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A MILNE PROBLEM FROM A BOSE CONDENSATE WITH EXCITATIONS

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Abstract. This paper deals with a half-space linearized problem for the distribution function of the excitations in a Bose gas close to equilibrium. Existence and uniqueness of the solution, as well as its asymptotic properties are proven for a given energy flow. The problem differs from the ones for the classical Boltzmann and related equations, where the hydrodynamic mass flow along the half-line is constant. Here it is no more constant. Instead we use the energy flow which is constant, but no more hydrodynamic.

1. **Introduction.** This paper studies a linearized half-line problem related to the kinetic equation for a gas of excitations interacting with a Bose condensate. Below the temperature T_c where Bose-Einstein condensation sets in, the system consists of a condensate and excitations. The condensate density n_c is modelled by a Gross-Pitaevskii equation. The excitations are described by a kinetic equation with a source term taking into account their interactions with the condensate,

$$\frac{\partial F}{\partial t} + p \cdot \nabla_x F = C_{12}(F, n_c). \tag{1}$$

With F the distribution function of the excitations, and n_c the density of the condensate, the collision operator in this model is

$$C_{12}(F,n_c)(p) = n_c \int \delta_0 \delta_3 \Big((1+F_1)F_2F_3 - F_1(1+F_2)(1+F_3) \Big) dp_1 dp_2 dp_3, \tag{2}$$

where $F(p_i)$ is denoted by F_i, and

$$\delta_0 = \delta(p_1 = p_2 + p_3, p_1^2 = p_2^2 + p_3^2 + n_c),$$
$$\delta_3 = \delta(p_1 = p) - \delta(p_2 = p) - \delta(p_3 = p).$$

This corresponds to the 'high temperature case' $|p| \gg \sqrt{n_c}$ in the superfluid rest frame with the temperature range close to $0.7T_c$, where the approximation $p^2 + n_c$ for the excitation energy is commonly used.

Multiplying (2) by $\log \frac{F}{1+F}$ and integrating in p, it follows that $C_{12}(F, n_c) = 0$ if and only if

$$\frac{F_1}{1+F_1} = \frac{F_2}{1+F_2}\frac{F_3}{1+F_3}, \quad p_1 = p_2 + p_3, p_1^2 = p_2^2 + p_3^2 + n_c.$$

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This implies that the kernel of C_{12} consists of the Planckian distribution functions

$$P_{\alpha,\beta}(p) = \frac{1}{e^{\alpha(p^2+n_c)+\beta\cdot p} - 1}, \quad p \in \mathbb{R}^3, \quad \text{for} \quad \alpha > 0,\ \beta \in \mathbb{R}^3.$$

We refer to [1] and references therein for a further discussion of the two-component model, and to [2] where its well-posedness and long time behaviour are studied close to equilibrium. In that context the linearized half-space problem of this paper is connected to boundary layer questions for (1), for which n_c may be taken as a constant n. Take $\alpha = 1$ and write $(|p|^2 + n) + \beta \cdot p = |p + \frac{\beta}{2}|^2 + n - \frac{|\beta|^2}{4}$. With the approximation (close to diffusive thermal equilibrium) $n - \frac{|\beta|^2}{4} = 0$, i.e. $|\beta| = 2\sqrt{n}$, the Planckian $P(p)$ takes the form

$$P(p) = \frac{1}{e^{|p-p_0|^2} - 1} \quad \text{with} \quad p_0 = -\frac{\beta}{2}.$$

Changing variables $p - p_0 \to p$ gives

$$P(p) = \frac{1}{e^{|p|^2} - 1}.$$

The Dirac measure δ_0 in (2) changes into $\delta_c = \delta(p_1 = p_2 + p_3 + p_0, p_1^2 = p_2^2 + p_3^2)$.

With $F = P(1 + f)$, the integrand of the collision operator becomes

$$\begin{aligned} &(1 + F_1)F_2F_3 - F_1(1 + F_2)(1 + F_3) \\ =& - (1 + P_2 + P_3)P_1f_1 + (P_3 - P_1)P_2f_2 + (P_2 - P_1)P_3f_3 \\ &+ P_2P_3f_2f_3 - P_1P_2f_1f_2 - P_1P_3f_1f_3. \end{aligned}$$

Here we have used that $(1 + P) = M^{-1}P$, where

$$M(p) = e^{-p^2},\ p \in \mathbb{R}^3,$$

and that $M(p_1) = M(p_2)M(p_3)$ when $p_1^2 = p_2^2 + p_3^2$. It follows that the linear term in the previous integrand gives the linearized operator

$$L(f) = \frac{n}{P} \int \delta_c\delta_3 \Big[- (1 + P_2 + P_3)P_1f_1 + (P_3 - P_1)P_2f_2 + (P_2 - P_1)P_3f_3 \Big] dp_1dp_2dp_3.$$

We shall here consider functions on a half-line in the x-direction, which in the variable $p = (p_x, p_y, p_z)$ are cylindrically symmetric functions of p_x and $p_r = \sqrt{p_y^2 + p_z^2}$. Assuming $p_0 = (0, p_{0y}, p_{0z})$, this changes the momentum conservation Dirac measure in L to $\delta(p_{1x} - p_{2x} - p_{3x})$. Being in the high temperature case, we introduce a cut-off at $\lambda > 0$ in the integrand of L, given by the characteristic function $\tilde{\chi}$ for the set of (p, p_1, p_2, p_3), such that

$$|p| \geq \lambda, \quad |p_1| \geq \lambda, \quad |p_2| \geq \lambda, \quad |p_3| \geq \lambda.$$

The Milne problem is

$$p_x\partial_x f = Lf, \quad x > 0,\ p_x \in \mathbb{R},\ p_r \in \mathbb{R}^+,\ |p| \geq \lambda, \tag{3}$$

$$f(0, p) = f_0(p_x, p_r), \quad p_x > 0, \quad |p| \geq \lambda, \tag{4}$$

where f_0 is given. The restriction $|p| \geq \lambda$ will be implicitly assumed below, and $\int dp$ will stand for $\int_{|p|\geq\lambda} dp$.

We shall prove in Section 2 (see Lemma 2.1) that the kernel of L is spanned by $|p|^2(1+P)$ and $p_x(1+P)$. For any measurable function $f(x,p)$ such that for almost all $x \in \mathbb{R}^+$,

$$\Big(p \to f(x,p)\Big) \in L^2_{p_r(1+|p|)^3 \frac{P}{1+P}}(\mathbb{R}\times\mathbb{R}^+),$$

where $|p| = \sqrt{p_x^2+p_r^2}$, let

$$f(x,p) = a(x)|p|^2(1+P) + b(x)p_x(1+P) + w(x,p) \tag{5}$$

be its orthogonal decomposition on the kernel of L and the orthogonal complement in $L^2_{p_r\frac{P}{1+P}}$, i.e.

$$\int p_x w(x,p)Pp_r dp_x dp_r = \int |p|^2 w(x,p)Pp_r dp_x dp_r = 0, \quad x \in \mathbb{R}^+. \tag{6}$$

Denote by D the function space

$$D = \Big\{ f;\ f \in L^\infty(\mathbb{R}^+; L^2_{p_r(1+|p|)^3\frac{P}{1+P}}(\mathbb{R}\times\mathbb{R}^+)),$$
$$p_x\partial_x f \in L^2_{loc}(\mathbb{R}^+; L^2_{p_r(1+|p|)^{-3}\frac{P}{1+P}}(\mathbb{R}\times\mathbb{R}^+))\Big\}.$$

The main result of this paper is the following.

