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# Asymptotic Techniques for Kinetic Problems of Boltzmann Type

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# ASYMPTOTIC TECHNIQUES FOR KINETIC PROBLEMS OF BOLTZMANN TYPE.

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**ABSTRACT**

The problems discussed in this work concern asymptotic techniques and detailed quantitative properties close to global equilibrium in classical kinetic theory. The discussion is mainly centered on a particular two-rolls model problem for the Boltzmann equation and hard forces, with the understanding that such a program can be applied in many other contexts for single and multi-component gases. The topics include asymptotic expansions, a priori estimates, existence results, fluid dynamic limits, bifurcations and stability questions.

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# 1 Background.

This section contains some background material for the following presentation, including references to more complete introductions for each separate topic.

The Boltzmann equation can formally and in a few notable cases rigorously, be derived from particle mechanics via the so called BBGKY hierarchy ([L], [IP] and others). With the $n$-particle Hamiltonian in a container $\Omega$,

$$H_N^\Omega = \sum_1^N \frac{p_i^2}{2} + \sum_{i<j=1}^N \Phi(q_i - q_j) + \sum_1^N u^\Omega(q_i),\ u^\Omega = 0 \text{ in } \Omega,\ = \infty \text{ outside } \Omega,$$

the Hamiltonian system for the $n$-particle evolution becomes

$$\frac{\partial P_i}{\partial t} = -\frac{\partial H_N^\Omega(X)}{\partial Q_i},\quad \frac{\partial Q_i}{\partial t} = \frac{\partial H_N^\Omega(X)}{\partial P_i},\quad t > 0,\quad X = (Q, P),$$
$$P_i(0) = p_i,\quad Q_i(0) = q_i,\quad i = 1, ..., N,$$

or in Poisson brackets

$$\frac{d}{dt} f(X) = \{f(X), H_N^\Omega\},$$

where

$$\{f(X), H_N^\Omega\} = \Big(\sum_{i=1}^N (P_i, \frac{\partial f(X)}{\partial Q_i}) - \sum_{i\neq j=1}^N (\frac{\partial}{\partial Q_i}\Phi(Q_i - Q_j), \frac{\partial f(X)}{\partial P_i})$$
$$-\sum_{i=1}^N (\frac{\partial}{\partial Q_i} u^\Omega(qQ_i), \frac{\partial f(X)}{\partial P_i})\Big).$$

This is the Liouville equation for the evolution of the phase space density $f_N$. Integrating away all but $s$ particles gives a hierarchy of equations, the BBGKY hierarchy, with the equation for the $s$-particle density

$$\frac{d}{dt} f_s = -\mathcal{H}_s f_s + ([\mathcal{H}, \int dx] f_{s+1}).$$

Here [ , ] denotes a certain commutator of operators. For finite $N$ the hierarchy is equivalent to the Liouville equation, but letting $N$ tend to infinity and $s$ run from one to infinity, it can also be used for a coarse grained description of states

of systems with infinitely many particles. In particular, $s = 1$ gives the Boltzmann equation under the hypothesis of molecular chaos (or factorization of $f_s$ into one-particle products) in the so-called Boltzmann Grad limit with the radius of the molecules and the size of the vessel appropriately scaled when $N \to \infty$. (For a broad discussion of this topic, see [CGP], [CIP].)
The n-particle evolution is reversible, whereas the limiting coarse-grained Boltzmann equation has an inbuilt arrow of time given by its negative entropy dissipation rate.

Velocities in the pair collisions of the Boltzmann equation in $I\!R^n$ - $(v, v_*)$ (before) $\to$ $(v', v'_*)$ (after) - are connected by

$$\mathbf{v}' = \frac{\mathbf{v} + \mathbf{v}_*}{\mathbf{2}} + \frac{|\mathbf{v} - \mathbf{v}_*|}{\mathbf{2}}\sigma,$$
$$\mathbf{v}'_* = \frac{\mathbf{v} + \mathbf{v}_*}{\mathbf{2}} - \frac{|\mathbf{v} - \mathbf{v}_*|}{\mathbf{2}}\sigma,$$

where $\sigma \in \mathcal{S}^{n-1}$, the unit sphere in $I\!R^n$ . The density of a rarefied gas is as usual modelled by nonnegative functions f(x,v) with $x$ the position and $v$ the velocity. With respect to the velocities of the two particles before collision $(v, v_*)$ and the ones after collision $(v', v'_*)$, we shall write

$f(v) = f,\ f(v_*) = f_*,\ f(v') = f',\ f(v'_*) = f'_*.$

The x-domain $\Omega$ will in our main example be the position space between two coaxial cylinders with inner normal $n(x)$. On the ingoing boundary $\partial\Omega^+ = \{(x,v) \in \partial\Omega \times I\!R^n; v \cdot n(x) > 0\}$ indata $f_b$ may be given, and a reflection operator $\mathcal{R}$ can be defined for diffuse reflection, e.g. the Maxwellian type

$$f(x,v) = cM(v) \int_{v' \cdot n(x) < 0} |v' \cdot n(x)| f(x, v') dv'.$$

Combining them leads to the mixed boundary conditions,

$$f = \Theta \mathcal{R} f + (1 - \Theta) f_b \qquad 0 \leq \Theta \leq 1. \tag{1.1}$$

The stationary Boltzmann equation in the domain $\Omega$ is

$$v \cdot \nabla_x f(x,v) = Q(f,f)(x,v) = Q^+(x,v) - Q^-(x,v) = Q^+(x,v) - f\nu(f)(x,v)$$
$$= \int_{I\!R^3} \int_{S^2} B(v - v_*, \omega)[f'f'^* - ff^*] d\omega dv_*, \quad x \in \Omega, v \in I\!R^n, \tag{1.2}$$

where $Q^+ - Q^-$ is the splitting into gain and loss parts of the collision operator

$Q$, and $\nu$ is the collision frequency. In this equation $v \cdot \nabla_x f(x,v)dxdv$ is the transport term, i.e. represents the net variation per unit time due to the free flow in and out of the volume element $dxdv$ centered at $(x,v)$ in phase space; $Q^-(x,v)dxdv$ represents the decrease per unit time of the number of particles in the same volume element by collisions with all other particles that are at the position $x$ at the same time; and $Q^+(x,v)dxdv$ represents the increase per unit time of the number of particles in the volume element as the result of collisions involving all particles at position $x$ with velocities $(v', v'_*)$. The kernel $B$ describes the specific collision process under study. A discussion of how to compute $B$ in particular cases can be found in [LaLi Section 18]. E. g. for interactions inversely proportional to some power of the distance, this function $B$ has a non-integrable singularity in the angular variable at grazing collisions. To remove such singularities, the Grad cut-off assumption is usually added, replacing the divergent angular dependence by an integrable one one, thereby guaranteeing separately convergent gain and loss terms.
Multiplying $Q(f,f)$ with a function $\psi(v)$, integrating with respect to velocity and changing variables, formally gives

$$\int_{I\!R^3} Q(f,f)(v)\psi(v)dv = \frac{1}{4}\int_{I\!R^3\times I\!R^3\times S^2} B(f'f'_* - ff_*)(\psi + \psi_* - \psi' - \psi'_*)dvdv_*dw.$$

In particular this integral vanishes for $\psi = 1, v, |v|^2$. In the cases of interest in these lectures, the formal calculations can be rigorously justified. Taking $\psi = \ln f$, we obtain the entropy dissipation rate

$$-e(f) = \int B(ff_* - f'f'_*)ln\frac{ff_*}{f'f'_*}dxdvdv_*dw.$$

The entropy dissipation rate is strictly negative except for Maxwell distributions $M_{\rho,u,\theta} = \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}}exp(-\frac{(v-u)^2}{2\theta})$, i.e. the equilibria for which the entropy dissipation vanishes. For additional general introductory material on kinetic theory, you may consult [C] or [CIP] and their references.

*Asymptotic studies* of the Boltzmann equation like this work, require scalings for collision terms, for variables, and for boundary values. The variables are first rescaled to make the equation non-dimensional. Physically motivated additional scalings in some parameter like the mean free path, may then be introduced for particular situations to obtain formal comparison between the kinetic models at leading order and corresponding gas dynamic ones. To go from the kinetic microscopic to the macroscopic fluid dynamic descriptions, the conserved fields have to be slowly varying on the kinetic scale and have reasonable space variations over macroscopic distances. To expose these fluid fields, power series expansions in the scaling variable are inserted into the kinetic equations and coupled with formal truncations. A rest term is added to the truncated expansion for questions of rigorous kinetic existence, and likewise for convergence issues when the mean free path tends to zero. Let us consider some examples.

