On a Boltzmann equation for Haldane statistics.

Leif ARKERYD [3] and Anne NOURI [4]

Abstract. The study of quantum quasi-particles at low temperatures including their statistics, is a frontier area in modern physics. In a seminal paper Haldane [10] proposed a definition based on a generalization of the Pauli exclusion principle for fractional quantum statistics. The present paper is a study of quantum quasi-particles obeying Haldane statistics in a fully non-linear kinetic Boltzmann equation model with large initial data on a torus. Strong L^1 solutions are obtained for the Cauchy problem. The main results concern existence, uniqueness and stabililty. Depending on the space dimension and the collision kernel, the results obtained are local or global in time.

1 Haldane statistics and the Boltzmann equation.

In a previous paper [2], we studied the Cauchy problem for a space-dependent anyon Boltzmann equation [5],

$$\partial_t f(t,x,v) + v_1 \partial_x f(t,x,v) = Q_\alpha(f)(t,x,v), \quad t \in \mathbb{R}_+,\ x \in [0,1],\ v=(v_1,v_2) \in \mathbb{R}^2,$$
$$f(0,x,v) = f_0(x,v).$$

The collision operator Q_α in [2] depends on a parameter $\alpha \in]0,1[$, and is given by

$$Q_\alpha(f)(v) = \int_{\mathbb{R}^2\times S^1} B(\mid v-v_* \mid, n)\big(f'f'_*F_\alpha(f)F_\alpha(f_*) - ff_*F_\alpha(f')F_\alpha(f'_*)\big)dv_*dn,$$

with the kernel B of Maxwellian type, f', f'_*, f, f_* the values of f at v', v'_*, v and v_* respectively, where

$$v' = v - (v-v_*, n)n, \quad v'_* = v_* + (v-v_*, n)n,$$

and the filling factor F_α

$$F_\alpha(f) = (1-\alpha f)^\alpha (1 + (1-\alpha) f)^{1-\alpha}.$$

Let us recall the definition of anyon. Consider the wave function $\psi(R,\theta,r,\varphi)$ for two identical particles with center of mass coordinates (R,θ) and relative coordinates (r,φ). Exchanging them, $\varphi \to \varphi + \pi$, gives a phase factor $e^{2\pi i}$ for bosons and $e^{\pi i}$ for fermions. In three or more dimensions those are all possibilities. Leinaas and Myrheim proved in 1977 [11], that in one and two dimensions any phase factor is possible in the particle exchange. This became an important topic after the first experimental confirmations in the early 1980-ies, and Frank Wilczek in analogy with the terms bos(e)-ons and fermi-ons coined the name any-ons for the new quasi-particles with any phase.

By moving from spin to a definition in terms of a generalized Pauli exclusion principle, Haldane [10]

---

[1] 2010 Mathematics Subject Classification. 82C10, 82C22, 82C40.

[2] Key words; anyon, Haldane statistics, low temperature kinetic theory, quantum Boltzmann equation.

[3] Mathematical Sciences, 41296 Göteborg, Sweden.

[4] Aix-Marseille University, CNRS, Centrale Marseille, I2M UMR 7373, 13453 Marseille, France.

extended this to a fractional exclusion statistics valid for any dimension. The conventional Bose-Einstein and Fermi-Dirac statistics are commonly associated with integer spin bosonic elementary particles resp. half integer spin fermionic elementary particles, whereas the Haldane fractional statistics is connected with quasi-particles corresponding to elementary excitations in many-body interacting quantum systems.

In this paper we consider the Cauchy problem associated to the Boltzmann equation in a torus $[0,1]^k$, $k\in\{1,2,3\}$, for quantum particles obeying the Haldane statistics;

$$\partial_t f(t,x,v)+\bar{v}\cdot\nabla_x f(t,x,v)=Q(f)(t,x,v),\ (t,x,v)\in\mathbb{R}_+\times[0,1]^k\times\mathbb{R}^3,\ v=(v_1,v_2,v_3)\in\mathbb{R}^3, \tag{1.1}$$

$$f(0,x,v)=f_0(x,v), \tag{1.2}$$

where

$$\bar{v}=(v_1)\ (\text{ resp. }\ \bar{v}=(v_1,v_2),\ \text{ resp. }\ \bar{v}=v)\ \text{for}\ k=1\ (\text{ resp. }\ k=2,\ \text{ resp. }\ k=3).$$

The collision operator Q is given by

$$Q(f)(v)=\int_{\mathbb{R}^3\times S^2}B(\mid v-v_*\mid,n)\big(f'f'_*F_\alpha(f)F_\alpha(f_*)-ff_*F_\alpha(f')F_\alpha(f'_*)\big)dv_*dn,\quad v\in\mathbb{R}^3.$$

Strong solutions to the space-homogeneous case were obtained in [1] for any dimension bigger than one in velocity. Strong solutions to the space-inhomogeneous case were obtained in [2] in a periodic slab for two-dimensional velocities. There the proof depends on the two-dimensional velocities setting. In the present paper we prove local in time well-posedness of the Cauchy problem for $k=1$ and collision kernels similar to those used in [2], and for $k\in\{1,2,3\}$ global in time well-posedness under the supplementary assumption of very soft potential at infinity [15]. The solutions conserve mass, momentum and energy.