Theorem 1.1. *For any $\mathcal{E}\in\mathbb{R}$ and $f_0 \in L^2_{p_r(1+|p|)^3\frac{P}{1+P}}(\mathbb{R}^+\times\mathbb{R}^+)$, there is a unique solution $f \in D$ to the Milne problem,*

$$p_x\partial_x f = Lf, \quad x>0,\ p_x\in\mathbb{R},\ p_r\in\mathbb{R}^+, \tag{7}$$

$$f(0,p) = f_0(p), \quad p_x > 0, \tag{8}$$

$$\int p_x|p|^2 f(x,p)P(p)dp = \mathcal{E}, \quad x\in\mathbb{R}^+. \tag{9}$$

Moreover, for the decomposition (5) of the solution, there are $(a_\infty, b_\infty)\in\mathbb{R}^2$ with

$$b_\infty = \frac{\mathcal{E}}{\gamma}, \quad \text{where} \quad \gamma = \int p_x^2|p|^2P(1+P)dp, \tag{10}$$

and a constant $c>0$, such that for any $\eta \in]0,c_1[$,

$$\int (1+|p|)^3 w^2(x,p)\frac{P}{1+P}dp + |a(x)-a_\infty|^2 + |b(x)-b_\infty|^2 \le ce^{-2\eta x}, \quad x\in\mathbb{R}^+. \tag{11}$$

Here with ν_0 defined by (2.4),

$$c_1 = \min\left\{\frac{\nu_0}{2}, \frac{\nu_0}{2c_2}\right\}, \quad c_2 = \frac{2}{\gamma}\Big(\int p_x^4P(1+P)dp\int p_x^2|p|^4P(1+P)dp\Big)^{\frac12}. \tag{12}$$

Remarks. This result should be compared to the analogous result concerning the Milne problem for the linearized Boltzmann operator around the absolute Maxwellian in [5]. In [5] the mass flow is constant and well-posedness for the Milne problem is proven for a given mass flow. In the present paper on the other hand, the mass flow may not be constant, since mass is not a hydrodynamic mode. But the energy flow is constant, and well-posedness here is proven for a fixed energy flow. That this energy flow is proportional to the asymptotic limit of the mass flow, is a new low temperature result.

A separate complication in the present case is that, whereas the given mass flow in [5] is a hydrodynamic component of the solution, here the energy flow is not in the kernel of L. Another differing aspect compared to classical kinetic theory, is that the collision frequency is asymptotically equivalent to $|p|^3$, when $p \to \infty$.

The interest in half space problems such as (7)-(8) is partly due to their role in the boundary layer behaviour of the solution of boundary-value problems of kinetic equations for small Knudsen numbers. This subject has received much attention for the Boltzmann equation ([13], [15], [16], [17], [18], [3]) and related equations ([7], [14]). Starting from the stationary Boltzmann equation in a half-space with given in-datum and a Maxwellian limit at infinity, the unknown is assumed to stay close to this Maxwellian, giving rise to the linearized stationary Boltzmann equation in a half space. A general treatment of the linearized problem for hard forces and hard spheres under null bulk velocity, is given in [12] and references therein. The case of a gas of hard spheres (resp. of hard or soft forces) and a null bulk velocity at infinity is independently treated in [5] (resp. in [11]). The case of a gas of hard spheres and a nonzero bulk velocity at infinity is considered in [9], positively answering a former conjecture [8]. The Milne problem for the Boltzmann equation with a force term is analyzed in [10]. Half-space problems in a discrete velocity frame are studied in [4]. For a review of mathematical results on the half-space problem for the linear and nonlinear Boltzmann equations, we refer to [6].

The plan of the paper is the following. In Section 2, the linearized collision operator L is studied, including a spectral inequality. In Section 3, Theorem 1.1 is proven.

2. The linearized collision operator.

Lemma 2.1. *L is a self-adjoint operator in $L^2_{\frac{P}{1+P}}$. Within the space of cylindrically invariant distribution functions, its kernel is the subspace spanned by $|p|^2(1+P)$ and $p_x(1+P)$.*

Proof. It follows from the equalities

$$P_2(1+P_2)(P_3-P_1) = P_3(1+P_3)(P_2-P_1) = P_1(1+P_2)(1+P_3),$$
$$P_1(1+P_2+P_3) = P_2P_3 = \frac{P_1(1+P_2)(1+P_3)}{1+P_1}, \quad p_1^2 = p_2^2 + p_3^2,$$

that for any functions f and g in $L^2_{\frac{P}{1+P}}$,

$$\int \frac{P}{1+P}(p)f(p)Lg(p)dp = -n\int \tilde{\chi}\delta_c P_1(1+P_2)(1+P_3)\Big(\frac{f_1}{1+P_1} - \frac{f_2}{1+P_2} - \frac{f_3}{1+P_3}\Big)\Big(\frac{g_1}{1+P_1} - \frac{g_2}{1+P_2} - \frac{g_3}{1+P_3}\Big)dp_1dp_2dp_3.$$

This proves the self-adjointness of L in $L^2_{\frac{P}{1+P}}$. Moreover, $Lf = 0$ for $f \in L^2_{\frac{P}{1+P}}$ implies that

$$\frac{f_1}{1+P_1} = \frac{f_2}{1+P_2} + \frac{f_3}{1+P_3}, \quad p_{1x} = p_{2x} + p_{3x}, \quad p_1^2 = p_2^2 + p_3^2.$$

It is a consequence of this Cauchy equation that the orthogonal functions

$$|p|^2(1+P) \text{ and } p_x(1+P) \tag{13}$$

span the kernel of L. □

The operator L splits into $K-\nu$, where

$$Kf(p) := \frac{2n}{P(p)}\Big(\int \tilde{\chi}\delta(p_x = p_{2x}+p_{3x}, p^2 = p_2^2+p_3^2)(P_3-P)P_2 f_2 dp_2 dp_3 \\ + \int \tilde{\chi}\delta(p_{1x} = p_x+p_{3x}, p_1^2 = p^2+p_3^2)(1+P+P_3)P_1 f_1 dp_1 dp_3 \\ + \int \tilde{\chi}\delta(p_{1x} = p_x+p_{3x}, p_1^2 = p^2+p_3^2)(P_1-P)P_3 f_3 dp_1 dp_3\Big) \tag{14}$$

and

$$\nu(p) := n\int \tilde{\chi}\delta(p_x = p_{2x}+p_{3x}, p^2 = p_2^2+p_3^2)(1+P_2+P_3)dp_2dp_3 \\ +2n\int \tilde{\chi}\delta(p_{1x} = p_x+p_{3x}, p_1^2 = p^2+p_3^2)(P_3-P_1)dp_1dp_3. \tag{15}$$

