t,x,y)\in I\!\!R\times\overline{\Omega}$. Therefore, the function $w_\infty(t,x,y)e^{-\Lambda t}$ does not depend on t.

Let us now define the function $\psi(x,y) = w_\infty(0,x,y)e^{-\Lambda\,(x\cdot e)/c}$. This function is positive and it is straighforward to check that, under the notations of Proposition 5.7, ψ is a $C^2(\overline{\Omega})$ solution of the following eigenvalue problem

$$\left\{\begin{array}{rcl} -L_{c,\Lambda/c,\zeta}\psi & = & 0 \ \text{ in } \Omega,\\ \nu A\left(\tilde{e}\dfrac{\Lambda}{c}\psi + \nabla\psi\right) & = & 0 \ \text{ on } \partial\Omega,\\ \psi \text{ is } L\text{-periodic w.r.t. } x, & & \end{array}\right.$$

where the function $\zeta(x,y) = f_u^+(x,y,0)$ is L-periodic with respect to x and of class $C^{0,\delta}(\overline{\Omega})$ from assumption (1.27). In other words, $\mu_{c,\zeta}(\Lambda/c) = 0$.

Since this function ζ is continuous, L-periodic w.r.t. x and positive everywhere, it follows then that there exists a real number $\zeta_0 > 0$ such that $\zeta(x,y) \ge \zeta_0 > 0$ for all $(x,y) \in \overline{\Omega}$. Parts 2) and 3) of Proposition 5.7 yield then that

$$0 = \mu_{c,\zeta}(\Lambda/c) \le \mu_{c,\zeta_0}(\Lambda/c) = -\zeta_0 + \Lambda + h(\Lambda/c).$$

Therefore, $\Lambda \ge \zeta_0 - h(\Lambda/c) > 0$ since $\zeta_0 > 0$ and h is a concave function such that $h(0) = h'(0) = 0$. □

Let us now complete this section with the following

Proposition 6.6 *Let f be a function satisfying (1.24-1.25), (1.27) and such that $f_u^+(x,y,0) > 0$ for all $(x,y) \in \overline{\Omega}$. Let (c,u) be a classical solution of (1.28). Then the function u is increasing with respect to the variable t.*

Proof. Let ϕ be the function defined as in the previous sections by $\phi(s,x,y) = u((s - x\cdot e)/c, x, y)$. Since $u_t(t,x,y)/u(t,x,y) = c\phi_s(ct + x\cdot e, x, y)/\phi(ct + x\cdot e, x, y)$ and $c > 0$, it follows from Lemma 6.5 that $\liminf_{s\to-\infty,\ (x,y)\in\overline{C}}\ \phi_s(s,x,y)/\phi(s,x,y) > 0$. Since ϕ is L-periodic with respect to x, there exists then $\underline{s} \in I\!R$ such that $\phi_s(s,x,y) > 0$ for all $s \le \underline{s}$ and for all $(x,y) \in \overline{\Omega}$. On the other hand, $\inf_{s\ge\underline{s}\ (x,y)\in\overline{\Omega}}\ \phi(s,x,y) > 0$ and $\phi(-\infty,x,y) = 0$ uniformly in $(x,y) \in \overline{\Omega}$. Therefore, there exists $B \in I\!R$ such that $-B \le \underline{s}$ and

$$\forall \tau \ge 0,\ \forall s \le -B,\ \forall (x,y) \in \overline{\Omega}, \quad \phi(s,x,y) \le \phi(s+\tau,x,y). \tag{6.13}$$

Next, even if it means increasing B, one can assume that $B \ge 0$ and $\phi(s,x,y) \ge 1 - \rho$ for all $s \ge B$, $(x,y) \in \overline{\Omega}$, where $\rho > 0$ is given as in (1.27). It follows then from (6.13) and from Lemma 3.4 applied in Σ_B^+ that $\phi(s,x,y) \le \phi(s+\tau,x,y)$ for all $\tau \ge 2B$ and for all $(s,x,y) \in I\!R \times \overline{\Omega}$.