2 The main results.

With $cos\theta=n\cdot\frac{v-v_*}{|v-v_*|}$, the kernel $B(|v-v_*|,n)$ will from now on be written $B(|v-v_*|,\theta)$ and be assumed measurable with

$$0\leq B\leq B_0, \tag{2.1}$$

for some $B_0>0$. It is also assumed for some $\gamma,\gamma'>0$, that

$$B(|v-v_*|,\theta)=0\ \text{if either}\ |\cos\theta|<\gamma',\quad\text{or}\ 1-|\cos\theta|<\gamma',\quad\text{or}\ |v-v_*|<\gamma, \tag{2.2}$$

together with the existence for any $\Gamma>0$ of a constant $c_\Gamma>0$ such that

$$\int\inf_{u\in[\gamma,\Gamma]}B(u,\theta)dn\geq c_\Gamma. \tag{2.3}$$

The initial datum $f_0(x,v)$,

$$\text{periodic in}\ x,\ \text{is a measurable function with values in}\ [0,\frac{1}{\alpha}], \tag{2.4}$$

and such that for some positive constants c_0 and $\tilde{c}_0$,

$$(1+|v|^2)f_0(x,v) \in L^1([0,1]^k \times \mathbb{R}^3), \tag{2.5}$$

$$\int \sup_{x\in[0,1]^k} f_0(x,v)dv = c_0, \tag{2.6}$$

$$\int \sup_{x\in[0,1]^k} |v|^2 f_0(x,v)dv = \tilde{c}_0, \tag{2.7}$$

$$\text{for any subset } X \text{ of } \mathbb{R}^3 \text{ of positive measure,} \quad \int_X \inf_{x\in[0,1]^k} f_0(x,v)dv > 0. \tag{2.8}$$

Denote by

$$f^\sharp(t,x,v) = f(t,x+t\bar{v},v) \quad (t,x,v) \in \mathbb{R}_+ \times [0,1]^k \times \mathbb{R}^3, \quad \bar{v} = (v_1,\cdots,v_k) \in \mathbb{R}^k. \tag{2.9}$$

Strong solutions to the Cauchy problem with initial value f_0 associated to the quantum Boltzmann equation (1.1) are considered in the following sense.

Definition 2.1 *f is a strong solution to (1.1) on the time interval I if*

$$f \in \mathcal{C}^1(I; L^1([0,1]^k \times \mathbb{R}^3)),$$

and

$$\frac{d}{dt} f^\sharp = \big(Q(f)\big)^\sharp, \quad \textit{on } I \times [0,1]^k \times \mathbb{R}^3. \tag{2.10}$$

The main results of the present paper are given in the following theorems.

Theorem 2.1
Under the assumptions (2.1)-(2.6) and (2.8), there is a time $T_0 > 0$, so that there exists a unique periodic in x, strong solution $f \in \mathcal{C}^1([0,T_0[; L^1([0,1] \times \mathbb{R}^3))$ of (1.1)-(1.2). It depends continuously in $\mathcal{C}([0,T_0[; L^1([0,1] \times \mathbb{R}^3))$ on the initial L^1-datum. It conserves mass, momentum and energy.

Theorem 2.2
Under the assumptions (2.1)-(2.8) and the supplementary assumption of very soft collision kernels at infinity,

$$B(u,\theta) = B_1(u)B_2(\theta) \quad \textit{with } |B_1(u)| \le c|u|^{-3-\eta} \textit{ for some } \eta > 0, \textit{and } B_2 \textit{ bounded}, \tag{2.11}$$

there exists a unique periodic in x, strong solution $f \in \mathcal{C}^1([0,\infty[; L^1([0,1]^k \times \mathbb{R}^3))$ of (1.1)-(1.2) for $k \in \{1,2,3\}$. For any $T > 0$ it continuously depends in $\mathcal{C}([0,T]; L^1([0,1]^k \times \mathbb{R}^3))$ on the initial L^1-datum. It conserves mass, momentum and energy.

Remarks.
Theorem 2.1 is restricted to the slab case, since its proof below uses an estimate for the Bony functional only valid in one space dimension.
Theorems 2.1 and 2.2 also hold with the same proofs in the fermion case where $\alpha = 1$, in particular giving strong solutions to the Fermi-Dirac equation.
Theorems 2.1 and 2.2 also hold with a limit procedure when $\alpha \to 0$ in the boson case where $\alpha = 0$, in particular giving strong solutions to the Boltzmann Nordheim equation [14]. It is the object of a separate paper [4] (see also [9], [13] and [7])

Theorems 2.1 and 2.2 also hold for $v \in \mathbb{R}^n$, $n \geq 3$.
The proofs in [2] strongly rely on the property that for any unit vector n with direct orthogonal unit vector $n_\perp$, either n_1 or $n_{\perp 1}$ is bigger that $\frac{1}{\sqrt{2}}$, where n_1 (resp. $n_{\perp 1}$) is the component of n (resp. $n_\perp$) along the x- axis. This allows to control the mass density of the solution from its Bony functional. This is no more the case in the three-dimensional velocity setting of the present paper. It is why our results are local in time under the same assumptions on the collision kernel B as in [2]. They are global in time under the supplementary assumption of a very soft potential at infinity.

The paper is organized as follows. Approximations are introduced in Section 3 for $k \in \{1,2,3\}$ together with for $k = 1$, a control of their Bony functional. Their mass density is uniformly controlled under the assumptions of Theorem 2.1 (resp. Theorem 2.2) in Section 4 (resp. Section 5). The well-posedness of the Cauchy problem is proven in Section 6. Conservation of mass, momentum and energy is proven in Section 7.