Lemma 2.2. *The operator K is compact from $L^2_{\nu\frac{P}{1+P}}$ in $L^2_{\nu^{-1}\frac{P}{1+P}}$. The collision frequency ν satisfies*

$$\nu_0(1+|p|)^3 \le \nu(p) \le \nu_1(1+|p|)^3, \quad p = (p_x, p_r) \in \mathbb{R}\times\mathbb{R}^+, \tag{16}$$

for some positive constants ν_0 and ν_1.

Proof of Lemma 2.2. $K = K_1+K_2+K_3$, where

$$K_1h(p) := 2\pi n\int k_1(p,p_2)h_2dp_2, \quad K_2h(p) = 2\pi n\int k_2(p,p_1)h_1dp_1,$$

$$K_3(p) = 2\pi n\int k_3(p,p_3)h_3dp_3,$$

$$k_1(p,p_2) := \frac{P_2}{\pi}\int \delta\left(p_x = p_{2x}+p_{3x}, p^2 = p_2^2+p_3^2\right)\frac{P_3-P}{P}dp_3 \\ = P_2\ \chi_{p^2-p_2^2-(p_x-p_{2x})^2>0}\left(\frac{1}{P(e^{p^2-p_2^2}-1)}-1\right),$$

$$k_2(p,p_1) := \frac{P_1}{\pi}\int \delta\left(p_{1x} = p_x+p_{3x}, p_1^2 = p^2+p_3^2\right)\frac{1+P+P_3}{P}dp_3 \\ = P_1\ \chi_{p_1^2-p^2-(p_{1x}-p_x)^2>0}\left(\frac{1}{P}+1+\frac{1}{P(e^{p_1^2-p^2}-1)}\right),$$

$$k_3(p,p_3) := \frac{P_3}{\pi}\int \delta\left(p_{1x} = p_x+p_{3x}, p_1^2 = p^2+p_3^2\right)\frac{P_1-P}{P}dp_1 \\ = P_3\ \chi_{p^2+p_3^2-(p_x-p_{3x})^2>0}\left(\frac{1}{P(e^{p^2+p_3^2}-1)}-1\right).$$

Let $m\in\mathbb{N}^*$. We treat separately the parts of K_1 with $\frac{P_3}{P}$ and with $\frac{P}{P}$, and notice that for $|p|, |p_2|, |p_3| \ge \lambda$, factors $P = \frac{M}{1-M}$ may be replaced by M for questions of boundedness and convergence to zero. For $m>\lambda$ fixed, split the domain of p_2 into $|p_2|<m$ and $|p_2|>m$. The mapping

$$h \to \int_{|p_2|<m} k_{11}(p,p_2)h_2dp_2,$$

where

$$k_{11}(p,p_2) = M_2M_3M^{-1}\chi_{p^2-p_2^2-(p_x-p_{2x})^2>0, \lambda^2<|p_2|^2<p^2-\lambda^2},$$

is compact from $L^2_{\nu\frac{P}{1+P}}$ into $L^2_{\nu^{-1}\frac{P}{1+P}}$. Indeed

$$\int_{|p_2|<m} \nu^{-1} M k_{11}^2(p,p_2)\nu_2^{-1} M_2^{-1} dp dp_2 < \infty.$$

The mapping $h \to \int_{|p_2|>m} k_{11}(p,p_2)h_2 dp_2$ tends to zero in $L^2_{\nu^{-1}\frac{P}{1+P}}$ when $m \to \infty$, uniformly for functions h with norm one in $L^2_{\nu\frac{P}{1+P}}$. Namely, it holds

$$\begin{aligned}
&\left(\int \nu^{-1} M (\int_{|p_2|>m} k_{11}(p,p_2)h_2 dp_2)^2 dp\right)^{\frac{1}{2}} \\
&\leq \int_{|p_2|>m} \left(\int \nu^{-1} M k_{11}^2(p,p_2) dp\right)^{\frac{1}{2}} h_2 dp_2 \\
&\leq \int_{|p_2|>m} \left(\int_{|p|>|p_2|} \nu^{-1} M dp\right)^{\frac{1}{2}} h_2 dp_2 \\
&\leq c \int_{|p_2|>m} \frac{M_2^{\frac{1}{2}}}{|p_2|} h_2 dp_2 \\
&\leq \frac{c}{m} \left(\int M_2 \nu_2 h_2^2 dp_2\right)^{\frac{1}{2}}.
\end{aligned}$$

The other term in K_1 only differs in the factor $\frac{P}{P} < \frac{P_3}{P}$. The compactness of K_1 follows.

An analogous splitting of K_2 with respect to velocities smaller and larger than m, gives for K_2 and $|p_1| < m$ that the dominating term corresponds to the factor $\frac{1}{P}$. The mapping becomes

$$h \to \int_{|p_1|<m} k_{21}(p,p_1)h_1 dp_1,$$

with

$$k_{21}(p,p_1) = M_1 M^{-1} \chi_{p_1^2 - p^2 - (p_{1x}-p_x)^2 > 0, \lambda^2 < p_1^2 - p^2}.$$

Since the kernel k_{21} is bounded on the domain of integration which is bounded, this mapping is compact. The mapping $h \to \int_{|p_1|>m} k_{21}(p,p_1)h_1 dp_1$ tends to zero in $L^2_{\nu^{-1}\frac{P}{1+P}}$ when $m \to \infty$, uniformly for functions h with norm one in $L^2_{\nu\frac{P}{1+P}}$. Here

$$\begin{aligned}
&\left(\int \nu^{-1} M \left(\int_{|p_1|>m} k_{21}(p,p_1)h_1 dp_1\right)^2 dp\right)^{\frac{1}{2}} \\
&\leq \int_{|p_1|>m} \left(\int_{p^2<p_1^2} \nu^{-1} M^{-1} dp\right)^{\frac{1}{2}} M_1^{\frac{1}{2}} \nu_1^{-\frac{1}{2}} (M_1\nu_1)^{\frac{1}{2}} h_1 dp_1 \\
&\leq c \int_{|p_1|>m} \nu_1^{-\frac{5}{6}} (M_1\nu_1)^{\frac{1}{2}} h_1 dp_1 \\
&\leq \left(\int_{|p_1|>m} \nu_1^{-\frac{5}{3}} dp_1\right)^{\frac{1}{2}} \left(\int M_1 \nu_1 h_1^2 dp_2\right)^{\frac{1}{2}},
\end{aligned}$$