In the *incompressible* case, the expansions and the limit-takings may be carried out starting from a (normalized) global Maxwellian $M = (2\pi)^{-\frac{3}{2}}e^{-\frac{v^2}{2}}$, and with the scaling $F = MG_\epsilon \geq 0$. A useful parameter is the Knudsen number, the ratio between the microscopic and macroscopic space units, such as the molecular mean free path (in ordinary air $10^{-5}$ cm) to a typical length scale for the flow, often based on the gradients occuring in the flows. With $\epsilon^j$ the Knudsen number or the mean free path, we get a Boltzmann equation in $G_\epsilon$,

$$\epsilon\partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon^j} J(G_\epsilon, G_\epsilon).$$

Here $J$ is the rescaled quadratic Boltzmann collision operator,

$$J(\Phi,\Psi)(v) := \frac{1}{2}\int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)M(v_*)(\Phi(v')\Psi(v'_*) + \Phi(v'_*)\Psi(v') \\ -\Phi(v_*)\Psi(v) - \Phi(v)\Psi(v_*))dv_*d\omega.$$

Also its linearization around 1 is an important operator in kinetic theory;

$$(L\Phi)(v) := \int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)M(v_*)(\Phi(v') + \Phi(v'_*) - \Phi(v_*) \\ -\Phi(v))dv_*d\omega = K(\Phi) - \nu\Phi.$$

With $G_\epsilon = 1 + \epsilon^m g_\epsilon$, the term of order $\epsilon^m$ denoted by $g_\epsilon$, determines the hydrodynamic fields $(\rho, u, \theta)$ representing the leading order density, velocity, and temperature fluctuations. The equation for the $g_\epsilon$ perturbation becomes

$$\epsilon\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon^j} L g_\epsilon = \epsilon^{m-j} J(g_\epsilon, g_\epsilon)$$

$$\Longrightarrow \quad \text{(formally when } \epsilon \to 0)$$

$$g_\epsilon \to \rho + u \cdot v + \theta(\frac{1}{2}v^2 - \frac{3}{2})$$

$$\nabla_x \cdot u = 0 \quad \text{(incompressibility)}, \quad \nabla_x(\rho + \theta) = 0 \quad \text{(Boussinesq relation)}$$

together with

$$j > 1, m = 1: \quad \partial_t u + u. \nabla_x u + \nabla_x p = 0, \quad \partial_t\theta + u \cdot \nabla_x \theta = 0 \quad \text{E.E.}$$

$$j = 1, m > 1: \quad \partial_t u + \nabla_x p = \mu\Delta_x u, \quad \partial_t\theta = \kappa\Delta_x\theta \quad \text{(Stokes' ekv.)}$$

$$j = 1, m = 1: \quad \partial_t u + u. \nabla_x u + \nabla_x p = \mu\Delta_x u, \quad \partial_t\theta + u \cdot \nabla_x\theta = \kappa\Delta_x\theta \quad \text{N.S.E.}$$

More generally we may start from a local Maxwellian

$$M_{\rho,u,\theta} = \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}} exp(-\frac{(v-u)^2}{2\theta}),$$

and are interested in solutions $f_\epsilon$ to the Boltzmann equation

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon} Q(f_\epsilon, f\epsilon),$$

where $f_\epsilon$ is a perturbation of a Maxwellian $M_{\rho,u,\theta}$, which corresponds to the solution of some *compressible* gas dynamic equation. Also $f_\epsilon$ is an approximate solution of order p if

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon} Q(f_\epsilon, f\epsilon) + \mathcal{O}(\epsilon^p).$$

Write $f_\epsilon$ as an asymptotic expansion plus a rest term,

$$f_\epsilon = \sum_{j=0}^{j_1} \epsilon^j f_j + \epsilon^{j_0} R.$$

This may be inserted into the Boltzmann equation and followed by a formal identification as equations of one order at a time (the Hilbert expansion), either just ending at some suitable order $j_1$, or ending by rigorously solving the rest term problem. The procedure in its simplest form is

$$\begin{aligned} \text{order} \quad -1: \quad & Q(f_0, f_0) = 0 \quad \Longrightarrow \quad f_0 = M_{\rho(x),u(x),\theta(x)}(v) \\ \text{order} \quad 0: \quad & \partial_t f_0 + v \cdot \nabla_x f_0 = Q(f_0, f_1) + Q(f_1, f_0). \end{aligned} \tag{1.3}$$

The expansion $\sum_{j=0}^{j_1} \epsilon^j f_j$ is of course not by itself a density solution of the Boltzmann equation, since it satisfies the Boltzmann equation only up to some order, and may by its essentially polynomial character become negative, whereas a real density should be everywhere positive.

As basis for the kernel of $L$ in $L^2_M(\mathbb{R}^3)$ (i.e. $L^2$ in velocity with Maxwellian weight function), we take $\psi_0 = 1, \psi_\theta = v_\theta, \psi_r = v_r, \psi_z = v_z, \psi_4 = \frac{1}{\sqrt{6}}(v^2 - 3)$. The right hand side in the zero order equation of (1.3) is orthogonal to the fluid dynamic $\psi_{..}$-moments, which span the kernel of $L$. A corresponding fluid dynamic projection gives the Euler equations of compressible gas dynamics

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\ \partial_t(\rho u) + \nabla_x(\rho u \otimes u) + \nabla_x(\rho\theta) &= 0, \\ \partial_t(\rho(\frac{1}{2}u^2 + \frac{3}{2}\theta)) + \nabla_x(\rho u(\frac{1}{2}u^2 + \frac{5}{2}\theta)) &= 0. \end{aligned}$$

Also mathematically interesting, but not implied by the formal asymptotics, is in what sense the leading order gas dynamics equations are limits of the kinetic ones. The Euler equations obviously do not depend on any detailed information about the Boltzmann equation, not even the cross section of the collisions. Composite molecules on the other hand, require additional terms for unavoidable rotational and vibrational modes of interaction. The Euler equations describe what happens at microscopic times of order $\epsilon^{-1}$.

To reach instead the compressible Navier-Stokes equations, one could perform the Chapman-Enskog variant of the Hilbert expansion, adding a kind of equation expansion. Low orders are then of most interest for obtaining/improving/varying fluid dynamic models. Up to the Navier-Stokes level all is simple. We may start from

$$f_\epsilon = M_{\rho_\epsilon,u_\epsilon,\theta_\epsilon}(1+\epsilon f_{1\epsilon}+\epsilon^2 f_{2\epsilon}), \tag{1.4}$$

and assume that $\rho_\epsilon, u_\epsilon, \theta_\epsilon$ solve the compressible Navier-Stokes system

$$\begin{aligned}
\partial_t\rho_\epsilon + \nabla_x\cdot(\rho_\epsilon u_\epsilon) &= 0,\\
\rho_\epsilon(\partial_t + u_\epsilon\cdot\nabla_x)u_\epsilon + \nabla_x(\rho_\epsilon\theta_\epsilon) &= \epsilon\nabla_x\cdot(\mu_\epsilon(Du_\epsilon)),\\
\frac{3}{2}\rho_\epsilon(\partial_t + u_\epsilon\cdot\nabla_x)\theta_\epsilon + \rho_\epsilon\theta_\epsilon\nabla_x . u_\epsilon &= \epsilon\frac{1}{2}\mu_\epsilon D(u_\epsilon):D(u_\epsilon) + \epsilon\nabla_x[\kappa_\epsilon\nabla_x\theta_\epsilon].
\end{aligned}$$

Inserting (1.4) into the Boltzmann equation, gives $f_{1\epsilon}$, $f_{2\epsilon}$, such that (1.4) becomes an approximative solution of the Boltzmann equation of order two (see [BGL]). The transport coefficients $\mu_\epsilon$ (viscosity) and $\kappa_\epsilon$ (thermal conductivity), are kinetically described by the collison operator dependent term $f_{1\epsilon}$ which contains the main contribution to the momentum and heat flow dissipation. The first order microscopic term is thus the main responsible for the conversion of mechanical work to heat and the transport of heat to the boundary. Adding a rest term, a true solution can be obtained for the Boltzmann equation. Conversely a solution to the Boltzmann equation may sometimes be used to derive rigorously a Navier-Stokes description from the Boltzmann one, that describes what happens for microscopic times of order $\epsilon^{-2}$. After mild changes in the set-up, extra terms may appear in the Navier-Stokes system (called ghost terms when their origin is not from leading order but comes from higher order terms). For a more extended introduction to such asymptotics, see Chapter 2 in [BGP] with references.
Proceeding beyond the Navier-Stokes level in the Chapman-Enskog procedure introduces undesired effects; well-posedness and the monotone entropy property may e.g. disappear. Among the many efforts to ameliorate this higher order situation, we mention two recent approaches, by M. Slemrod [S] using certain rational approximations, and A. Bobylev's operator calculus with projections [B], both delivering well-posed alternative equations.

This work will focus on stationary aspects. Stationary solutions are of importance in their own right, but also as time-asymptotics, and in rarefied gas dynamics. The latter deals with gas flows, where Navier Stokes type equations are not valid in some significant region of the flow field. The broad picture is one of normal regions where the gas flow follows the macroscopic fluid equations, plus thin shock layers, boundary layers, and initial layers, where matching conditions are sought between different fluid regions or between fluid regions and boundaries.
We shall here concentrate on the boundary layer case for a situation where the gas is contained between two concentric rotating cylinders, and also consider its scaling limit for vanishing Knudsen number. That two-rolls set-up is a classical problem on the fluid dynamics side with a surprisingly varied bifurcation behaviour, when the rotation rates of the cylinders change, which is well demonstrated in the experimental work of Andereck, Liu and Swinney [ALS]. An interesting question is how much of the bifurcations survive on the kinetic side. One may crudely expect that, as soon as there is a rigorous enough mathematical analysis of the fluid behavior, then the result should somehow carry over to the kinetic side. This work demonstrates how the leading order fluid terms dominate the higher order behaviour, when the solutions are close to equilibrium.
Systematic asymptotic studies close to equilibrium started already in the 1960-ies with Grad [G], Kogan [K], and Guiraud [Gu] among the pioneers, and with the main arguments based on fixed points and contraction mapping techniques. Two main approaches are presently in use, one based on energy methods in Sobolev spaces (i.e. involving $L^p$-estimates of derivatives). The other employs the setting of mixed weighted $L^p$-spaces, where precise spectral aspects are readily available. We shall here use the latter approach to study certain fully nonlinear stationary kinetic problems between rotating cylinders, including fluid limits when instabilities (bifurcations) arise. Part of the results were first published in [AN2] and [AN3]. Among the new results are in particular the stability properties discussed in Section 5.

# 2 A kinetic gas between two coaxial cylinders.

In this section asymptotic expansions are introduced and discussed for three archetypical two-rolls situations.