3 Preliminaries on solution approximations and the Bony functional.

Let $k \in \{1,2,3\}$. For any $j \in \mathbb{N}^*$, denote by ψ_j, the cut-off function with

$$\psi_j(r) = 0 \quad \text{if } r > j^2 \quad \text{and} \quad \psi_j(r) = 1 \quad \text{if } r \leq j^2,$$

and set

$$\chi_j(v, v_*) = \psi_j(|v^2| + |v_*|^2).$$

Let F_j be the C^1 function defined on $[0, \frac{1}{\alpha}]$ by

$$F_j(y) = \frac{1 - \alpha y}{(\frac{1}{j} + 1 - \alpha y)^{1-\alpha}}(1 + (1-\alpha)y)^{1-\alpha}.$$

Denote by Q_j (resp. Q_j^+, and Q_j^- to be used later), the operator

$$Q_j(f)(v) := \int B(|v - v_*|, \theta)\chi_j(v, v_*)\Big(f' f'_* F_j(f) F_j(f_*) - f f_* F_j(f') F_j(f'_*)\Big) dv_* dn,$$

$$\text{(resp. its gain part } Q_j^+(f)(v) := \int B(|v - v_*|, \theta)\chi_j(v, v_*) f' f'_* F_j(f) F_j(f_*) dv_* dn, \tag{3.1}$$

$$\text{and its loss part } Q_j^-(f)(v) := \int B(|v - v_*|, \theta)\chi_j(v, v_*) f f_* F_j(f') F_j(f'_*) dv_* dn\text{)}. \tag{3.2}$$

For $j \in \mathbb{N}^*$, let a mollifier φ_j be defined by $\varphi_j(x, v) = j^{3+k}\varphi(jx, jv)$, where

$$\varphi \in C_0^\infty(\mathbb{R}^{3+k}), \quad \textit{support}(\varphi) \subset [0,1]^k \times \{v \in \mathbb{R}^3; |v| \leq 1\},$$

$$\varphi \geq 0, \quad \int_{[0,1]^k \times \mathbb{R}^3} \varphi(x, v) dx dv = 1.$$

Let

$$f_{0,j} \text{ be the restriction to } [0,1]^k \times \{v; |v| \leq j\} \text{ of } \big(\min\{f_0, \frac{1}{\alpha} - \frac{1}{j}\}\big) * \varphi_j. \tag{3.3}$$

The following lemma concerns a corresponding approximation of (1.1)-(1.2) for $k \in \{1,2,3\}$.

Lemma 3.1
For any $T > 0$*, there is a unique solution* $f_j \in C\big([0,T]\times[0,1]^k; L^1(\{v; |v| \leq j\})\big)$ *to*

$$\partial_t f_j + \bar{v}\cdot\nabla_x f_j = Q_j(f_j), \quad f_j(0,\cdot,\cdot) = f_{0,j}. \tag{3.4}$$

There is $\eta_j > 0$ *such that* f_j *takes its values in* $[0, \frac{1}{\alpha} - \eta_j]$*.*
The solution conserves mass, momentum and energy.

Proof of Lemma 3.1.
Let $T > 0$ be given. We shall first prove by contraction that for $T_1 > 0$ and small enough, there is a unique solution

$$f_j \in C\big([0,T_1]\times[0,1]^k; L^1(\{v; |v| \leq j\})\big) \cap \{f; f \in [0, \frac{1}{\alpha}]\}$$

to (3.4). Let the map $\mathcal{C}$ be defined on periodic in x functions in

$$C\big([0,T]\times[0,1]^k; L^1(\{v; |v| \leq j\})\big) \cap \{f; f \in [0, \frac{1}{\alpha}]\}$$

by $\mathcal{C}(f) = g$, where

$$\begin{aligned}\partial_t g + \bar{v}\cdot\nabla_x g &= (1-\alpha g)\Big(\frac{1+(1-\alpha)f}{\frac{1}{j}+1-\alpha f}\Big)^{1-\alpha}\int B\chi_j f' f'_* F_j(f_*)dv_* dn\\ &\quad - g\int B\chi_j f_* F_j(f')F_j(f'_*)dv_* dn,\\ g(0,\cdot,\cdot) &= f_{0,j}.\end{aligned}$$

The previous linear partial differential equation has a unique periodic solution

$$g \in C([0,T]\times[0,1]^k; L^1(\{v; |v| \leq j\})).$$

For f with values in $[0, \frac{1}{\alpha}]$, g takes its values in $[0, \frac{1}{\alpha}]$. Indeed, denoting by

$$\bar{\sigma}_f := \alpha\Big(\frac{1+(1-\alpha)f}{\frac{1}{j}+1-\alpha f}\Big)^{1-\alpha}\int B\chi_j f' f'_* F_j(f_*)dv_* dn + \int B\chi_j f_* F_j(f')F_j(f'_*)dv_* dn,$$

and

$$g^\sharp(t,x,v) = g(t, x+t\bar{v}, v),$$

it holds that

$$\begin{aligned}g^\sharp(t,x,v) &= f_{0,j}(x,v)e^{-\int_0^t \bar{\sigma}^\sharp_f(r,x,v)dr}\\ &\quad + \int_0^t ds\Big(\big(\frac{1+(1-\alpha)f}{\frac{1}{j}+1-\alpha f}\big)^{1-\alpha}\int B\chi_j f' f'_* F_j(f_*)dv_* dn\Big)^\sharp(s,x,v)e^{-\int_s^t \bar{\sigma}^\sharp_f(r,x,v)dr}\\ &\geq f_{0,j}(x,v)e^{-\int_0^t \bar{\sigma}^\sharp_f(r,x,v)dr} \geq 0,\end{aligned}$$