which again tends to zero, uniformly in h when $m \to \infty$. In K_3 the dominating term corresponds to the factor $\frac{P}{P}$. For the kernel

$$k_{31}(p, p_3) = M_3 \chi_{p^2+p_3^2-(p_x+p_{3x})^2>0, |p_3|>\lambda},$$

it holds that

$$\int_{|p_3|<m} \nu^{-1} M k_{31}^2(p, p_3) \nu_3^{-1} M_3^{-1} dp dp_3 < \infty,$$

and

$$\begin{aligned}
&\left(\int \nu^{-1} M \left(\int_{|p_3|>m} k_{31}(p, p_3) h_3 dp_3 \right)^2 dp \right)^{\frac{1}{2}} \\
&\leq \int_{|p_3|>m} \left(\int \nu^{-1} M k_{31}^2(p, p_3) dp \right)^{\frac{1}{2}} h_3 dp_3 \\
&\leq \int_{|p_3|>m} M_3 h_3 dp_3 \left(\int \nu^{-1} M dp \right)^{\frac{1}{2}} \\
&\leq c \left(\int_{|p_3|>m} M_3 \nu_3^{-1} dp_3 \right)^{\frac{1}{2}} \left(\int M_3 \nu_3 h_3^2 dp_3 \right)^{\frac{1}{2}}.
\end{aligned}$$

This ends the proof of the compactness of K.

The function ν is bounded from below by a positive constant, since

$$P_3 - P_1 > 0, \quad p_1^2 = p^2 + p_3^2.$$

For $|p| > \lambda$ the first term of $\nu(p)$ belongs to the interval with end points

$$2\pi^2 n \int_{p_{2r}>0, p_{2r}^2+2(p_{2x}-\frac{1}{2}p_x)^2<\frac{1}{2}p_x^2+p_r^2} p_{2r} dp_{2r} dp_{2x}$$

and

$$2\pi^2 n \left(1 + \frac{2}{e^{\lambda^2}-1} \right) \int_{p_{2r}>0, p_{2r}^2+2(p_{2x}-\frac{1}{2}p_x)^2<\frac{1}{2}p_x^2+p_r^2} p_{2r} dp_{2r} dp_{2x}.$$

With the change of variables $(x, y) := (p_{2x}, p_{2r}^2)$,

$$\begin{aligned}
&2 \int_{p_{2r}>0, p_{2r}^2+2(p_{2x}-\frac{1}{2}p_x)^2<\frac{1}{2}p_x^2+p_r^2} p_{2r} dp_{2r} dp_{2x} \\
&= \int_{y>0, y+2(x-\frac{1}{2}p_x)^2<\frac{1}{2}p_x^2+p_r^2} dx dy \\
&= \int_0^{\frac{1}{2}p_x^2+p_r^2} \int_{(x-\frac{1}{2}p_x)^2<\frac{1}{4}(p_x^2+2p_r^2-2y)} dx dy \\
&= \int_0^{\frac{1}{2}p_x^2+p_r^2} \sqrt{p_x^2+2p_r^2-2y} dy \\
&= \frac{1}{3}(p_x^2+2p_r^2)^{\frac{3}{2}} \\
&\sim |p|^3.
\end{aligned}$$

The second term of $\nu(p)$ is bounded. Indeed,

$$
\begin{aligned}
0 &\leq \frac{1}{2\pi^2}\int \delta(p_{1x}=p_x+p_{3x}, p_1^2=p^2+p_3^2)(P_3-P_1)dp_1dp_3\\
&\leq \int_{p_{1r}>0}\left(\frac{1}{e^{p_1^2-p^2}-1}-P_1\right)\Big(\int_0^{+\infty}\delta(p_{3r}^2=p_{1r}^2-p^2-p_x^2+2p_xp_{1x})\\
&\qquad\qquad p_{3r}dp_{3r}\Big)p_{1r}dp_{1r}dp_{1x}\\
&\leq \iint_0^{+\infty}\frac{1}{e^{x^2+y-p^2}-1}\chi_{y>p^2+p_x^2-2xp_x}dydx\\
&= \sum_{k\geq 1} e^{kp^2}\int e^{-kx^2}\int_0^{+\infty} e^{-ky}\chi_{y>p^2+p_x^2-2xp_x}dydx\\
&\leq \sum_{k\geq 1}\frac{1}{k}\int e^{-k(x-p_x)^2}dx\\
&= \sqrt{\pi}\sum_{k\geq 1}\frac{1}{k^{\frac{3}{2}}}.
\end{aligned}
$$

□

Denote by $(\cdot,\cdot)$ the scalar product in $L^2_{\frac{P}{1+P}}$, and by $\tilde{P}$ the orthogonal projection on the kernel of L.

Lemma 2.3. *L satisfies the spectral inequality,*

$$
-(Lf,f) \geq \nu_0\left((1+|p|)^3(I-\tilde{P})f,(I-\tilde{P})f\right),\quad f\in L^2_{(1+|p|)^3\frac{P}{1+P}}. \tag{17}
$$

Proof. For the compact, self-adjoint operator K, the spectrum behaves similarly to the classical Boltzmann case. Namely, there is no eigenvalue $\alpha>1$ for $\frac{K}{\nu}$. Else there is $f\neq 0$ such that $Lf=(\alpha-1)\nu f$ and so $(Lf,f)>0$. But

$$
(Lf,f) = -n\int \tilde{\chi}\delta_c\left(\frac{f_1}{1+P_1}-\frac{f_2}{1+P_2}-\frac{f_3}{1+P_3}\right)^2 dp_1dp_2dp_3 \leq 0.
$$

In the complement of the kernel of L, the eigenvalues of $\frac{K}{\nu}$ are bounded from above by $\alpha_0<1$. Spanning $L^2_{(1+|p|)^3\frac{P}{1+P}}$ with the corresponding eigenfunctions of $\frac{K}{\nu}$ and the kernel of L, we obtain the spectral inequality

$$
(Lf,f)\leq(\alpha_0-1)(\nu(I-\tilde{P})f,(I-\tilde{P})f),\quad f\in L^2_{(1+|p|)^3\frac{P}{1+P}}.
$$

From here, (17) follows by (16).

3. The Milne problem.

The Milne problem. This section gives the proof of Theorem 1.1.