Consider the stationary Boltzmann equation in the space $\Omega$ between two coaxial cylinders with radii $r_A < r_B$. Denote by $(r, \theta, z)$ and $(v_r, v_\theta, v_z)$ respectively, the cylindrical spatial coordinates and the corresponding velocity coordinates. Let us start with parameter ranges where the system stays axially and rotationally uniform, the interesting solutions then being positive functions $f(r, v_r, v_\theta, v_z)$. In these coordinates the Boltzmann equation may be written

$$v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f = \frac{1}{\epsilon^j} Q(f, f), \qquad (2.1)$$
$$r \in (r_A, r_B), \quad (v_r, v_\theta, v_z) \in I\!R^3.$$

Here

$$N f := v_\theta^2 \frac{\partial f}{\partial v_r} - v_\theta v_r \frac{\partial f}{\partial v_\theta}.$$

In the collision term $Q$ the kernel $B = |v - v_*|^\beta b(\theta)$, where $b \in L^1_+(S^2)$, and $0 \leq \beta \leq 1$ in the hard force setting of these lectures.
The Knudsen number $k = \epsilon^j$ will be considered for various j's. As boundary conditions, functions $f_b$ are given on the ingoing cylinder boundary $\partial\Omega^+$, i.e. $\{(r_A, v); v_r > 0\}$ and $\{(r_B, v); v_r < 0\}$. For the axially homogeneous case we may assume that the solutions are even in the $v_z$-variable. The most general we are then able to say about the solvability of the problem is

**Theorem 2.1** [AN1] *Let $\beta$ be the power of the relative velocity in the Boltzmann collision kernel. Given $m = \int_{r_A}^{r_B} \int_{I\!R^3} (1 + |v|)^\beta f dx dv$ and ingoing boundary values $f_b$ with finite flow of mass, energy and entropy, then there exists a weak $L^1$-solution to the Boltzmann equation for hard forces in the two-rolls domain with $\beta$-moment $m$ and the indata profile $k f_b$ for some $k$ depending on $m$.*

Thus for mere existence it is enough to require that the flows of mass, energy and entropy are finite for $f_b$. Also the mixed boundary conditions (1.1) can be handled. Results in this generality are based on weak $L^1$ compactness coming from the entropy dissipation control. It gives on the other hand no information about uniqueness, isolated solutions, fluid limits with extra terms, or possible ghost effects. Such results have instead to be based on the asymptotic methods inititated by Grad [G], Kogan [K] and Guiraud [Gu] a full generation ago. But still today many, if not most, important problems are open when it comes to rigorous mathematical analysis. The 1993 monograph by Maslova [M] is probably still the best introduction to the rigorous mathematics in the area. The present frontiers reached by rigorous mathematics unfortunately lag far behind what has been obtained in the approach by formal asymptotics and scientific computing. There two recent monographs by Sone, [S1] and [S2] give a good picture of the state of the art. In [S1] one also finds a thorough discussion about the asymptotic expansions for the two-rolls problems of this lecture series, including many aspects not covered here.

For the asymptotic problems in the domain between the two rotating cylinders, our main concern in this work will be with (multiple) isolated solutions, bifurcations and strict positivity, when the boundary indata are given as Maxwellians $M_\alpha$ with known boundary pressure $P_\alpha$, temperature $T_\alpha$, and rotation rate $v_{\theta\alpha}$, where $\alpha = A$ for the inner and $B$ for the outer cylinder. Split the solution to the BE (2.1) as $f = M(1+\varphi+\epsilon^{j_0}R) = M(1+\Phi)$ with $\varphi$ an asymptotic expansion,

$$\varphi = \sum_1^{j_1} \epsilon^j \Phi^j, \quad M = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{v^2}{2}\right), \tag{2.2}$$

and with R, the rest term, in turn split into

$$R = P_0 R + (I - P_0) R = R_{||} + R_{\perp}.$$

The projection $P_0$ represents the fluid dynamic part. The asymptotic expansion (2.2) has boundary values equal the corresponding terms up to a suitable order in the $\epsilon$-expansions of the boundary Maxwellians $M_\alpha$. The remaining part of the boundary values are taken care of by the rest term.

As orthonormal basis for $P_0$ in $L^2_M(I\!R^3)$ (i.e. $L^2$ with weight function $M$), we take $\psi_0 = 1, \psi_\theta = v_\theta, \psi_r = v_r, \psi_z = v_z, \psi_4 = \frac{1}{\sqrt{6}}(v^2-3)$.

The new unknown $\Phi(r, z, v_r, v_\theta, v_z)$ should solve

$$v_r \frac{\partial \Phi}{\partial r} + v_z \frac{\partial \Phi}{\partial z} + \frac{1}{r} N\Phi = \frac{1}{\epsilon}(L\Phi + J(\Phi, \Phi)).$$

Here $J$ is the rescaled quadratic Boltzmann collision operator,

$$J(\Phi,\psi)(v) := \frac{1}{2}\int_{I\!R^3\times S^2} B(v-v_*,\omega) M(v_*)(\Phi(v')\psi(v'_*) + \Phi(v'_*)\psi(v')$$
$$-\Phi(v_*)\psi(v) - \Phi(v)\psi(v_*)) dv_* d\omega,$$

and $L$ is this operator linearized around 1,

$$(L\Phi)(v) := \int_{I\!R^3\times S^2} B(v-v_*,\omega)M(v_*)(\Phi(v')+\Phi(v'_*)-\Phi(v_*)$$
$$-\Phi(v))dv_*d\omega = K(\Phi)-\nu\Phi.$$

By a change of variables

$$(\varphi, Lf) := \int MfL\varphi dv = \int MLf\varphi dv$$
$$= -\frac{1}{4}\int(\varphi'_*+\varphi'-\varphi_*-\varphi)(f'_*+f'-f_*-f)BMM_*dvdv_*d\omega.$$

In particular we notice that $\varphi = f$ gives $(f, Lf) \leq 0$. Taking $\varphi$ as a fluid moment $\psi_j$, implies that $(\psi_j, Lf) = 0$ for all $f$, hence that the fluid dynamic functions are in the kernel of $L$. There are no others since the only solutions to the equation of Cauchy type $f + f_* - f' - f'_* = 0$ are the fluid moments, as first shown already by Boltzmann. Hence the kernel of $L$ is spanned by the fluid moments. Moreover,

**Lemma 2.2** *There exists a positive constant $c$ such that*

$$-(f, Lf) \geq c\int(\nu^{\frac{1}{2}}(I-P_0)f)^2Mdv.$$

Proof of Lemma 2.2 We give the proof from [M]. Set

$$\bar{K} = \nu^{-\frac{1}{2}}K\nu^{-\frac{1}{2}},$$
$$\bar{\lambda} = \sup\{\lambda; \bar{K}f = \lambda f \quad \text{with} \quad P_0\nu^{-\frac{1}{2}}f = 0, \int(\nu^{-\frac{1}{2}}f)^2Mdv = 1\}.$$

The compactness of $\bar{K}$ (cf proof of Lemma 3.2 below) together with $(f, Lf) \leq 0$ imply that with $\bar{\lambda} < 1$

$$((I-P_0)f, K(I-P_0)f) \leq \bar{\lambda}\int((I-P_0)f)^2\nu Mdv,$$

and so

$$(f, Lf) \leq (\bar{\lambda}-1)\int((I-P_0)f)^2\nu Mdv. \quad \Box$$

Lengthy elementary computations show that $L(v_\theta v_r\bar{B}) = v_\theta v_r$, $L(v_r\bar{A}) = v_r(v^2 - 5)$ for some functions $\bar{B}(|v|)$ and $\bar{A}(|v|)$, with $v_\theta v_r\bar{B}(|v|)$ and $v_r\bar{A}(|v|)$ bounded in the $L^2_M$-norm (cf [BGP] Lemma 2.2.3).

Our basic **Case 1** will be this two-rolls set-up with $j = 1$ in (2.1) and $j_0 = 1$, $j_1 = 2$ in (2.2) with given Maxwellian ingoing, axially uniform boundary data, modelling for instance when the cylinder surfaces are of ice in the form of the solid phase of the gas between them. We assume that (no essential restriction) the inner cylinder is rotating with velocity $\epsilon u_{\theta A}$, the outer cylinder is not rotating and the temperature and saturated pressure are the same at the two cylinders. Then

$$\begin{aligned}\gamma^+ f(r_A, z, v) &= \frac{1}{(2\pi)} e^{-\frac{1}{2}(v_r^2 + (v_\theta - \epsilon u_{\theta A})^2 + v_z^2)}, \quad v_r > 0,\\ \gamma^+ f(r_B, z, v) &= \frac{1}{(2\pi)} e^{-\frac{1}{2}v^2}, \quad v_r < 0.\end{aligned} \tag{2.3}$$

We shall keep the same boundary values in the following Case 2-3. To simplify the exposition in these lectures, we shall take $u_{\theta A} = U_{\theta A}(r_B - r_A)$ with $U_{\theta A}$ fixed. This will allow for additional conditions on the size of $r_B - r_A$ when needed in the convergence studies. An alternative would be to have $r_B - r_A$ fixed (even large) and introduce more extended asymptotic expansions.
An axially homogeneous solution $M(1 + \Phi)$ will be determined for (2.1), (2.3), with in Case 1 an approximate asymptotic expansion $\varphi$ of order 2 with boundary values of first and second orders being $\Phi_{Ai}, \Phi_{Bi}(= 0)$, $1 \le i \le 2$,

$$\Phi_{A1} = \epsilon u_{\theta A} v_\theta$$
$$\Phi_{A2} = \frac{\epsilon^2}{2} u_{\theta A}^2 (-1 + v_\theta^2),$$

plus a rest term $\epsilon R$,

$$\Phi(r, v) = \varphi(r, v) + \epsilon R(r, v),$$

and

$$\varphi(r, v) = \epsilon \Phi_{H1}(r, v) + \epsilon^2 (\Phi_{H2}(r, v) + \Phi_{K2A}(\frac{r - r_A}{\epsilon}, v) + \Phi_{K2B}(\frac{r - r_B}{\epsilon}, v)). \tag{2.4}$$

Here the Hilbert expansion term $\Phi_{H2}$ cannot by itself satisfy all boundary conditions. To remedy that, second order additional Knudsen boundary layer terms $\Phi_{K2.}$ are inserted.