and

$$\begin{aligned}(1-\alpha g)^\sharp(t,x,v) &= (1-\alpha f_{0,j})(x,v)e^{-\int_0^t \bar{\sigma}^\sharp_f(r,x,v)dr}\\ &\quad + \int_0^t \Big(\int B\chi_j f_* F_j(f')F_j(f'_*)dv_* dn\Big)^\sharp(s,x,v)e^{-\int_s^t \bar{\sigma}^\sharp_f(r,x,v)dr}ds\\ &\geq (1-\alpha f_{0,j})(x,v)e^{-\int_0^t \bar{\sigma}^\sharp_f(r,x,v)dr} \geq 0.\end{aligned}$$

$\mathcal{C}$ is a contraction on $C([0,T_1]\times[0,1]^k;L^1(\{v;|v|\leq j\}))\cap\{f;f\in[0,\frac{1}{\alpha}]\}$, for $T_1>0$ small enough only depending on j, since the derivative of the map F_j is bounded by $(3j\alpha^{\alpha-1}+1)j^{1-\alpha}$ on $[0,\frac{1}{\alpha}]$. Let f_j be its fixed point, i.e. the solution of (3.4) on $[0,T_1]$. The argument can be repeated and the solution continued up to $t=T$. By Duhamel's form for f_j (resp. $1-\alpha f_j$),

$$f_j^\sharp(t,x,v)\geq f_{0,j}(x,v)e^{-\int_0^t\bar{\sigma}^\sharp_{f_j}(r,x,v)dr}\geq 0,\quad (t,x)\in[0,T]\times[0,1]^k,\ |v|\leq j,$$

(resp.

$$\begin{aligned}(1-\alpha f_j)^\sharp(t,x,v)&\geq(1-\alpha f_{0,j})(x,v)e^{-\int_0^t\bar{\sigma}^\sharp_{f_j}(r,x,v)dr}\\&\geq\frac{1}{je^{cj^4T}},\quad (t,x)\in[0,T]\times[0,1]^k,\ |v|\leq j).\end{aligned}$$

Consequently, for some $\eta_j>0$, there is a periodic in x solution

$$f_j\in C([0,T]\times[0,1]^k;L^1(\{v;|v|\leq j\}))$$

to (3.4) with values in $[0,\frac{1}{\alpha}-\eta_j]$.
If there were another nonnegative local solution $\tilde{f}_j$ to (3.4), defined on $[0,T']$ for some $T'\in]0,T]$, then by the exponential form it would strictly stay below $\frac{1}{\alpha}$. The difference $f_j-\tilde{f}_j$ would for some constant $c_{T'}$ satisfy

$$\int|(f_j-\tilde{f}_j)^\sharp(t,x,v)|dxdv\leq c_{T'}\int_0^t|(f_j-\tilde{f}_j)^\sharp(s,x,v)|dsdxdv,\ t\in[0,T'],\quad (f_j-\tilde{f}_j)^\sharp(0,x,v)=0,$$

implying that the difference would be identically zero on $[0,T']$. Thus f_j is the unique solution on $[0,T]$ to (3.4), and has its range contained in $[0,\frac{1}{\alpha}-\eta_j]$. ■

Denote by M_j the mass density

$$M_j(t)=\int\sup_{(s,x)\in[0,t]\times[0,1]}f_j^\sharp(s,x,v)dv.\tag{3.5}$$

In Lemma 3.2 the tails for large velocities of the mass are controlled with respect to the mass density.

Lemma 3.2
Given $T>0$, the solution f_j of (3.4) satisfies

$$\int_0^1\int_{|v|>\lambda}|v|\sup_{t\in[0,T]}f_j^\sharp(t,x,v)dvdx\leq\frac{c_T}{\lambda}M_j(T),\quad j\in\mathbb{N},$$

where c_T only depends on T and $\int|v|^2f_0(x,v)dxdv$.

Proof of Lemma 3.2.
Denote f_j by f for simplicity. By the non-negativity of f,

$$\sup_{t\in[0,T]}f^\sharp(t,x,v)\leq f_0(x,v)+\int_0^T(Q_j^+(f))^\sharp(s,x,v)ds,\tag{3.6}$$

where $Q_j^+(f)$ is defined in (3.1). Integration with respect to (x,v) for $|v|>\lambda$, gives

$$\int_0^1 \int_{|v|>\lambda} |v| \sup_{t\in[0,T]} f^\sharp(t,x,v)dvdx \leq \int\int_{|v|>\lambda} |v| f_0(x,v)dvdx + \int_0^T \int_{|v|>\lambda} B\chi_j$$
$$|v|f(s,x+sv_1,v')f(s,x+sv_1,v'_*)F_j(f)(s,x+sv_1,v)F_j(f)(s,x+sv_1,v_*)dvdv_*dndxds.$$