Let

$$
\tilde{f}=f-\frac{\mathcal{E}}{\gamma}p_x(1+P),\quad \tilde{f}_0(p)=f_0(p)-\frac{\mathcal{E}}{\gamma}p_x(1+P).
$$

Solving the Milne problem (7)-(8)-(9) for the unknown f is equivalent to solving

$$
p_x\partial_x\tilde{f}=L\tilde{f},\quad x>0,\ p_x\in\mathbb{R},\ p_r\in\mathbb{R}^+, \tag{18}
$$

$$
\tilde{f}(0,p)=\tilde{f}_0(p),\quad p_x>0, \tag{19}
$$

$$
\int p_x|p|^2\tilde{f}(x,p)P(p)dp=0,\quad x\in\mathbb{R}^+, \tag{20}
$$

for the unknown $\tilde{f}$.We first study the behaviour of a solution $\tilde{f}$ to the Milne problem (18)-(19)-(20), when $x \to +\infty$.

Set

$$\tilde{f}(x,p) = (a(x)|p|^2 + \tilde{b}(x)p_x)(1+P) + w(x,p),$$

with

$$\int p_x w P dp = \int |p|^2 w P dp = 0,$$

an orthogonal decomposition of $\tilde{f}$. Denote by

$$W(x) = \frac{1}{2}\int p_x \tilde{f}^2(x,p)\frac{P}{1+P}dp \tag{21}$$

the linearized entropy flux of $\tilde{f}$. It holds that

$$W(0) \leq \frac{1}{2}\int_{p_x>0} p_x \tilde{f}_0^2(p)\frac{P}{1+P}dp. \tag{22}$$

By (20)

$$\begin{aligned} W(x) =&\frac{1}{2}\int p_x \tilde{f}^2(x,p)\frac{P}{1+P}dp - \frac{1}{\gamma}\int p_x^2 \tilde{f}(x,p)P(p)dp \int p_x|p|^2\tilde{f}(x,p)P(p)dp \\ =&\frac{1}{2}\int p_x w^2(x,p)\frac{P}{1+P}dp + a\int p_x|p|^2 wPdp + \tilde{b}\int p_x^2 wPdp + a\tilde{b}\gamma \\ &-\frac{1}{\gamma}\Big(\int p_x^2 wPdp + a\gamma\Big)\Big(\int p_x|p|^2 wPdp + \tilde{b}\gamma\Big), \end{aligned}$$

i.e.

$$W(x) = \frac{1}{2}\int p_x w^2(x,p)\frac{P}{1+P}dp - \frac{1}{\gamma}\int p_x^2 wPdp\int p_x|p|^2 wPdp. \tag{23}$$

This differs from ([BCN]), where the linearized entropy flux of the solution is equal to the linearized entropy flux of its non-hydrodynamic component.The expression (23) for W in terms of w is important in the proof.

Multiplying (18) by $\tilde{f}\frac{P}{1+P}$, integrating on $(0,X)\times\mathbb{R}\times\mathbb{R}^+$ (resp. $\mathbb{R}\times\mathbb{R}^+$), and using (17), gives

$$W(X) + \nu_0\int_0^X\int(1+|p|)^3 w^2(x,p)\frac{P}{1+P}dpdx \leq W(0), \quad X>0, \tag{24}$$

and

$$W'(x) + \nu_0\int(1+|p|)^3 w^2(x,p)\frac{P}{1+P}dp \leq 0. \tag{25}$$

Since $\tilde{f}\in D$, it holds that $W\in L^\infty(\mathbb{R}^+)$. Then by (24) and (23), $W\in L^1(\mathbb{R}^+)$. By (25), W is a non-increasing function. Hence it tends to zero, when x tends to $+\infty$ and is a nonnegative function. Let $\eta\in]0,c_1[$. Multiply (25) by $e^{2\eta x}$, so that

$$\big(W(x)e^{2\eta x}\big)' - 2\eta W(x)e^{2\eta x} + \nu_0 e^{2\eta x}\int(1+|p|)^3 w^2(x,p)\frac{P}{1+P}dp \leq 0.$$

By the Cauchy-Schwartz inequality,

$$\Big|\int p_x^2 w(x,p)Pdp\int p_x|p|^2 w(x,p)Pdp\Big| \leq \frac{\gamma c_3}{2}\int w^2(x,p)\frac{P}{1+P}dp.$$

Hence,

$$\big(W(x)e^{2\eta x}\big)' + e^{2\eta x}\int\Big(\nu_0(1+|p|)^3 - \eta(p_x+c_3)\Big)w^2(x,p)\frac{P}{1+P}dp \leq 0, \quad x\geq 0. \tag{26}$$

By the definition (12) of c_1, the nonnegativity of W and (22), it holds that

$$\int_0^\infty e^{2\eta x}\int(1+|p|)^3 w^2(x,p)\frac{P}{1+P}dpdx\le c, \tag{27}$$

for some constant c. Moreover, by (26) and (25),

$$0\le W(x)\le W(0)e^{-2\eta x}\le ce^{-2\eta x},\quad x\ge 0. \tag{28}$$

(27) implies that $\tilde f(x,\cdot)$ converges to a hydrodynamic state when $x\to+\infty$. In order to prove the exponential point-wise decay of $\int(1+|p|)^3 w^2(x,p)\frac{P}{1+P}dp$ in (11), let $0<Y<X$ be given and introduce a smooth cutoff function $\Phi(x)$ such that

$$\Phi(x)=0,\ x\in\Big[0,\frac{Y}{2}\Big[\cup\,]X+1,+\infty[,\quad \Phi(x)=1,\ x\in[Y,X].$$

Denote by $\varphi(x)=e^{\eta x}\Phi(x)$. Then,

$$p_x\partial_x^2(\varphi\tilde f)=L(\partial_x(\varphi\tilde f))+\varphi' Lw+p_x\varphi''\tilde f. \tag{29}$$

Multiply (29) by $\partial_x(\varphi\tilde f)\frac{P}{1+P}$, integrate over $\mathbb{R}_{p_x}\times\mathbb{R}^+_{p_r}$ and use (17). Hence,

$$\begin{aligned}&\frac12\frac{d}{dx}\int p_x(\partial_x(\varphi\tilde f))^2\frac{P}{1+P}dp+\nu_0\int(1+|p|)^3(\partial_x(\varphi w))^2\frac{P}{1+P}dp\\&\le\int\partial_x(\varphi\tilde f)\Big(\varphi' Lw+p_x\varphi''\tilde f\Big)\frac{P}{1+P}dp\\&=\varphi'\int\partial_x(\varphi w)Lw\frac{P}{1+P}dp+\varphi\varphi''\int p_x\tilde f\partial_x\tilde f\frac{P}{1+P}dp+\varphi'\varphi''\int p_x\tilde f^2\frac{P}{1+P}dp,\end{aligned}$$

i.e.

$$\begin{aligned}&\frac12\frac{d}{dx}\int p_x(\partial_x(\varphi\tilde f))^2\frac{P}{1+P}dp+\nu_0\int(1+|p|)^3(\partial_x(\varphi w))^2\frac{P}{1+P}dp\\&\qquad\le\varphi'\int\partial_x(\varphi w)Lw\frac{P}{1+P}dp+(\varphi\varphi'' W)'+(\varphi'\varphi''-\varphi\varphi^{(3)})W.\end{aligned}$$