In the asymptotic expansion the Hilbert terms $\Phi_{H1}$ and $\Phi_{H2}$ satisfy

$$L\Phi_{H1} = L\Phi_{H2} + J(\Phi_{H1}, \Phi_{H1}) - v \cdot \nabla_x \Phi_{H1} = 0.$$

Here $L\Phi_{H1} = 0$ implies that $\Phi_{H1} = a_1(r) + d_1(r)v^2 + b_1(r)v_\theta + c_1(r)v_r$. The $v_z$-term is zero due to the symmetry imposed. For compatibility reasons the hydrodynamic moments of the second equation are zero (cf. also (2.21), (2.41) below). In particular the 1-moment gives for $\Phi_{H1}$ that

$$c_1' + \frac{c_1}{r} = 0,$$

hence $c_1(r) = \frac{c}{r}$, where due to the boundary conditions $c = 0$.
Set $w_1 = \int v_r^2 v_\theta^2 \bar{B} M dv$, $w_2 = \int v_r^2 \bar{A} M dv$, $w_3 = \int v_r^2 v^2 \bar{A} M dv$. It is also consistent (and implied by the fluid dynamic projection equations) to take $a_1 = d_1 = 0$, giving

$$\Phi_{H1}(r, v) = b_1(r)v_\theta. \tag{2.5}$$

Then similarly

$$\Phi_{H2}(r, v) = a_2 + d_2 v^2 + b_2 v_\theta + c_2 v_r + \frac{1}{2} b_1^2 v_\theta^2 + (b_1' - \frac{1}{r} b_1) v_r v_\theta \bar{B}, \tag{2.6}$$

where by fluid dynamic projections and after some computations,

$$b_1(r) = \frac{u_{\theta A}}{r_B^2 - r_A^2}(\frac{r_B^2}{r} - r),$$

$$(a_2 + 5d_2)' + b_1 b_1' - \frac{1}{r} b_1^2 = 0, \quad c_2(r) = \frac{\gamma_2}{r}, \tag{2.7}$$

$$b_2'' + \frac{1}{r} b_2' - \frac{1}{r^2} b_2 = -\frac{1}{w_1}(b_1' + \frac{1}{r} b_1) c_2, \tag{2.8}$$

$$\begin{aligned}(w_3 - 5w_2)(d_2'' + \frac{1}{r} d_2') = (b_1(b_1' - \frac{1}{r} b_1))' \int M v_r (v^2 - 5)(\tilde{L}^{-1}(2\tilde{J}(v_\theta, v_r v_\theta \bar{B})\\ - v_r(v_\theta^2 - 1))dv + (b_1 b_1' - \frac{1}{r} b_1^2) \int M (v^2 - 5) N(\tilde{L}^{-1}(2\tilde{J}(v_\theta, v_r v_\theta \bar{B})\\ -(v_r(v_\theta^2 - 1))dv,\end{aligned} \tag{2.9}$$

for some constant $\gamma_2$. With the term $(b_1' - \frac{1}{r} b_1) v_r v_\theta \bar{B}$, the function $\Phi_{H2}$ of (2.6) cannot satisfy the boundary conditions $\Phi_{A2}$ (resp. $\Phi_{B2}$) at $r_A$ (resp. $r_B$ ). That is instead handled by adding Knudsen boundary layers as will be discussed in next lemma. Inserting $1 + \Phi$ into the rescaled Boltzmann equation gives the pure $\varphi$-part

$$l = \frac{1}{\epsilon}(L\varphi + J(\varphi, \varphi) - \epsilon v \cdot \nabla_x \varphi), \tag{2.10}$$

which is of $\epsilon$-order two, provided the Knudsen terms satisfy

$$L\Phi_{K2A} = v_r \frac{\partial \Phi_{K2A}}{\partial r}, \quad L\Phi_{K2B} = v_r \frac{\partial \Phi_{K2B}}{\partial r}.$$

Denote by $\eta = \frac{r - r_A}{\epsilon}$ and $\mu = \frac{r - r_B}{\epsilon}$.

**Lemma 2.3** *There exist a second-order Hilbert term $\Phi_{H2}$ defined by (2.6) with $a_2$, $d_2$, $b_2$, $c_2$ satisfying (2.7-9), and Knudsen terms $\Phi_{K2A}(\eta, v)$, $\Phi_{K2B}(\mu, v)$ such that*

$$\begin{aligned} v_r \frac{\partial \Phi_{K2A}}{\partial \eta} &= L\Phi_{K2A}, \\ \Phi_{K2A}(0,v) = \Phi_{A2}(v) - \Phi_{H2}(r_A, v), &\quad v_r > 0, \\ \lim_{\eta \to +\infty} \Phi_{K2A}(\eta, v) &= 0, \end{aligned} \tag{2.11}$$

*and*

$$\begin{aligned} v_r \frac{\partial \Phi_{K2B}}{\partial \mu} &= L\Phi_{K2B}, \\ \Phi_{K2B}(0,v) = \Phi_{B2}(v) - \Phi_{H2}(r_B, v), &\quad v_r < 0, \\ \lim_{\mu \to -\infty} \Phi_{K2B}(\mu, v) &= 0. \end{aligned} \tag{2.12}$$

To prove this lemma, we need some properties of the Milne half-space problem:

$$\begin{aligned} v_r \frac{\partial \psi}{\partial \eta} &= L\psi, \quad \eta > 0, \\ \psi(0,v) &= g, \quad v_r > 0, \\ \int M v_r \psi(\eta, v) dv &= m, \quad \eta > 0. \end{aligned} \tag{2.13}$$

Set $I\!R^3_+ = I\!R^3 \cap \{v_r > 0\}$ and take $b_\psi = (a(r), b(r), c(r), d(r))$ as the coefficients of the fluid dynamic moments of $\psi$ (the $v_z$-moment in our present setting is identically zero by symmetry). The following results about the Milne problem were proved in [BCN] and [GP].

**Theorem 2.4** *Let $m \in I\!R$ and $g \in L^2_{v_r M}(I\!R^3_+)$. There exists a unique solution $\psi$ to (2.13), which belongs to $L^\infty(r > 0; L^2_{|v_r|M} \cap L^2_{\nu M}) \cap L^2_{\nu M}(r > 0)$ and has $b_\psi \in L^\infty(I\!R_+)$. If $M^{\frac{1}{2}}\varphi = O(|v|^{-n})$ for all $n > 0$, then $b_\infty = lim_{r\to\infty} b_\psi$ exists, and $|b_\psi - b_\infty| = O(r^{-n})$ for any $n > 0$.*

Proof of Lemma 2.3 It follows from Theorem 2.4, that there are unique solutions $\psi$, $\psi_{2A}$ and $\psi_{2B}$ to

$$\begin{aligned} v_r \frac{\partial \psi}{\partial \eta} &= L\psi, \\ \psi(0,v) &= 0, \quad v_r > 0, \\ \int M v_r \psi(\eta, v) dv &= 1, \end{aligned}$$

$$v_r \frac{\partial \psi_{2A}}{\partial \eta} = L\psi_{2A},$$
$$\psi_{2A}(0,v) = -(b_1' - \frac{1}{r}b_1)(r_A)v_r v_\theta \bar{B}, \quad v_r > 0,$$
$$\int M v_r \psi_{2A}(\eta, v)dv = 0,$$

$$v_r \frac{\partial \psi_{2B}}{\partial \eta} = L\psi_{2B},$$
$$\psi_{2B}(0,v) = -(b_1' - \frac{1}{r}b_1)(r_B)v_r v_\theta \bar{B}, \quad v_r > 0,$$
$$\int M v_r \psi_{2B}(\eta, v)dv = 0.$$

Moreover,

$$\lim_{\eta\to+\infty} \psi(\eta, v) = a_\infty + d_\infty v^2 + b_\infty v_\theta + v_r,$$
$$\lim_{\eta\to+\infty} \psi_{2A}(\eta, v) = a_{2A,\infty} + d_{2A,\infty} v^2 + b_{2A,\infty} v_\theta,$$
$$\lim_{\eta\to+\infty} \psi_{2B}(\eta, v) = a_{2B,\infty} + d_{2B,\infty} v^2 + b_{2B,\infty} v_\theta,$$

for some constants $a_\infty$, $d_\infty$, $b_\infty$, $a_{2A,\infty}$, $d_{2A,\infty}$, $b_{2A,\infty}$, $a_{2B,\infty}$, $d_{2B,\infty}$ and $b_{2B,\infty}$. Choose

$$a_2(r_A) = \gamma_2 a_\infty + a_{2A,\infty} - \frac{1}{2}(u_{\theta A})^2, \tag{2.14}$$
$$a_2(r_B) = -\frac{\gamma_2}{r_B} a_\infty + a_{2B,\infty}, \tag{2.15}$$

$$d_2(r_A) = \gamma_2 d_\infty + d_{2A,\infty}, \tag{2.16}$$
$$d_2(r_B) = -\frac{\gamma_2}{r_B} d_\infty + d_{2B,\infty}, \tag{2.17}$$

$$b_2(r_A) = \gamma_2 b_\infty + b_{2A,\infty}, \tag{2.18}$$
$$b_2(r_B) = \frac{\gamma_2}{r_B} b_\infty - b_{2B,\infty}, \tag{2.19}$$

Then

$$\Phi_{K2A} = \gamma_2(\psi - a_\infty - d_\infty v^2 - b_\infty v_\theta - v_r) + \psi_{2A} - a_{2A,\infty} - d_{2A,\infty} v^2 - b_{2A,\infty} v_\theta,$$

and

$$\Phi_{K2B}(\mu, v) = -\frac{\gamma_2}{r_B}(\psi(-\mu, -v) - a_\infty - d_\infty v^2 + b_\infty v_\theta + v_r) + \psi_{2B}(-\mu, -v) - a_{2B,\infty} - d_{2B,\infty} v^2 + b_{2B,\infty} v_\theta,$$

satisfy (2.11-12). The first equation in (2.7) defines $a_2 + 5d_2$ if and only if

$$(a_2 + 5d_2)(r_B) - (a_2 + 5d_2)(r_A) = \frac{1}{2}u_{\theta A}^2 + \int_{r_A}^{r_B} \frac{1}{s} b_1^2(s)ds,$$

i.e.