Here in the last integral, either $|v'|$ or $|v'_*|$ is the largest and larger than $\frac{\lambda}{\sqrt{2}}$. The two cases are symmetric, and we discuss the case $|v'|\geq |v'_*|$. After a translation in x, the integrand of the r.h.s of the former inequality is estimated from above by

$$c|v'|f^\#(s,x,v') \sup_{(t,x)\in[0,T]\times[0,1]} f^\#(t,x,v'_*).$$

The change of variables $(v,v_*,n)\rightarrow(v',v'_*,-n)$ and the integration over

$$(s,x,v,v_*,n)\in[0,T]\times[0,1]\times\{v\in\mathbb{R}^3; |v|>\frac{\lambda}{\sqrt{2}}\}\times\mathbb{R}^3\times\mathcal{S}^2,$$

give the bound

$$\frac{c}{\lambda}\Big(\int_0^T\int |v|^2 f^\#(s,x,v)dxdvds\Big)\Big(\int \sup_{(t,x)\in[0,T]\times[0,1]} f^\#(t,x,v_*)dv_*\Big)$$
$$\leq \frac{cTM_j(T)}{\lambda}\int |v|^2 f_0(x,v)dxdv.$$

The lemma follows. ■

For $k=1$ there is a Bony type inequality available (cf [6] [8]) as follows.

Lemma 3.3
For any $n\in\mathcal{S}^2$, denote by n_1 the component of n along the x-axis. It holds that

$$\int_0^t\int n_1^2[(v-v_*)\cdot n]^2 B\chi_j f_j f_{j*} F_j(f'_j)F_j(f'_{j*})dvdv_*dndxds \leq c'_0(1+t), \quad t>0,\ j\in\mathbb{N}^*, \quad (3.7)$$

with c'_0 only depending on $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.

Proof of Lemma 3.3.
Denote f_j by f. The integral over time of the first component of momentum $\int v_1 f(t,0,v)dv$ (resp. $\int v_1^2 f(t,0,v)dv$) is first controlled. Let $\beta\in C^1([0,1])$ be such that $\beta(0)=-1$ and $\beta(1)=1$. Multiply (3.4) for $k=1$ by $\beta(x)$ (resp. $v_1\beta(x)$) and integrate over $[0,t]\times[0,1]\times\mathbb{R}^3$. It gives

$$\int_0^t\int v_1 f(\tau,0,v)dvd\tau = \frac{1}{2}\Big(\int\beta(x)f_{0,j}(x,v)dxdv - \int\beta(x)f(t,x,v)dxdv$$
$$+\int_0^t\int\beta'(x)v_1 f(\tau,x,v)dxdvd\tau\Big),$$

$\Big($ resp.

$$\int_0^t\int v_1^2 f(\tau,0,v)dvd\tau = \frac{1}{2}\Big(\int\beta(x)v_1 f_{0,j}(x,v)dxdv - \int\beta(x)v_1 f(t,x,v)dxdv$$
$$+\int_0^t\int\beta'(x)v_1^2 f(\tau,x,v)dxdvd\tau\Big)\Big).$$

Consequently, using the conservation of mass and energy of f,

$$|\int_0^t \int v_1 f(\tau,0,v)dvd\tau| + \int_0^t \int v_1^2 f(\tau,0,v)dvd\tau \leq c(1+t). \tag{3.8}$$

Let

$$\mathcal{I}(t) = \int_{x<y} (v_1 - v_{*1}) f(t,x,v) f(t,y,v_*) dxdydvdv_*.$$

It results from

$$\begin{aligned}\mathcal{I}'(t) = &-\int (v_1 - v_{*1})^2 f(t,x,v) f(t,x,v_*) dxdvdv_* \\ &+ 2\int v_{*1}(v_{*1} - v_1) f(t,0,v_*) f(t,x,v) dxdvdv_*,\end{aligned}$$

and the conservations of the mass, momentum and energy of f that

$$\begin{aligned}&\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f(s,x,v) f_*(s,x,v_*) dvdv_*dxds \\ &\leq 2\int f_0(x,v)dxdv \int |v_1| f_0(x,v)dv + 2\int f(t,x,v)dxdv \int |v_1| f(t,x,v)dxdv \\ &+ 2\int_0^t \int v_{*1}(v_{*1} - v_1) f(\tau,0,v_*) f(\tau,x,v) dxdvdv_*d\tau \\ &\leq 2\int f_0(x,v)dxdv \int (1+|v|^2) f_0(x,v)dv + 2\int f(t,x,v)dxdv \int (1+|v|^2) f(t,x,v)dxdv \\ &+ 2\int_0^t (\int v_{*1}^2 f(\tau,0,v_*)dv_*)d\tau \int f_0(x,v)dxdv \\ &- 2\int_0^t (\int v_{*1} f(\tau,0,v_*)dv_*)d\tau \int v_1 f_0(x,v)dxdv \\ &\leq c\Big(1 + \int_0^t \int v_1^2 f(\tau,0,v)dvd\tau + |\int_0^t \int v_1 f(\tau,0,v)dv|\Big).\end{aligned}$$

And so, by (3.8),

$$\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f(\tau,x,v) f(\tau,x,v_*) dxdvdv_*d\tau \leq c(1+t). \tag{3.9}$$

Here, c is a constant depending only on $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.
Denote by $u_1 = \frac{\int v_1 f dv}{\int f dv}$. It holds

$$\begin{aligned}&\int_0^t \int_0^1 \int (v_1 - u_1)^2 B\chi_j f f_* F_j(f') F_j(f'_*)(s,x,v,v_*,n) dvdv_*dndxds \\ &\qquad \leq c\int_0^t \int_0^1 \int (v_1 - u_1)^2 f f_*(s,x,v,v_*) dvdv_*dxds \\ &\qquad = \frac{c}{2}\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f f_*(s,x,v,v_*) dvdv_*dxds \\ &\qquad \leq c(1+t). \end{aligned} \tag{3.10}$$