Integrate the last inequality on $[0,+\infty[$, so that

$$\begin{aligned}&\nu_0\int_0^{+\infty}\int(1+|p|)^3(\partial_x(\varphi w))^2\frac{P}{1+P}dpdx\\&\le\int_0^{+\infty}\varphi'(x)\int\partial_x(\varphi w)Lw\frac{P}{1+P}dpdx+\int_0^{+\infty}(\varphi'\varphi''-\varphi\varphi^{(3)})W(x)dx\\&\le\|\varphi'\|_\infty^2\,\frac{\alpha}{2}\int_0^{+\infty}\int(1+|p|)^3(\partial_x(\varphi w))^2\frac{P}{1+P}dpdx\\&+\frac{1}{2\alpha}\int_0^{+\infty}\int\frac{1}{(1+|p|)^3}(Lw)^2\frac{P}{1+P}dpdx\\&+\int_0^{+\infty}(\varphi'\varphi''-\varphi\varphi^{(3)})W(x)dx\\&\le\|\varphi'\|_\infty^2\,\frac{\alpha}{2}\int_0^{+\infty}\int(1+|p|)^3(\partial_x(\varphi w))^2\frac{P}{1+P}dpdx\\&+\frac{c}{2\alpha}\int_0^{+\infty}\int(1+|p|)^3w^2\frac{P}{1+P}dpdx\\&+\int_0^{+\infty}(\varphi'\varphi''-\varphi\varphi^{(3)})W(x)dx,\quad\alpha>0.\end{aligned}$$

Choose $\alpha < \frac{\nu_0}{\|\varphi'\|^2_\infty}$. Use (27), the Cauchy-Schwartz inequality, and the exponential decay of W expressed in (28) in the W-term. It then holds

$$\int_0^{+\infty}\int (1+|p|)^3(\partial_x(\varphi w))^2\frac{P}{1+P}dpdx \leq c,$$

for some positive constant c. Finally,

$$\begin{aligned}&e^{2\eta X}\int (1+|p|)^3 w^2(X,p)\frac{P}{1+P}dp\\ =&2\int_0^X\int (1+|p|)^3(\partial_x(\varphi w))^2\frac{P}{1+P}dpdx \leq c, X>0.\end{aligned} \tag{30}$$

The exponential decay of $(a,\tilde{b})$ to some limit $(a_\infty,\tilde{b}_\infty)$ when x tends to $+\infty$, can be proved as follows. The solution $\tilde{f}(x,p)=(a(x)|p|^2+\tilde{b}(x)p_x)(1+P)+w(x,p)$ is solution to (18) if and only if

$$\left(a'p_x|p|^2+\tilde{b}'p_x^2\right)(1+P)+p_x\partial_x w = Lw.$$

Multiply the former equation by p_xP (resp. $|p|^2P$) and integrate with respect to p, so that

$$\left(a+\frac{1}{\gamma}\int p_x^2 w(\cdot,p)Pdp\right)' = \left(\tilde{b}+\frac{1}{\gamma}\int p_x|p|^2w(\cdot,p)Pdp\right)' = 0.$$

Denote by

$$a_\infty := a(0)+\frac{1}{\gamma}\int p_x^2 w(0,p)Pdp, \quad \tilde{b}_\infty := \tilde{b}(0)+\frac{1}{\gamma}\int p_x|p|^2w(0,p)Pdp.$$

By the Cauchy-Schwartz inequality and (30),

$$\begin{aligned}|a(x)-a_\infty| &= \tfrac{1}{\gamma}\left|\int p_x^2 w(x,p)Pdp\right| \leq c\left(\int (1+|p|)^3 w^2(x,p)\frac{P}{1+P}dp\right)^{\frac12} \leq ce^{-\eta x},\\ \left|\tilde{b}(x)-\tilde{b}_\infty\right| &= \frac{1}{\gamma}\left|\int p_x|p|^2 w(x,p)Pdp\right| \leq c\Big(\int (1+|p|)^3 w^2(x,p)\frac{P}{1+P}dp\Big)^{\frac12} \leq ce^{-\eta x}.\end{aligned} \tag{31}$$

By (28)

$$\lim_{x\to+\infty}\int p_x\tilde{f}^2(x,p)\frac{P}{1+P}dp = 0. \tag{32}$$

By (30)

$$\lim_{x\to+\infty}\int (1+|p|)^3 w^2(x,p)\frac{P}{1+P}dp = 0. \tag{33}$$

Using the decomposition $\tilde{f}=(a|p|^2+\tilde{b}p_x)(1+P)+w$ of $\tilde{f}$ into its hydrodynamic and non hydrodynamic components, and setting

$$a_\infty = \lim_{x\to+\infty} a(x), \quad \tilde{b}_\infty = \lim_{x\to+\infty}\tilde{b}(x),$$

it follows from (32)-(33) that

$$\lim_{x\to+\infty}\int p_x\Big((a(x)|p|^2+\tilde{b}(x)p_x)(1+P)\Big)^2\frac{P}{1+P}dp = 0, \quad \text{i.e.} \quad a_\infty\tilde{b}_\infty = 0.$$

Below this will be improved to (10) $\tilde{b}_\infty = 0$.

But first we prove the existence of a solution $\tilde{f}\in D$ to the Milne problem (18)-(19)-(20). It will be obtained as the limit when $l\to+\infty$ of the sequence $(\tilde{f}_l)_{l\in\mathbb{N}^*}$

of solutions to the stationary linearized equation on the slab $[0,l]$ with specular reflection at $x=l$, i.e.

$$p_x\partial_x \tilde{f}_l = L\tilde{f}_l, \quad x\in[0,l],\ p_x\in\mathbb{R},\ p_r\in\mathbb{R}^+, \tag{34}$$

$$\tilde{f}_l(0,p)=\tilde{f}_0(p), \quad p_x>0, \tag{35}$$

$$\tilde{f}_l(l,p_x,p_r)=\tilde{f}_l(l,-p_x,p_r), \quad p_x<0. \tag{36}$$

Switch from given in-data and no inhomogeneous term, to zero indata and an inhomogeneous term. Let $\epsilon>0$ be given. Let the subspace $D(A)$ of $L^2_{p_r(1+|p|)^3\frac{P}{1+P}}((0,l)\times\mathbb{R}\times\mathbb{R}^+)$ be defined by

$$D(A)=\{g\in L^2_{p_r(1+|p|)^3\frac{P}{1+P}}((0,l)\times\mathbb{R}\times\mathbb{R}^+);\ p_x\partial_x g\in L^2_{p_r(1+|p|)^3\frac{P}{1+P}}((0,l)\times\mathbb{R}\times\mathbb{R}^+), g(0,p)=0, p_x>0, \quad g(l,p_x,p_r)=g(l,-p_x,p_r), p_x<0\}.$$