$$\gamma_2 = \frac{r_B}{(r_B+1)(a_\infty + 5d_\infty)}\Big(a_{2B,\infty} - a_{2A,\infty} \\ +5d_{2B,\infty} - 5d_{2A,\infty} - \int_{r_A}^{r_B} \frac{1}{s} b_1^2(s)ds\Big). \tag{2.20}$$

This fixes $\gamma_2$, hence $c_2$ and $a_2 + 5d_2$. Finally the second-order differential equations (2.8-9) together with the boundary conditions (2.16-19) define $b_2$ and $d_2$. □

**Case 2.** If the Knudsen number is decreased by choosing $j = 4$ in (2.1), but still keeping the rotation velocity of the inner cylinder of order $\epsilon$, then the boundary layer depth (of order $\epsilon$) is no longer of the same order as the Knudsen number (i.e. $\epsilon^4$). That gives rise to additional technical difficulties. In particular we now have to introduce an additional so-called suction boundary layer from first order in $\epsilon$, and then from third order on also retain the previous Knudsen terms. For convenience we take $r_A = 1$ below in Case 2-3.

An asymptotic expansion $\varphi$ of order 4 will thus be determined,

$$\varphi(r,v) = \epsilon\Big(\Phi_{H1}(r,v) + \Phi_{W1}(\frac{r-r_B}{\epsilon},v)\Big) + \epsilon^2\Big(\Phi_{H2}(r,v) + \Phi_{W2}(\frac{r-r_B}{\epsilon},v)\Big) \\ +\epsilon^3\Big(\Phi_{H3}(r,v) + \Phi_{W3}(\frac{r-r_B}{\epsilon},v) + \Phi_{K3A}(\frac{r-1}{\epsilon^4},v) + \Phi_{K3B}(\frac{r-r_B}{\epsilon^4},v)\Big) \\ +\epsilon^4\Big(\Phi_{H4}(r,v) + \Phi_{W4}(\frac{r-r_B}{\epsilon},v) + \Phi_{K4A}(\frac{r-1}{\epsilon^4},v) + \Phi_{K4B}(\frac{r-r_B}{\epsilon^4},v)\Big). \tag{2.21}$$

The successive asymptotic computations order by order, allow us to require by (hydrodynamic) orthogonality that

$$\int \Phi_{H1}(.,v)(1,v_r,v^2)M(v)dv = \int \Phi_{W1}(.,v)(1,v_r,v^2)M(v)dv \\ = \int \Phi_{H2}(.,v)v_rM(v)dv = 0, \tag{2.22}$$

$$\lim_{\frac{r-r_B}{\epsilon}\to-\infty} \Phi_{Wi}(\frac{r-r_B}{\epsilon}, v) = 0, \quad 1 \le i \le 4, \tag{2.23}$$

$$\lim_{\frac{r-1}{\epsilon^4}\to+\infty} \Phi_{KiA}(\frac{r-1}{\epsilon^4}, v) = 0, \qquad \lim_{\frac{r-r_B}{\epsilon^4}\to-\infty} \Phi_{KiB}(\frac{r-r_B}{\epsilon^4}, v) = 0, \quad 3 \le i \le 4. \tag{2.24}$$

Here $(\epsilon\Phi_{H1}+\epsilon^2\Phi_{H2}+\epsilon^3\Phi_{H3}+\epsilon^4\Phi_{H4})(r,v)$ denotes the Hilbert terms up to fourth order. The sum $(\epsilon\Phi_{W1}+\epsilon^2\Phi_{W2})(\frac{r-r_B}{\epsilon}, v)$ consists of correction terms allowing the boundary conditions to be satisfied to first and second order. They correspond to suction boundary layer terms at $r_B$. At third and fourth orders, supplementary boundary layers of Knudsen type described by

$$\epsilon^3(\Phi_{K3A}(\frac{r-1}{\epsilon^4}, v) + \Phi_{K3B}(\frac{r-r_B}{\epsilon^4}, v)) + \epsilon^4(\Phi_{K4A}(\frac{r-1}{\epsilon^4}, v) + \Phi_{K4B}(\frac{r-r_B}{\epsilon^4}, v)),$$

are also required in order to get all the boundary conditions satisfied.
Let $\psi(\eta, v)$ be the solution to the half-space problem

$$\begin{aligned} &v_r\frac{\partial\psi}{\partial\eta} = L\psi, \quad \eta > 0, \quad v \in I\!R^3, \\ &\psi(0, v) = 0, \quad v_r > 0, \\ &\int \psi(\eta, v)v_r M(v)dv = 1, \quad \eta > 0. \end{aligned} \tag{2.25}$$

From Theorem 2.4 about the Milne problem, it follows that there are constants $A$, $D$, and $E$, such that

$$\lim_{\eta\to+\infty} \psi(\eta, v) = A + Dv^2 + Ev_\theta + v_r. \tag{2.26}$$

Let the nondimensional density, perturbed temperature and saturated pressure at $r_B$ be

$$\omega_B = \frac{\epsilon^2}{1+\tau_B}(P_{SB2} - \tau_{B2}), \quad \tau_B = \epsilon^2\tau_{B2}, \quad P_{SB} = \epsilon^2 P_{SB2}.$$

We may here in Case 2 couple the angular velocity to the Knudsen number through

$$P_{SB2} - \frac{r_{B2}^2 - 1}{r_B^2}u_{\theta A1}^2 = \Delta\epsilon.$$

The boundary condition at $r_B$ in (2.3) is replaced by

$$\gamma^+ f(r_B, z, v) = (2\pi)^{-\frac{3}{2}}\frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}}e^{-\frac{v^2}{2(1+\tau_B)}}, \quad v_r < 0.$$

For the third order asymptotic term that will lead to a bifurcation of the radial velocity - see (2.40) below- if $A + 5D < 0$.

**Lemma 2.5**
*Assume that*

$$(A+5D)<0,$$

*and set*

$$\Delta_{bif} := -\Big(2w_1\frac{r_B+1}{r_B^3}(A+5D)(3u_{\theta A1}^2)\Big)^{\frac{1}{2}}.$$

*For $\Delta > \Delta_{bif}$, there is no solution $\varphi$ to the family defined in (2.21-24).*
*For $\Delta = \Delta_{bif}$, there is a unique solution $\varphi$ to the family defined in (2.21-24).*
*For $\Delta < \Delta_{bif}$, there are two solutions $\varphi$ to the family defined in (2.21-24).*

Proof of Lemma 2.5. Define $Y := \frac{r-r_B}{\epsilon}$, and let the expansions

$\sum_{k\geq 1}\epsilon^k\Phi_{Hk}(r,v)$ and $\sum_{k\geq 1}\epsilon^k(\Phi_{Hk}(r_B+\epsilon Y,v)+\Phi_{Wk}(Y,v))$

formally satisfy the Boltzmann equation order by order. Then,

$$\begin{aligned}&L\Phi_{H1} = L\Phi_{H2}+J(\Phi_{H1},\Phi_{H1}) = L\Phi_{H3}+2J(\Phi_{H1},\Phi_{H2})\\ &= L\Phi_{H4}+2J(\Phi_{H1},\Phi_{H3})+J(\Phi_{H2},\Phi_{H2})=0,\end{aligned} \tag{2.27}$$

$$v_r\frac{\partial\Phi_{Hk-4}}{\partial r}+\frac{1}{r}N\Phi_{Hk-4} = L\Phi_{Hk}+\sum_{j=1}^{k-1}J(\Phi_{Hj},\Phi_{Hk-j}),\quad k\geq 5, \tag{2.28}$$

and

$$\begin{aligned}&L\Phi_{W1} = L\Phi_{W2}+J(\Phi_{W1},2\Phi_{H1}(r_B,.)+\Phi_{W1})\\ &= L\Phi_{W3}+2J(\Phi_{H1}(r_B,.)+\Phi_{W1},\Phi_{W2})+2J(\Phi_{W1},\Phi_{H2}(r_B,.)+Y\Phi'_{H1}(r_B,.))\\ &= L\Phi_{W4}+2J(\Phi_{W3},\Phi_{H1}(r_B,.)+\Phi_{W1})+J(\Phi_{W2},\Phi_{W2}+2\Phi_{H2}(r_B,.)\\ &+2Y\Phi'_{H1}(r_B,.))+2J(\Phi_{W1},\Phi_{H3}(r_B,.)+Y\Phi'_{H2}(r_B,.)\\ &+\frac{Y^2}{2}\Phi''_{H1}(r_B,.))-v_r\frac{\partial\Phi_{W1}}{\partial Y}=0,\end{aligned} \tag{2.29}$$

$$\begin{aligned}&v_r\frac{\partial\Phi_{Wk-3}}{\partial Y}+\frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i(\frac{Y}{r_B})^iN(\Phi_{Hk-4-i}(r_B,.)+\Phi_{Wk-4-i})\\ &= L\Phi_{Wk}+\sum_{j=1}^{k-1}J(2\Phi_{Hj}(r_B,.)+\Phi_{Wj},\Phi_{Wk-j}),\quad k\geq 5.\end{aligned} \tag{2.30}$$