Multiply equation (3.4) for f by v_1^2, integrate and use that $\int v_1^2 Q_j(f)dv = \int (v_1-u_1)^2 Q_j(f)dv$ and (3.10). It results

$$\begin{aligned}
&\int_0^t \int (v_1-u_1)^2 B\chi_j f'f'_* F_j(f)F_j(f_*)dvdv_*dndxds = \int v_1^2 f(t,x,v)dxdv \\
&-\int v_1^2 f_{0,j}(x,v)dxdv + \int_0^t \int (v_1-u_1)^2 B\chi_j ff_* F_j(f')F_j(f'_*)dxdvdv_*dnds \\
&< c'(1+t),
\end{aligned}$$

where c' is a constant only depending on $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.

After a change of variables the left hand side can be written

$$\begin{aligned}
&\int_0^t \int (v'_1-u_1)^2 B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds \\
&= \int_0^t \int (c_1 - n_1[(v-v_*)\cdot n])^2 B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds,
\end{aligned}$$

where $c_1 = v_1 - u_1$. Expand $(c_1 - n_1[(v-v_*)\cdot n])^2$, remove the positive term containing c_1^2.

The term containing $n_1^2[(v-v_*)\cdot n]^2$ is estimated as follows;

$$\begin{aligned}
&\int_0^t \int n_1^2[(v-v_*)\cdot n]^2 B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds \\
&\leq c'(1+t) + 2\int_0^t \int (v_1-u_1)n_1[(v-v_*)\cdot n]B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds \\
&\leq c'(1+t) + 2\int_0^t \int \Big(v_1 \sum_{l=2}^{3}(v_l - v_{*l})n_1 n_l\Big) B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds,
\end{aligned}$$

since

$$\int u_1(v_l - v_{*l})n_1 n_l \chi_j B ff_* F_j(f')F_j(f'_*)dvdv_*dndx = 0, \qquad l = 2,3,$$

by an exchange of the variables v and v_*. Moreover, exchanging first the variables v and v_*,

$$\begin{aligned}
&2\int_0^t \int v_1 \sum_{l=2}^{3}(v_l - v_{*l})n_1 n_l B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds \\
&\quad= \int_0^t \int (v_1 - v_{*1}) \sum_{l=2}^{3}(v_l - v_{*l})n_1 n_l B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds \\
&\quad\leq \frac{1}{\beta^2}\int_0^t \int (v_1 - v_{*1})^2 B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds \\
&\qquad+ \frac{\beta^2}{4}\int_0^t \int \sum_{l=2}^{3}(v_l - v_{*l})^2 n_1^2 n_l^2 B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds \\
&\quad\leq \frac{2c'}{\beta^2}(1+t) + \frac{\beta^2}{4}\int_0^t \int n_1^2 \sum_{l=2}^{3}(v_l - v_{*l})^2 n_l^2 B\chi_j ff_* F_j(f')F_j(f'_*)dvdv_*dndxds,
\end{aligned}$$

for any $\beta > 0$. It follows that

$$\int_0^t \int n_1^2[(v-v_*)\cdot n]^2 B\chi_j f f_* F_j(f')F_j(f'_*)dvdv_*dndxds \leq c'_0(1+t),$$

with c'_0 only depending on $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$. This completes the proof of the lemma. ∎

4 Control of the mass density under the assumptions of Theorem 2.1.

Let $k=1$. Lemmas 4.1 to 4.3 are devoted to the local in time uniform control with respect to j of the mass density defined in (3.5).

Lemma 4.1
For any $\epsilon > 0$, there exists a constant c'_1 only depending on $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$, such that

$$\int \sup_{s\in[0,t]} f_j^\sharp(s,x,v)dxdv \leq c'_1\Big((1+\frac{1}{\epsilon^2})(1+t)+\epsilon t M_j(t)\Big), \quad t>0, \quad j\in\mathbb{N}^*. \tag{4.1}$$

Proof of Lemma 4.1.
Denote f_j by f for simplicity. By (3.6),

$$\sup_{s\in[0,t]} f^\sharp(s,x,v) \leq f_0(x,v)$$
$$+\int_0^t \int B\chi_j f(r,x+rv_1,v')f(r,x+rv_1,v'_*)F_j(f)^\sharp(r,x,v)F_j(f)(r,x+rv_1,v_*)dndv_*dr. \tag{4.2}$$

For any $(v,v_*)\in\mathbb{R}^3\times\mathbb{R}^3$, let $\mathcal{N}_\epsilon$ be the set of $n\in\mathcal{S}^2$ with $\max\{n_1,n_{\perp 1}\}<\epsilon$, where $n_\perp$ is the unit vector in the direction $v-v'_*$ (orthogonal to n) in the plane defined by $v-v_*$ and n, and n_1 is the component of n along the x-axis.
Let $\mathcal{N}^c_\epsilon$ be the complement of $\mathcal{N}_\epsilon$ in $\mathcal{S}^2$. Denote by

$$\mathcal{I}_\epsilon(t) = \int_0^t \int\int_{\mathcal{N}_\epsilon} B\chi_j f(r,x+rv_1,v')f(r,x+rv_1,v'_*)$$
$$F_j(f)^\sharp(r,x,v)F_j(f)(r,x+rv_1,v_*)dndvdv_*dxdr.$$