The operator A defined on $D(A)$ by

$$(Ag)(x,p)=\epsilon g(x,p)+p_x\partial_x g(x,p)$$

is m-accretive since $I-\frac{1}{2\epsilon}A$ is bijective. Indeed, for any $f\in L^2_{p_r(1+|p|)^3\frac{P}{1+P}}((0,l)\times\mathbb{R}\times\mathbb{R}^+)$, there is a unique $g\in D(A)$ such that

$$\left(I-\frac{1}{2\epsilon}A\right)g=f \quad \text{i.e.} \quad \frac{1}{2}g-\frac{1}{2\epsilon}p_x\partial_x g=f. \tag{37}$$

Here g is explicitly given by

$$g(x,p)=-\frac{2\epsilon}{p_x}\int_0^x f(y,p)e^{\epsilon\frac{x-y}{p_x}}dy, \quad p_x>0,$$

$$g(x,p)=\frac{2\epsilon}{p_x}\Big(\int_0^l f(y,p)e^{\epsilon\frac{x+y-2l}{p_x}}dy+\int_x^l f(y,p)e^{\epsilon\frac{x-y}{p_x}}dy\Big), \quad p_x<0.$$

It belongs to $L^2_{p_r(1+|p|)^3\frac{P}{1+P}}((0,l)\times\mathbb{R}\times\mathbb{R}^+)$ since multiplying (37) by $2g(1+|p|)^3\frac{P}{1+P}$, then integrating on $[0,l]\times\mathbb{R}^3$ implies that

$$\begin{aligned}\int g^2(x,p)(1+|p|)^3\frac{P}{1+P}dxdp+\frac{1}{2\epsilon}\int_{p_x<0}|p_x|(1+|p|)^3g^2(0,p)\frac{P}{1+P}dp\\ =2\int f(x,p)g(x,p)(1+|p|)^3\frac{P}{1+P}dp\\ \le\int f^2(x,p)(1+|p|)^3\frac{P}{1+P}dxdp\\ +\int g^2(x,p)(1+|p|)^3\frac{P}{1+P}dxdp.\end{aligned}$$

It then follows from (37) that $p_x\partial_x g\in L^2_{p_r(1+|p|)^3\frac{P}{1+P}}((0,l)\times\mathbb{R}\times\mathbb{R}^+)$.

Since $-L$ is an accretive operator, from here by an m-accretive study of $A-L$, there exists a solution

$$\tilde{f}_\epsilon\in L^2_{p_r(1+|p|)^3\frac{P}{1+P}}((0,l)\times\mathbb{R}\times\mathbb{R}^+)$$

to

$$\epsilon\tilde{f}_\epsilon+p_x\partial_x\tilde{f}_\epsilon=L\tilde{f}_\epsilon, \quad x>0,\ p_x\in\mathbb{R},\ p_r\in\mathbb{R}^+, \tag{38}$$

$$\tilde{f}_\epsilon(0,p)=\tilde{f}_0(p), \quad p_x>0,$$

$$\tilde{f}_\epsilon(l,p_x,p_r)=\tilde{f}_\epsilon(l,-p_x,p_r), \quad p_x<0.$$

In order to prove that there is a converging subsequence of $(\tilde{f}_\epsilon)$ when ϵ tends to zero, split $\tilde{f}_\epsilon$ into its hydrodynamic and non-hydrodynamic parts as

$$\tilde{f}_\epsilon(x,p) = (a_\epsilon(x)|p|^2 + b_\epsilon(x)p_x)(1+P) + w_\epsilon(x,p),$$

with

$$\int p_x w_\epsilon P dp = \int |p|^2 w_\epsilon P dp = 0. \tag{39}$$

Multiply (38) by $\tilde{f}_\epsilon \frac{P}{1+P}$, integrate w.r.t. $(x,p) \in [0,l] \times \mathbb{R} \times \mathbb{R}^+$ and use the spectral inequality (17), so that (w_ϵ) is uniformly bounded in $L^2_{p_r(1+|p|)^3 \frac{P}{1+P}}([0,l] \times \mathbb{R} \times \mathbb{R}^+)$. Notice that the boundary term at l vanishes. And so, up to a subsequence, (w_ϵ) weakly converges in $L^2_{p_r(1+|p|)^3 \frac{P}{1+P}}([0,l] \times \mathbb{R} \times \mathbb{R}^+))$ to some function w. Moreover, the same argument as for getting (30) can be used here, so that

$$e^{2\eta x} \int (1+|p|)^3 w_\epsilon^2(x,p) \frac{P}{1+P} dp \leq c, \quad x \in [0,l]. \tag{40}$$

Expressing $\int_{p_x>0} p_x \tilde{f}_\epsilon(0,p) \frac{P}{1+P} dp$ (resp. $\int_{p_x>0} p_x |p|^2 \tilde{f}_\epsilon(0,p) \frac{P}{1+P} dp$) in terms of $a_\epsilon(0)$, $b_\epsilon(0)$ and $w_\epsilon(0,\cdot)$ leads to

$$\begin{aligned} &a_\epsilon(0) \int_{p_x>0} p_x |p|^2 P dp + b_\epsilon(0) \int_{p_x>0} p_x^2 P dp \\ &= \int_{p_x>0} p_x \tilde{f}_0(p) \frac{P}{1+P} dp - \int_{p_x>0} p_x w_\epsilon(0,p) \frac{P}{1+P} dp, \end{aligned}$$

and

$$\begin{aligned} &a_\epsilon(0) \int_{p_x>0} p_x |p|^4 P dp + b_\epsilon(0) \int_{p_x>0} p_x^2 |p|^2 P dp \\ &= \int_{p_x>0} p_x |p|^2 \tilde{f}_0(p) \frac{P}{1+P} dp - \int_{p_x>0} p_x |p|^2 w_\epsilon(0,p) \frac{P}{1+P} dp. \end{aligned}$$

By the Cauchy-Schwartz inequality and (40) taken at $x=0$, it follows that

$$\int_{p_x>0} p_x w_\epsilon(0,p) \frac{P}{1+P} dp \quad \text{and} \quad \int_{p_x>0} p_x |p|^2 w_\epsilon(0,p) \frac{P}{1+P} dp$$

are bounded. Consequently, $(a_\epsilon(0), b_\epsilon(0))$ is uniformly bounded with respect to ϵ. Moreover, f_ϵ solves (38) if and only if

$$\epsilon\Big((a_\epsilon |p|^2 + b_\epsilon p_x)(1+P) + w_\epsilon\Big) + (a'_\epsilon p_x |p|^2 + b'_\epsilon p_x^2)(1+P) + p_x \partial_x w_\epsilon = L w_\epsilon. \tag{41}$$