Similarly to (2.5), by (2.27) $\Phi_{H1}(r,v)=b_1(r)v_\theta$ for some function $b_1$, and $\Phi_{Hi}, i\geq 2$ split into a fluid dynamical part $a_i(r)+d_i(r)v^2+b_i(r)v_\theta+c_i(r)v_r$ and a non-fluid-dynamic part involving Hilbert terms of lower order than $i$. In particular for $1 \leq j \leq 4$ we get

$$\Phi_{H1}(r,v) = b_1(r)v_\theta,$$
$$\Phi_{H2} = a_2 + d_2v^2 + b_2v_\theta + \frac{1}{2}b_1^2v_\theta^2,$$
$$\Phi_{H3} = a_3 + d_3v^2 + b_3v_\theta + c_3v_r + b_1d_2v_\theta v^2 + b_1b_2v_\theta^2 + \frac{1}{6}b_1^3v_\theta^3,$$
$$\Phi_{H4} = a_4 + d_4v^2 + b_4v_\theta + c_4v_r + (b_1d_3 + b_2d_2)v_\theta v^2 + (b_1b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2a_2)v_\theta^2$$
$$+b_1c_3v_rv_\theta + \frac{1}{2}b_1^2b_2v_\theta^3 + \frac{1}{2}d_2^2v^4 + \frac{1}{24}b_1^4v_\theta^4 + \frac{1}{2}b_1^2d_2v_\theta^2v^2.$$

Equations (2.28) have solutions if and only if the following compatibility conditions hold,

$$\int \Big(v_r\frac{\partial\Phi_{Hi}}{\partial r} + \frac{1}{r}N\Phi_{Hi}\Big)(1, v^2-5, v_\theta, v_r)M(v)dv = 0, \quad i \geq 1.$$

They provide first-order differential equations for the functions $a_i(r)$, $b_i(r)$, $c_i(r)$ and $d_i(r)$, $i \geq 1$. In particular,

$$(rb_1)' = 0, \quad (10d_2 + b_1^2)' = 0, \tag{2.31}$$
$$(r^2c_3b_2)' = w_1r^2(b_1' - \frac{1}{r}b_1)' + (2w_1 - w_2)r(b_1' - \frac{1}{r}b_1),$$
$$(a_2 + 5d_2 + \frac{1}{2}b_1^2)' = \frac{1}{r}b_1^2, \tag{2.32}$$
$$(a_3 + 5d_3 + b_1b_2)' = \frac{2}{r}b_1b_2, \tag{2.33}$$
$$(rc_3)' = 0, \tag{2.34}$$
$$(a_4 + 5d_4 + b_1b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2a_2 + \frac{35}{2}d_2^2 + \frac{7}{2}b_1^2d_2)'$$
$$= \frac{2}{r}(b_1b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2a_2) + \frac{1}{2r}b_1^4 + \frac{7}{r}b_1^2d_2, \tag{2.35}$$
$$(rc_4)' = 0.$$

Together with the boundary condition at $r_A$ of first and second orders, this fixes

$$\Phi_{H1}(r,v) = \frac{u_{\theta A}}{r}v_\theta, \quad \Phi_{H2} = -\frac{u_{\theta A}^2}{2r^2} + \frac{u_{\theta A}^2}{10}(1 - \frac{1}{r^2})v^2 + \frac{u_{\theta A}^2}{2r^2}v_\theta^2,$$

and $c_3(r) = \frac{u_3}{r}$, for some constant $u_3 \neq 0$. Moreover, (2.22) and (2.29-30) give that $\Phi_{W1}(Y,v) = z_1(Y)v_\theta$, for some function $z_1$, and that $\Phi_{Wi}, i \geq 2$ split into a fluid dynamical part $x_i(Y) + y_i(Y)v^2 + z_i(Y)v_\theta + t_i(Y)v_r$ and a non-fluid-dynamic part involving Hilbert terms of lower order than $i$. Notice that $\Phi_{W4}$ is the sum of $z_1' v_\theta v_r \bar{B}$ and a polynomial in the $v$-variable with bounded coefficients in the r-variable. More precisely,

$$\Phi_{W2} = x_2 + y_2 v^2 + z_2 v_\theta + (b_1(r_B) z_1 + \frac{1}{2} z_1^2) v_\theta^2,$$
$$\Phi_{W3} = x_3 + y_3 v^2 + z_3 v_\theta + t_3 v_r + (b_1(r_B) y_2 + z_1 y_2 + z_1 d_2(r_B)) v_\theta v^2$$
$$+ (b_1(r_B) z_2 + z_1 z_2 + z_1 b_2(r_B) + Y b_1'(r_B) z_1) v_\theta^2$$
$$+ (\frac{1}{2} b_1^2(r_B) z_1 + \frac{1}{2} b_1(r_B) z_1^2 + \frac{1}{6} z_1^3) v_\theta^3,$$
$$\Phi_{W4} = x_4 + y_4 v^2 + z_4 v_\theta + t_4 v_r + z_1' v_r v_\theta \bar{B}(v) + \dots \quad .$$

Equations (2.29) have solutions if and only if the following compatibility (orthogonality) conditions hold,

$$\int \Big( v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i (\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B, .)$$
$$+ \Phi_{Wk-4-i} \Big) (v^2 - 5, v_\theta) M(v) dv = 0, \quad k \geq 5, \tag{2.36}$$

and

$$\int \Big( v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i (\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B, .) +$$
$$\Phi_{Wk-4-i} \Big) (1, v_r) M(v) dv = 0, \quad k \geq 5. \tag{2.37}$$

Equations (2.36) (resp. (2.37)) provide second-order (resp. first-order) differential equations for $y_i$ and $z_i$ (resp. $x_i + 5y_i$ and $t_i$). In particular,

$$w_1 z_1'' - \frac{u_3}{r_B} z_1' = 0,$$
$$(x_2 + 5y_2 + b_1(r_B) z_1 + \frac{1}{2} z_1^2)' = 0,$$
$$w_2 y_2'' + \frac{10}{r_B} y_2' + A_1 = 0, \quad w_1 z_2'' - \frac{u_3}{r_B} z_2' + A_1 = 0,$$
$$t_3' = 0,$$
$$(x_3 + 5y_3 + b_1(r_B) z_2 + z_1 z_2 + z_1 b_2(r_B) + Y b_1'(r_B) z_1)' =$$
$$\frac{1}{r_B} (2 b_1(r_B) z_1 + z_1^2), \tag{2.38}$$
$$w_2 y_3'' + \frac{10}{r_B} y_3' + A_2 = 0,$$
$$w_1 z_3'' - \frac{u_3}{r_B} z_3' + \Big( (b_1(r_B) + z_1)(c_5(r_B) + t_5) \Big)' + A_2 = 0,$$
$$t_4' + \frac{1}{r_B} (t_3 + c_3(r_B)) + c_3'(r_B) = 0,$$
$$(x_4 + 5y_4)' + A_3 = 0. \tag{2.39}$$

Here $A_i, 1 \leq i \leq 3$, denote terms involving Hilbert and suction coefficients up to order $i$. Together with the boundary conditions at first and second orders, and the conditions (2.23), this fixes

$$\Phi_{W1}(Y,v) = -\frac{u_{\theta A1}}{r_B} e^{\frac{u_3 Y}{w_1 r_B}} \cdot v_\theta,$$

as well as $\Phi_{W2}$ in terms of $u_3$, and implies that $t_3 = t_4 = 0$. Then, giving the value 0 to any coefficient of order bigger than 5 in the second-order differential equations satisfied by $y_i$ and $z_i$, $3 \leq i \leq 4$ and taking into account (2.21-24) fixes the functions $y_i$ and $z_i$, $3 \leq i \leq 4$ in terms of $u_i$. A Knudsen analysis at third and fourth orders makes the first-order differential equations satisfied by $x_3 + 5y_3$ and $x_4 + 5y_4$ compatible with (2.23) at third and fourth orders. Finally $u_3$ must solve the equation

$$u_3^2(A+5D)\frac{r_B+1}{r_B} - \Delta u_3 + \frac{w_1}{2r_B^2}(-3u_{\theta A1}^2) = 0. \tag{2.40}$$

A study of the positive roots $u_3$ to (2.40) leads to the three cases described in the theorem for $\Delta$ with respect to $\Delta_{bif}$ . That proof requires a non-degeneracy in the Milne asymptotics (2.26),

$$A + 5D < 0. \tag{2.41}$$

The condition is expected to hold on physical grounds and has been verified numerically for hard spheres and Maxwellian molecules. A mathematical proof of (2.41) related to the numerical approach seems feasible, but has not been undertaken . □

**Case 3.** The techniques developed for the previous particular two-rolls situations, also hold the key to resolving other and sometimes more famous problems. This third example is such a generalization. The density $f$ will now be allowed to depend on the axial variable $z$, assuming periodicity in the axial direction. The previous transport term in (2.1) is then extended to include also a $z$-derivative term $v_z \frac{\partial f}{\partial z}$. We consider the Knudsen number $\epsilon^j$ for $j = 1$ in (2.1), and keep the earlier ingoing Maxwellian data. For small enough parameters, there is an axially uniform solution as in the Case 1. This axially homogeneous cylindrical Couette flow of Case 1 will bifurcate into axially periodic ones - Taylor rolls - when the rotation of the inner cylinder is started from rest and then is being sufficiently increased. The equations for the successive terms in the asymptotic expansions are now no longer ordinary but partial differential equations, which here may be solved by elementary and explicit Fourier methods. We shall only allow bifurcations to a fixed axial period, for convenience taken as $c(r_B - r_A)$, and carry out the computations when $c = 1$. Denote the first (lowest) bifurcation velocity value by $u_{\theta Ab}$, and require that all functions have the symmetry $f(r, z, v_r, v_\theta, v_z) = f(r, -z, v_r, v_\theta, -v_z)$. The first order asymptotic expansion term $\Phi_{H1}$, should satisfy $L\Phi_{H1} = 0$, i.e. belong to the kernel of $L$, hence

$$\Phi_{H1}(r, v) = a_1(r, z) + d_1(r, z)v^2 + b_1(r, z)v_\theta + c_1(r, z)v_r + e_1(r, z)v_z. \quad (2.42)$$

The fluid dynamic orthogonality arguments leading to (2.5) in Case 1, here imply that in a one-sided neighbourhood of $u_{\theta Ab}$, the first order coefficients may satisfy a steady (secondary) Taylor Couette fluid flow problem ((4.9) below) with corresponding boundary values. This fluid bifurcation problem was first rigorously studied in [V] using topological Leray Schauder degree, to be followed over the years by a number of alternative treatments and expansions - see [CI] for properties, references and an overview. It follows from that theory that the coefficients in (2.42) are smooth functions with uniform bounds in a neighbourhood of $u_{\theta Ab}$.