(3.7) also holds with n_1 replaced by $n_{\perp 1}$. Integrating (4.2) with respect to (x,v) and using (2.2) and Lemma 3.3 leads to

$$\begin{aligned}
&\int \sup_{s\in[0,t]} f^\sharp(s,x,v)dxdv \leq \int f_0(x,v)dxdv + \mathcal{I}_\epsilon(t)\\
&+\int_0^t\int\int_{\mathcal{N}_\epsilon^c} B\chi_j f(r,x+rv_1,v')f(r,x+rv_1,v'_*)\\
&\qquad\qquad F_j(f)^\sharp(r,x,v)F_j(f)(r,x+rv_1,v_*)dvdv_*dndxdr\\
&=\int f_0(x,v)dxdv + \mathcal{I}_\epsilon(t) + \int_0^t\int\int_{\mathcal{N}_\epsilon^c} B\chi_j ff_*F_j(f')F_j(f'_*)dvdv_*dndxdr\\
&\leq \int f_0(x,v)dxdv + \mathcal{I}_\epsilon(t) + \frac{1}{(\gamma\gamma'\epsilon)^2}\int_0^t\int (n_1^2+n_{\perp 1}^2)[(v-v_*)\cdot n]^2 B\chi_j ff_*\\
&\qquad\qquad F_j(f')F_j(f'_*)dvdv_*dndxdr\\
&\leq \int f_0(x,v)dxdv + \mathcal{I}_\epsilon(t) + \frac{2c'_0}{(\gamma\gamma'\epsilon)^2}(1+t).
\end{aligned}\tag{4.3}$$

Moreover,

$$\mathcal{I}_\epsilon(t) \leq 2\pi B_0\epsilon\, t \parallel F_\alpha \parallel_\infty^2 M_j(t)\int f_0(x,v)dxdv.$$

And so, (4.1) holds with

$$c'_1 = \max\{\int f_0(x,v)dxdv,\, \frac{2c'_0}{(\gamma\gamma')^2},\, 2\pi B_0 \parallel F_\alpha \parallel_\infty^2 \int f_0(x,v)dxdv\}.$$

Lemma 4.2
There is c'_2 only depending on $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$ such that, for any $\delta\in]0,1[$,

$$\sup_{x_0\in[0,1]}\int_{|x-x_0|<\delta}\sup_{s\in[0,t]} f_j^\sharp(s,x,v)dxdv \leq c'_2\Big(\delta^{\frac{2}{5}} + t^{\frac{8}{11}}(1+t)^{\frac{3}{11}}\big(1+M_j(t)\big)\Big),\quad t>0,\quad j\in\mathbb{N}^*. \tag{4.4}$$

<u>Proof of Lemma 4.2.</u>
Denote f_j by f for simplicity. For $s\in[0,t]$ it holds,

$$f^\sharp(s,x,v) = f^\sharp(t,x,v) - \int_s^t Q_j(f)^\sharp(r,x,v)dr \leq f^\sharp(t,x,v) + \int_s^t (Q_j^-(f))^\sharp(r,x,v)dr,$$

where Q_j^- is defined in (3.2). And so

$$\begin{aligned}
&\sup_{s\in[0,t]} f^\sharp(s,x,v) \leq f^\sharp(t,x,v)\\
&+\int_0^t\int B\chi_j f^\sharp(r,x,v)f(r,x+rv_1,v_*)F_j(f)(r,x+rv_1,v')F_j(f)(r,x+rv_1,v'_*)dv_*dndr.
\end{aligned}\tag{4.5}$$

Denote by

$$\begin{aligned}
\mathcal{J}_\epsilon(t) = \sup_{x_0\in[0,1]}\int_0^t\int_{|x-x_0|<\delta}\int\int_{\mathcal{N}_\epsilon} &B\chi_j f^\sharp(r,x,v)f(r,x+rv_1,v_*)\\
&F_j(f)(r,x+rv_1,v')F_j(f)(r,x+rv_1,v'_*)dvdv_*dndxdr.
\end{aligned}$$

Integrating (4.5) with respect to (x,v), using Lemma 3.3, the $\frac{1}{\alpha}$ (resp. $\alpha^{\alpha-1}$) bound from above of f (resp. $F_j(y)$,$y\in[0,\frac{1}{\alpha}]$), gives for any $x_0\in[0,1]$, $\lambda>0$ and $\Lambda>0$ that

$$\begin{aligned}
&\int_{|x-x_0|<\delta}\sup_{s\in[0,t]}f^\sharp(s,x,v)dxdv\leq\int_{|x-x_0|<\delta}f^\sharp(t,x,v)dxdv+\mathcal{J}_\epsilon(t)\\
&+\frac{1}{(\lambda\gamma'\epsilon)^2}\int_0^t\int_{|v-v_*|\geq\lambda}(n_1^2+n_{\perp1}^2)[(v-v_*)\cdot n]^2B\chi_jff_*F_j(f')F_j(f'_*)dvdv_*dndxds\\
&+c\int_0^t\int_{|v-v_*|<\lambda}B\chi_jf^\sharp(s,x,v)f(s,x+sv_1,v_*)dvdv_*dndxds\\
\leq&\int_{|x-x_0|<\delta}f^\sharp(t,x,v)dxdv+\mathcal{J}_\epsilon(t)+\frac{c_0'(1+t)}{(\lambda\gamma'\epsilon)^2}+ct\lambda^3\int f_0(x,v)dxdv\\
\leq&\frac{1}{\Lambda^2}\int v^2f_0dxdv+c\delta\Lambda^3+\mathcal{J}_\epsilon(t)+\frac{c_0'(1+t)}{(\lambda\gamma'\epsilon)^2}+ct\lambda^3\int f_0(x,v)dxdv\\
\leq&c\big(\delta^{\frac{2}{5}}+t^{\frac{2}{5}}\epsilon^{-\frac{6}{5}}(1+t)^{\frac{3}{5}}\big)+\mathcal{J}_\epsilon(t),
\end{aligned}\tag{4.6}$$