Let now define

$$\tau^* = \inf\ \{\tau > 0,\ \phi(s,x,y) \le \phi(s+\tau',x,y) \text{ for all } \tau' \ge \tau \text{ and for all } (s,x,y) \in I\!R \times \overline{\Omega}\}.$$

By using (6.13) and adapting the proof of Lemma 3.5, one concludes that $\tau^* = 0$. In other words, the function ϕ is nondecreasing in s. Hence, the function u is nondecreasing in time t. As already underlined, it follows from the strong maximum principle and Hopf lemma that $u(t+h,x,y) > u(t,x,y)$ for all $h > 0$ and for all $(t,x,y) \in I\!R \times \overline{\Omega}$, which means that u is increasing with respect to t. Since the restriction of the function $(x,y,\tau) \mapsto f(x,y,\tau)$ to $\overline{\Omega} \times [0,1]$ is of class C^1 with respect to τ, one can even say, by differentiating the equation satisfied by u, that $u_t(t,x,y) > 0$ for all $(t,x,y) \in I\!R \times \overline{\Omega}$. □

7 Appendix : uniform pointwise gradient estimates for a general class of elliptic equations; proof of Lemma 5.13

The uniform gradients estimates stated in Lemma 5.13 are a consequence of the following more general result :

Theorem 7.1 *Let $\omega \subset \mathbb{R} \times \mathbb{R}^N$ be an open set and let*

$$X = (t, x) = (t, x_1, \cdots, x_N)$$

be the generic notation for the points in $\mathbb{R} \times \mathbb{R}^N$. Let $(\alpha^{ij})_{1\le i,j\le N}$ be a $C^1(\overline{\omega})$ matrix field such that

$$\exists \sigma > 0, \quad \forall X \in \overline{\omega}, \quad \forall \xi \in \mathbb{R}^N, \qquad \sum_{1\le i,j\le N} \alpha^{ij}(X)\xi_i\xi_j \ge \sigma|\xi|^2.$$

Let $(\beta^i)_{1\le i\le N}$ be a $C^1(\overline{\omega})$ vector field. Let $M > 0$ and let $(X, u) \mapsto f(X, u)$ be a $C^1(\overline{\omega}\times[M, M])$ function. Let $b \ge 0$ be such that $\|\alpha^{ij}\|_{C^1(\overline{\omega})} \le b$ for all $1 \le i, j \le N$, $\|\beta^i\|_{C^1(\overline{\omega})} \le b$ for all $1 \le i \le N$ and $\|f\|_{C^1(\overline{\omega}\times[-M,M])} \le b$.

1) There exists a constant $C_1 = C_1(M, b, N, \sigma)$, which only depends on M, b, N and σ such that, for any $0 < \varepsilon \le 1$ and for any function $u \in C^3(\omega)$ such that $|u| \le M$ and

$$\varepsilon u_{tt} - u_t + \sum_{1\le i,j\le N} \alpha^{ij}(X)u_{ij} + \sum_{1\le i\le N} \beta^i(X)u_i + f(X, u) = 0 \quad \text{in } \omega,$$

then

$$\forall X \in \omega, \quad |\nabla_x u(X)| \le C_1 \left(d(X, \partial\omega)^{-1} + 1\right) \tag{7.1}$$

where $u_t = \partial_t u$, $u_i = \partial_{x_i} u$, $u_{ij} = \partial_{x_i x_j} u$ and $d(X, \partial\omega)$ denotes the euclidian distance in $\mathbb{R}^{N+1}$ of X to $\partial\omega$, under the convention that $d(X, \partial\omega) = +\infty$ if $\partial\omega$ is empty.