Denote by the index $b$ when an axially homogeneous term $\Phi_{Hj}$ is evaluated at the first bifurcation velocity $u_{\theta A} = u_{\theta Ab}$, and let $\delta^2$ denote the deviation from this bifurcation value. With $\Phi_{Hj} = \Phi_{Hjb} + \delta\Phi_j^1$, and $\Phi_1^1$ given by the smooth perturbation to the fluid Taylor Couette problem, we can successively construct higher order terms in the asymptotic expansion. E.g. for $j = 2, 3$, the perturbations $\Phi_2^1(x, v, \delta)$ and $\Phi_3^1(x, v, \delta)$ should satisfy

$$\tilde{L}\Phi_2^1 + g_{1\perp} - v_r\frac{\partial\Phi_1^1}{\partial r} - v_z\frac{\partial\Phi_1^1}{\partial z} - Nh_1 = 0, \quad (2.43)$$

$$\tilde{L}\Phi_3^1 + g_{2\perp} - v_r\frac{\partial\Phi_2^1}{\partial r} - v_z\frac{\partial\Phi_2^1}{\partial z} - Nh_2 = 0, \quad (2.44)$$

with

$$g_{1\perp} = 2\tilde{J}(\Phi_{H1b}, \Phi_1^1) + \delta\tilde{J}(\Phi_1^1, \Phi_1^1), \quad h_1 = \frac{1}{r}\Phi_1^1,$$

$$g_{2\perp} = 2\tilde{J}(\Phi_{H1}, \Phi_2^1) + 2\tilde{J}(\Phi_{H2b}, \Phi_1^1), \quad h_2 = \frac{1}{r}\Phi_2^1.$$

The locally uniform smoothness of $\Phi_{H1}$ (for small $\delta$), implies by (2.43) spacewise smoothness for $\Phi_{2,\perp}^1$ uniformly for small $\delta$. We may also prove by Fourier techniques, that the fluid dynamic moments of $\Phi_2^1$ and its derivatives are uniformly

bounded in $L^\infty$ in a $\delta^2$-neighbourhood of the bifurcation point $u_{\theta Ab}$ for small enough $\epsilon$. The procedure may be repeated for the $\Phi_3^1$-term.
To provide the correct boundary values for the problem, we add boundary layer corrections to $\Phi_2^1$ and $\Phi_3^1$ of Knudsen type. Our previous boundary layer analysis based on [GP] applies, when the equations are taken in Fourier space for the periodic $z$-variable. This is so since at the crucial steps in the decay study for the Milne problem in [GP], the relevant squared $L^2$-integrals in velocity space of the Fourier coefficients can be added to give (by Parseval's identity) analogous estimates for the corresponding squared $L^2$-norms with respect to $z$ of $\Phi_2^1$ and $\Phi_3^1$. This also holds for their $z$-derivatives, which in turn via Sobolev embedding leads to uniform bounds with respect to $z$ for the Knudsen layer terms.

For the interested reader we end this section with a proof of the appearance of this Taylor bifurcation in the present context. Extend the asymptotic expansion of Case 1 by third and fourth order terms $\Phi^3(r,v)$ and $\Phi^4(r,v)$, and denote it by

$$\epsilon b_1 v_\theta + \epsilon^2(\varphi_{2u} + \Phi_{K2A}(\eta,v) + \Phi_{K2B}(\mu,v))$$
$$+\epsilon^3(\varphi_{3u} + \Phi_{K3A}(\eta,v) + \Phi_{K3B}(\mu,v)) + \epsilon^4(\varphi_{4u} + \Phi_{K4A}(\eta,v) + \Phi_{K4B}(\mu,v)),$$

where $\varphi_{2u} = \Phi_{H2}$ of Case 1. This expansion is uniform with respect to the variable $z$, and $\eta = \frac{r-1}{\epsilon}$, $\mu = \frac{r-r_B}{\epsilon}$. Consider the following $z$-periodic perturbation $\varphi(r,z,v)$ of the $z$-homogeneous expansion,

$$\varphi(r,z,v) = \epsilon\Big(b_1 v_\theta + \delta cos\alpha z(Uv_\theta + Vv_r) + \delta(sin\alpha z)Wv_z + \delta^2 U_{20} v_\theta\Big)$$
$$+\epsilon^2\Big(\varphi_{2u} + \Phi_{K2A} + \Phi_{K2B} + \delta(cos\alpha z)(\varphi_{11}^2 + \Phi_{K21A}(\eta,v) + \Phi_{K21B}(\mu,v))$$
$$+\delta(sin\alpha z)(\psi_{11}^2 + \psi_{K21A} + \psi_{K21B}) + \delta^2(\varphi_{20}^2 + \Phi_{K20A} + \Phi_{K20B})$$
$$+\delta^2(cos2\alpha z)(\varphi_{22}^2 + \Phi_{K22A} + \Phi_{K22B})$$
$$+\delta^2(sin2\alpha z)(\psi_{22}^2 + \psi_{K22A} + \psi_{K22B})\Big)$$
$$+\epsilon^3\Big(\varphi_{3u} + \Phi_{K3A} + \Phi_{K3B} + \delta(cos\alpha z)(\varphi_{11}^3 + \Phi_{K31A} + \Phi_{K31B})$$
$$+\delta(sin\alpha z)(\psi_{11}^3 + \psi_{K31A} + \psi_{K31B})\Big)$$
$$+\epsilon^4(\phi_{4u} + \Phi_{K4A} + \Phi_{K4B}).$$

Here all coefficient functions are taken with respect to space as functions of $r$ only. Look for boundary conditions where the rotational velocity of first order in $\epsilon$, $b_1 + \delta(cos\alpha z)U + \delta^2 U_{20}$, at $r_A = 1$ deviates from $b_1$ by a $\delta^2$-order term $\Delta u_{\theta A}$. All the unknowns $U$, $V$, $W$, ... should then vanish at $r_A$ and $r_B$, except $U_{20}$, for which

$$U_{20}(r_A) = \Delta u_{\theta A}, \quad U_{20}(r_B) = 0.$$

**Lemma 2.6** *Let*

$$l = \frac{1}{\epsilon}(L\varphi + J(\varphi, \varphi) - \epsilon v \cdot \nabla_x \varphi).$$

*If* $\delta \le \epsilon$ *and if* $(U, V)$ *are solutions to*

$$L_\theta(U) - q_\theta V = 0, \quad L_r(V) + q_r U = 0,$$
$$U(r) = V(r) = V'(r) = 0 \text{ at } r = r_A, r = r_B, \tag{2.45}$$

*where*

$$L_\theta(U) = U'' + \frac{1}{r}U' - (\frac{1}{r^2} + \alpha^2)U, \quad L_r(V) = V^{(4)} + \frac{2}{r}V^{(3)} - (\frac{3}{r^2} + 2\alpha^2)V''$$
$$+(\frac{3}{r^3} - \frac{2\alpha^2}{r})V' + (-\frac{3}{r^4} + \frac{2\alpha^2}{r^2} + \alpha^4)V,$$
$$q_\theta = \frac{2u_{\theta A}}{w_1(r_B^2 - 1)}, \quad q_r = -\frac{2\alpha^2 u_{\theta A}}{w_1(r_B^2 - 1)}(\frac{r_B^2}{r^2} - 1),$$

*then the function* $\varphi$ *can be taken z-dependent, and so that* $l = l_\perp$ *is of order* $\epsilon^4$ *in* $\tilde{L}^\infty$.

The function $\varphi$ is the asymptotic expansion for an axially periodic solution bifurcating from the axially homogeneous one at $u_{\theta A} = u_{\theta Ab}$.

Proof of Lemma 2.6 Replacing in $l$, $\varphi$ by its expansion implies that

$$l = \epsilon\delta cos\alpha z\Big(L(\varphi_{11}^2 - b_1 U v_\theta^2 - b_1 V v_r v_\theta) - (U' - \frac{1}{r}U)v_r v_\theta$$
$$-(V'v_r^2 + \frac{1}{r}Vv_\theta^2 + \alpha W v_z^2) + L\Phi_{K21A} - v_r\frac{\partial \Phi_{K21A}}{\partial \eta} + L\Phi_{K21B} - v_r\frac{\partial \Phi_{K21B}}{\partial \mu}\Big)$$
$$+\epsilon\delta sin\alpha z\Big(L(\psi_{11}^2 - b_1 W v_\theta v_z) + \alpha U v_\theta v_z + (\alpha V - W')v_r v_z$$
$$+L\psi_{K21A} - v_r\frac{\partial \psi_{K21A}}{\partial \eta} + L\psi_{K21B} - v_r\frac{\partial \psi_{K21B}}{\partial \mu}\Big)$$
$$+\epsilon\delta^2\Big(L(\varphi_{20}^2 - \frac{1}{4}U^2 v_\theta^2 - \frac{1}{4}V^2 v_r^2 - \frac{1}{2}UVv_r v_\theta - \frac{1}{4}W^2 v_z^2 - b_1 U_{20} v_\theta^2)$$
$$-(U_{20}' - \frac{1}{r}U_{20})v_r v_\theta + L\Phi_{K20A} - v_r\frac{\partial \Phi_{K20A}}{\partial \eta} + L\Phi_{K20B} - v_r\frac{\partial \Phi_{K20B}}{\partial \mu}\Big)$$
$$+\epsilon\delta^2 cos2\alpha z\Big(L(\varphi_{22}^2 - \frac{1}{4}U^2 v_\theta^2 - \frac{1}{4}V^2 v_r^2 - \frac{1}{2}UVv_r v_\theta + \frac{1}{4}W^2 v_z^2)$$
$$+L\Phi_{K22A} - v_r\frac{\partial \Phi_{K22A}}{\partial \eta} + L\Phi_{K22B} - v_r\frac{\partial \Phi_{K22B}}{\partial \mu}\Big)$$
$$+\epsilon\delta^2 sin2\alpha z\Big(L(\psi_{22}^2 - UWv_\theta v_z - VWv_r v_z) + L\psi_{K22A} - v_r\frac{\partial \psi_{K22A}}{\partial \eta}$$