Multiplying the previous equation by $p_x P$ (resp. $(p^2+n)P$) and integrating w.r.t. p, implies that

$$\begin{aligned} &\epsilon b_\epsilon \int p_x^2 P(1+P) dp + \gamma a'_\epsilon + \Big(\int p_x^2 w_\epsilon P dp\Big)' = 0, \\ &\epsilon a_\epsilon \int |p|^4 P(1+P) dp + \gamma b'_\epsilon + \Big(\int p_x |p|^2 w_\epsilon P dp\Big)' = 0. \end{aligned}$$

Consequently, denoting by

$$\alpha = \sqrt{\int p_x^2 P(1+P) dp} \quad \text{and} \quad \beta = \sqrt{\int |p|^4 P(1+P) dp},$$

it holds that

$$\gamma a_\epsilon(x) = -\int p_x^2 w_\epsilon(x,p)Pdp + (\gamma a_\epsilon(0) + \int p_x^2 w_\epsilon(0,p)Pdp)e^{\frac{\alpha\beta\epsilon}{\gamma}x}$$
$$+\,\epsilon\int_0^x \Big(\int p_x\frac{\alpha^2}{\gamma}(p^2+n)w_\epsilon(y,p)Pdp\Big)e^{\frac{\alpha\beta\epsilon}{\gamma}(x-y)}dy, \quad x\in[0,l],$$
$$\gamma b_\epsilon(x) = -\int p_x|p|^2 w_\epsilon(x,p)Pdp + (\gamma b_\epsilon(0) + \int p_x|p|^2 w_\epsilon(0,p)Pdp)e^{-\frac{\alpha\beta\epsilon}{\gamma}x}$$
$$+\,\epsilon\int_0^x \Big(\int \frac{\beta^2}{\gamma}p_x^2 w_\epsilon(y,p)Pdp\Big)e^{-\frac{\alpha\beta\epsilon}{\gamma}(x-y)}dy, \quad x\in[0,l].$$

Together with the bounds of $(a_\epsilon(0), b_\epsilon(0))$ and (40), this implies that (a_ϵ) (resp. (b_ϵ)) is bounded in L^2. And so, up to a subsequence, $\tilde{f}_\epsilon$ weakly converges in $L^2_{p_r(1+|p|)^3\frac{P}{1+P}}((0,l)\times\mathbb{R}\times\mathbb{R}^+)$ to a solution $\tilde{f}_l$ of (34)-(35)-(36).

Similar arguments can be used in order to prove that up to a subsequence, $(\tilde{f}_l)$ converges to a solution $\tilde{f}$ of the Milne problem (18)-(19)-(20) when l tends to $+\infty$. Indeed, if $\tilde{f}_l$ admits the decomposition

$$\tilde{f}_l(x,p) = (a_l(x)|p|^2 + b_l(x)p_x)(1+P) + w_l(x,p),$$

with

$$\int p_x w_l P dp = \int |p|^2 w_l P dp = 0,$$

then the sequence (w_l) is bounded in $L^2_{p_r(1+|p|)^3\frac{P}{1+P}}(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+)$ and pointwise in x as in (40). And so, up to a subsequence, (w_l) converges weakly in $L^2_{p_r(1+|p|)^3\frac{P}{1+P}}(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+)$ and also weak star in x, weak in p in $L^\infty(\mathbb{R}^+; L^2_{p_r(1+|p|)^3}(\mathbb{R}\times\mathbb{R}^+))$. The sequences (a_l) and (b_l) satisfy

$$\Big(\gamma a_l + \int p_x^2 w_l P dp\Big)' = 0, \quad \Big(\gamma b_l + \int p_x|p|^2 w_l P dp\Big)' = 0,$$

so that

$$\gamma a_l(x) = -\int p_x^2 w_l(x,p)Pdp + \gamma a_l(0) + \int p_x^2 w_l(0,p)Pdp,$$
$$\gamma b_l(x) = -\int p_x|p|^2 w_l(x,p)Pdp + \gamma b_l(0) + \int p_x|p|^2 w_l(0,p)Pdp.$$

It follows that the sequences (a_l) and (b_l) are uniformly bounded on $\mathbb{R}^+$, and so, up to a subsequence, converge weak star in x. The limit of $(\tilde{f}_l)$ is a weak solution to the problem. This weak solution belongs to D.

We can now prove that $\tilde{b}_\infty = 0$. For this we notice that the discussion of this section up to (28) included, also holds for $\tilde{f}_l$, W being nonnegative on $[0,l]$ because it is non increasing and vanishes at l. The discussion from (29) leading up to (32) is valid as well. But for f_l it holds that $\tilde{b}_l(l) = 0$, and so (31) taken at $x = l$ leads to $|\tilde{b}_{l\infty}| \leq ce^{-\eta l}$.

Take $\beta \geq \alpha \gg 0$. Using (31) again implies that for all $l > \beta$,

$$|\tilde{b}_l(x)| \leq |\tilde{b}_l(x) - \tilde{b}_{l\infty}| + ce^{-\eta\alpha} \leq 2ce^{-\eta\alpha}, \quad x \geq \alpha.$$

It follows that

$$|\tilde{b}(x)| \leq 2ce^{-\eta\alpha}, \quad x \geq \alpha.$$

Hence

$$\lim_{x\to\infty}\tilde{b}(x)=0=\tilde{b}_\infty.$$

The uniqueness of the solution of the Milne problem (18)-(19)-(20) can be proven as follows. Let $\tilde{f}\in D$ be solution to the Milne problem (18)-(19)-(20) with zero indatum at $x=0$ and zero energy flow. Let

$$\tilde{f}(x,p)=a(x)|p|^2(1+P)+b(x)p_x(1+P)+w(x,p)$$

be its orthogonal decomposition. By (28)

$$\lim_{x\to+\infty}\int p_x\tilde{f}^2(x,p)\frac{P}{1+P}dp=0. \tag{42}$$

Multiply the equation

$$p_x\partial_x\tilde{f}=L\tilde{f}, \tag{43}$$

by $\tilde{f}\frac{P}{1+P}$, integrate over $]0,+\infty[\times\mathbb{R}^3$ and use the spectral inequality. Then,

$$\begin{aligned}&\frac{1}{2}\int_{p_x<0}|p_x|\tilde{f}^2(0,p)\frac{P}{1+P}dp+\nu_0\int_0^{+\infty}\int w^2(x,p)\frac{P}{1+P}dpdx\\&\leq-\frac{1}{2}\lim_{x\to+\infty}\int p_x\tilde{f}^2(x,p)\frac{P}{1+P}dp\\&=0.\end{aligned}$$

And so,

$$\tilde{f}(0,\cdot)=0,\quad w(\cdot,\cdot)=0.$$

Equation (43) reduces to

$$\partial_x\tilde{f}=0,$$

so that together with $\tilde{f}(0,\cdot)=0$, it holds that $a(\cdot)=b(\cdot)=0$. Hence $\tilde{f}$ is identically zero. □

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