2) Let Σ be a smooth (at least globally of class C^3) subset of $\partial\omega$ and assume that Σ is open relatively to $\partial\omega$, in the sense that for each $X \in \Sigma$, there exists $r_X > 0$ such that $Y \in \Sigma$ whenever $Y \in \partial\omega \cap B_{r_X}(X)$, where $B_{r_X}(X)$ is the open ball with radius r_X and center X. Let $\nu = \nu(X)$ be the unit outward normal to ω on Σ. Assume that the t-component of $\nu(X)$ is zero for all $X \in \Sigma$. Assume moreover that there exists $\eta > 0$ such that, for all $X \in \Sigma$, the connected component of $B_\eta(X)\backslash\Gamma$ containing $X - r\nu(X)$ for $r > 0$ small enough is included in ω, where Γ is the connected component of $\partial\omega \cap B_\eta(X)$ containing X. Let μ be a $C^3(\Sigma)$ unit vector field whose t-component is zero, whose $C^3(\Sigma)$ norm is finite, and assume there exists $\gamma > 0$ such that $\mu(X) \cdot \nu(X) \ge \gamma > 0$ for all $X \in \Sigma$.

Then there exists a constant $C_2 = C_2(M, b, N, \sigma, \eta, \gamma, \Sigma, \mu)$, which only depends on M, b, N, σ, η, γ, on the bounds of the derivatives up to the third order of the functions representing Σ, and on the $C^3(\Sigma)$ norm of μ, such that, for any $0 < \varepsilon \le 1$ and for any function $u \in C^2(\omega \cup \Sigma) \cap C^3(\omega)$ such that $|u| \le M$ and

$$\begin{cases} \varepsilon u_{tt} - u_t + \displaystyle\sum_{1\le i,j\le N} \alpha^{ij}(X)u_{ij} + \sum_{1\le i\le N} \beta^i(X)u_i + f(X, u) &= 0 \quad \text{in } \omega, \\ \mu \cdot \nabla_X u &= 0 \quad \text{on } \Sigma, \end{cases}$$

then

$$\forall X \in \omega \cup \Sigma, \quad |\nabla_x u(X)| \le C_2 \left(d(X, \partial\omega\backslash\Sigma)^{-1} + 1\right), \tag{7.2}$$

under the convention that $d(X, \partial\omega\backslash\Sigma) = +\infty$ if $\partial\omega\backslash\Sigma = \emptyset$.

Remark 7.2 Part 2) is clearly stronger than part 1), which corresponds to the case where $\Sigma = \emptyset$. But for the sake of clarity we chose to write two different results for the interior estimates and for the estimates up to the boundary.

As far as we know, for this type of regularizing problems, estimates of the type (7.1) or (7.2) up to the boundary have not been yet obtained in this full generality, with explicit dependance on the distance to $\partial\omega\backslash\Sigma$. Theorem 7.1 is of independent interest and it is the purpose of the paper [11]. Its proof is based on the maximum principle applied to Bernstein-type functions involving $|\nabla_x u|^2$ and u^2.

Let us complete this paper with the

Proof of Lemma 5.13. Remember that each function u^ε solves (5.32) and that, under the assumptions of Lemma 5.13, $\varepsilon/(c^\varepsilon)^2 \to 0^+$ as $\varepsilon \to 0^+$ since $c^\varepsilon \to c > 0$. Therefore, $0 < \varepsilon/(c^\varepsilon)^2 \le 1$ for ε small enough.

From the regularity assumptions on A, q and f, each u^ε is of class $C^3(I\!\!R\times\Omega)\cap C^2(I\!\!R\times\overline{\Omega})$. On the other hand, the domain Ω is of class C^3 and it is L-periodic with respect to x. Therefore, the domain $I\!\!R\times\Omega$ is globally C^3 and its boundary $I\!\!R\times\partial\Omega$ satisfies the assumptions of part 2) of Theorem 7.1. Furthermore, the vector fields ν and $A\nu$ are also globally $C^3(I\!\!R\times\partial\Omega)$.

Since the condition $\nu A\nabla_{x,y}u^\varepsilon = 0$ is satisfied on the whole straight boundary $I\!\!R\times\Omega$ and since the functions u^ε are globally bounded, the conclusion of Lemma 5.13 follows. □

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