$$+L\psi_{K22B} - v_r \frac{\partial \psi_{K22B}}{\partial \mu}\Big)$$

$$+\epsilon^2 \delta cos\alpha z\Big(L\varphi_{11}^3 + 2J(b_1 v_\theta, \varphi_{11}^2) + 2J(\varphi_{2u}, Uv_\theta + Vv_r) - (v_r \frac{\partial \varphi_{11}^2}{\partial r} + \frac{1}{r} N\varphi_{11}^2 + \alpha \psi_{11}^2 v_z)$$

$$+L\Phi_{K31A} + 2J(b_1 v_\theta, \Phi_{K21A}) + 2J(Uv_\theta + Vv_r, \Phi_{K2A}) - N\Phi_{K21A} - v_r \frac{\partial \Phi_{K31A}}{\partial \eta}$$

$$+L\Phi_{K31B} + 2J(b_1 v_\theta, \Phi_{K21B}) + 2J(Uv_\theta + Vv_r, \Phi_{K2B}) - \frac{1}{r_B} N\Phi_{K21B} - v_r \frac{\partial \Phi_{K31B}}{\partial \mu}\Big)$$

$$+\epsilon^2 \delta sin\alpha z\Big(L\psi_{11}^3 + 2J(b_1 v_\theta, \psi_{11}^2) + 2J(\varphi_{2u}, Wv_z) - (v_r \frac{\partial \psi_{11}^2}{\partial r} + \frac{1}{r} N\psi_{11}^2 - \alpha \varphi_{11}^2 v_z)$$

$$+L\psi_{K31A} + 2J(b_1 v_\theta, \psi_{K21A}) + 2J(Wv_z, \Phi_{K2A}) - N\psi_{K21A} - v_r \frac{\partial \psi_{K31A}}{\partial \eta}$$

$$+L\psi_{K31B} + 2J(b_1 v_\theta, \psi_{K21B}) + 2J(Wv_z, \Phi_{K2B}) - \frac{1}{r_B} N\psi_{K21B} - v_r \frac{\partial \psi_{K31B}}{\partial \mu}\Big)$$

$$+O(\epsilon^4).$$

The compatiblity conditions in the $\epsilon\delta cos\alpha z$ term write

$$\alpha W = -V' - \frac{1}{r} V. \tag{2.46}$$

And so $\varphi_{11}^2$ can be taken as

$$\varphi_{11}^2 = a_{11}^2 + d_{11}^2 v^2 + b_{11}^2 v_\theta + c_{11}^2 v_r + e_{11}^2 v_z + b_1 U v_\theta^2 + b_1 V v_r v_\theta$$
$$+(U' - \frac{1}{r} U) v_r v_\theta \bar{B} + \frac{1}{r} V (v_\theta^2 - v_r^2) \bar{B} + \alpha W (v_z^2 - v_r^2) \bar{B},$$

for some functions $a_{11}^2$, $d_{11}^2$, $b_{11}^2$, $c_{11}^2$ and $e_{11}^2$. Moreover,

$$\psi_{11}^2 = \alpha_{11}^2 + \delta_{11}^2 v^2 + \beta_{11}^2 v_\theta + \gamma_{11}^2 v_r + \eta_{11}^2 v_z + b_1 W v_\theta v_z - \alpha U v_\theta v_z \bar{B}$$
$$-\alpha V v_r v_z \bar{B} + W' v_r v_z \bar{B},$$

for some functions $\alpha_{11}^2$, $\delta_{11}^2$, $\beta_{11}^2$, $\gamma_{11}^2$ and $\eta_{11}^2$. Then, the compatibility conditions of the $\epsilon^2\delta cos\alpha z$-term of $l$ are

$$(c_{11}^2)' + \frac{1}{r}(c_{11}^2) + \alpha \eta_{11}^2 = 0, \tag{2.47}$$

$$\frac{1}{w_1}(a_{11}^2 + 5d_{11}^2 + b_1 U)' = \alpha W' + \frac{2\alpha}{r} W + \frac{2}{r} V' + (\frac{2}{r^2} + \alpha^2) V + \frac{2}{w_1 r} b_1 U, \tag{2.48}$$

$$(b_1 V)' + \frac{2}{r} b_1 V + w_1 (U' - \frac{1}{r} U)' + \frac{2w_1}{r}(U' - \frac{1}{r} U) + \alpha b_1 W - \alpha^2 w_1 U = 0, \tag{2.49}$$

$$\alpha_{11}^2 + 5\delta_{11}^2 = 0. \tag{2.50}$$

Taking (2.46) into account in (2.49) implies that

$$L_\theta U + q_\theta V = 0.$$

The compatibility conditions of the $\epsilon^2\delta sin\alpha z$-term of $l$ are

$$(\gamma_{11}^2)' + \frac{1}{r}(\gamma_{11}^2) - \alpha e_{11}^2 = 0, \qquad (2.51)$$

$$(\alpha_{11}^2 + 5\delta_{11}^2)' = 0, \qquad (2.52)$$

$$\frac{1}{w_1}(a_{11}^2 + 5d_{11}^2 + b_1 U) = W'' + \frac{1}{r}W' - 2\alpha^2 W - \alpha(V' + \frac{1}{r}V). \qquad (2.53)$$

Differentiating (2.53) with respect to the variable $r$ and taking (2.48) and (2.46) into account, implies that

$$L_r V + q_r U = 0.$$

It follows that the coefficients $\varphi_{20}^2$, $\varphi_{22}^2$, $\psi_{22}^2$, $\varphi_{11}^3$, $\psi_{11}^3$, as well as the Knudsen terms can be defined so that $l$ be of order 4 provided (2.45) holds. □

**Lemma 2.7** *Let $\alpha > 0$ be given. There are nonnegative functions $u_1$ and $v_1$, and $u_{\theta A} = u_{\theta Ab} > 0$, such that for $r_B - r_A$ small enough, the problem (2.45) has the solutions $\{(U, V) = x(u_1, v_1); x \in I\!R\}$.*

Proof of Lemma 2.7 The equation $L_\theta U = 0$ is disconjugate on $[1, r_B]$ for any $r_B > 1$ since

$$\int_1^{r_B} (ry'^2 + (\frac{1}{r} + \alpha^2)y^2)dr$$

is nonneggative ([Co]). Hence there is a continuous Green function $G$ such that for any continuous founction $f$, the problem

$$L_\theta U = f, \quad U(1) = U(r_B) = 0,$$

has the unique solution

$$U(r) = \int_1^{r_B} G(r, s)f(s)ds.$$

Moreover,

$$G(r, s)(r - 1)(r - r_B) \geq 0, \quad (r, s) \in [1, r_B]^2,$$

so that $G$ is non positive. It also satisfies

$$rG(r, s) = sG(s, r), \quad (r, s) \in [1, r_B]^2,$$

since

$$\int_1^{r_B} rL_\theta(U)X dr = \int_1^{r_B} rL_\theta(X)U dr.$$

By [Co] the equation

$$L_r(V) = 0, \quad V(1) = V(r_B) = V'(1) = V'(r_B) = 0,$$

is disconjugate on $[1, r_B]$ for $r_B - 1$ small enough. Hence there is a $C^2$ Green function $H$ such that for any continuous function $f$, the problem

$$L_r V = f, \quad V(1) = V(r_B) = V'(1) = V'(r_B) = 0,$$

has the unique solution

$$V(r) = \int_1^{r_B} H(r,s) f(s) ds.$$

Moreover,

$$H(r,s)(r-1)^2(r-r_B)^2 \geq 0, \quad (r,s) \in [1, r_B]^2,$$

so that $H$ is nonnegative. It also satisfies

$$rH(r,s) = sH(s,r), \quad (r,s) \in [1, r_B]^2,$$

since

$$\int_1^{r_B} rL_r(V)Y dr = \int_1^{r_B} rL_r(Y)V dr.$$

And so, solving (2.45) comes back to finding $u_{\theta Ab} := U_{\theta Ab}(r_B - 1)$ and $V \geq 0$ such that

$$KV = \Big(\frac{w_1(r_B^2 - 1)}{4\alpha u_{\theta Ab}}\Big)^2 V, \tag{2.54}$$

where $K$ is the operator defined by

$$KV(r) = -\int_1^{r_B} \int_1^{r_B} H(r,s)(\frac{r_B^2}{s^2} - 1)G(s,t)V(t)dtds.$$

$K$ is compact in $L^2(1, r_B)$. It maps the cone of the nonneggative functions of $L^2$ into its interior, since $G$ is nonpositive, $H$ is nonnegative, and neither $G$ nor $H$ are identically zero. And so the Krein-Rutman theorem applies. There is an eigenvector $v_1 \geq 0$ corresponding to a positive eigenvalue of $K$, $\Big(\frac{w_1(r_B^2-1)}{2\alpha u_{\theta Ab}}\Big)^2 = \Big(\frac{w_1(r_B+1)}{2\alpha U_{\theta Ab}}\Big)^2$ with algebraic and geometric multiplicity equal to one. Denote by

$$u_1(r) = -q_0 \int_1^{r_B} G(r,s)v_1(s)ds, \quad r \in [1, r_B].$$

Then any $(xu_1, xv_1)$, $x \in I\!R_+$ is solution to (2.45). $\square$