for an appropriate choice of (Λ,λ). Moreover,

$$\mathcal{J}_\epsilon(t)\leq 2\pi B_0\epsilon t\parallel F_\alpha\parallel_\infty^2 M_j(t)\int f_0(x,v)dxdv.$$

Taking $\epsilon=\tilde{c}\big(\frac{1+t}{t}\big)^{\frac{3}{11}}M^{-\frac{5}{11}}$ with $\tilde{c}$ suitably chosen, leads to

$$\int_{|x-x_0|<\delta}\sup_{s\in[0,t]}f^\sharp(s,x,v)dxdv\leq c\big(\delta^{\frac{2}{5}}+t^{\frac{8}{11}}(1+t)^{\frac{3}{11}}M_j(t)^{\frac{6}{11}}\big).$$

The lemma follows. ■

Lemma 4.3
There is $T>0$ such that the solutions f_j of (3.4) satisfy

$$\int\sup_{(t,x)\in[0,T]\times[0,1]}f_j^\sharp(t,x,v)dv\leq 2c_0,\quad j\in\mathbb{N}^*,$$

with c_0 defined in (2.6).

Proof of Lemma 4.3.
Denote by $E(x)$ the integer part of $x\in\mathbb{R}$, $E(x)\leq x<E(x)+1$. As in (3.6),

$$\begin{aligned}
&\sup_{s\in[0,t]}f^\sharp(s,x,v)\leq f_0(x,v)\\
&+\int_0^t\int B\chi_jf(s,x+sv_1,v')f(s,x+sv_1,v'_*)(F_j(f))^\sharp(s,x,v)F_j(f)(s,x+sv_1,v_*)dv_*dnds\\
&\leq f_0(x,v)+\parallel F_\alpha\parallel_\infty^2(A_1+A_2+A_3+A_4),
\end{aligned}\tag{4.7}$$

where, for $\epsilon > 0$, $\delta > 0$ and λ that will be fixed later,

$$A_1 = \int_0^t \int_{|n_1| \geq \epsilon,\, t|v_1 - v_1'| > \delta} B\chi_j \sup_{\tau \in [0,t]} f^{\#}(\tau, x + s(v_1 - v_1'), v') \sup_{\tau \in [0,t]} f^{\#}(\tau, x + s(v_1 - v_{*1}'), v_*') dv_* dn ds,$$

$$A_2 = \int_0^t \int_{|n_1| \geq \epsilon,\, t|v_1 - v_1'| < \delta,\, |v'| < \lambda} B\chi_j \sup_{\tau \in [0,t]} f^{\#}(\tau, x + s(v_1 - v_1'), v') \times \\ \times \sup_{\tau \in [0,t]} f^{\#}(\tau, x + s(v_1 - v_{*1}'), v_*') dv_* dn ds,$$

$$A_3 = \int_0^t \int_{|n_1| \geq \epsilon,\, t|v_1 - v_1'| < \delta,\, |v'| > \lambda} B\chi_j \sup_{\tau \in [0,t]} f^{\#}(\tau, x + s(v_1 - v_1'), v') \times \\ \times \sup_{\tau \in [0,t]} f^{\#}(\tau, x + s(v_1 - v_{*1}'), v_*') dv_* dn ds,$$

$$A_4 = \int_0^t \int_{|n_1| < \epsilon} B\chi_j \sup_{\tau \in [0,t]} f^{\#}(\tau, x + s(v_1 - v_1'), v') \sup_{\tau \in [0,t]} f^{\#}(\tau, x + s(v_1 - v_{*1}'), v_*') dv_* dn ds.$$

In A_1, A_2 and A_3, bound the factor $\sup_{\tau \in [0,t]} f^{\sharp}(\tau, x + s(v_1 - v_{*1}'), v_*')$ by its supremum over $x \in [0,1]$, and make the change of variables

$$s \to y = x + s(v_1 - v_1'),$$

with Jacobian

$$\frac{Ds}{Dy} = \frac{1}{|v_1 - v_1'|} = \frac{1}{|v - v_*|\, |(n, \frac{v - v_*}{|v - v_*|})|\, |n_1|} \leq \frac{1}{\epsilon \gamma \gamma'}\,.$$

Consequently,

$$\begin{aligned} &\sup_{x \in [0,1]} A_1(t,x,v) \\ &\leq \sup_{x \in [0,1]} \int_{t|v_1 - v_1'| > \delta} \frac{B\chi_j}{|v_1 - v_1'|} \Big(\int_{y \in (x, x + t(v_1 - v_1'))} \sup_{\tau \in [0,t]} f^{\#}(\tau, y, v') dy \Big) \\ &\qquad \sup_{(\tau, X) \in [0,t] \times [0,1]} f^{\#}(\tau, X, v_*') dv_* dn \\ &\leq \int_{t|v_1 - v_1'| > \delta} \frac{B\chi_j}{|v_1 - v_1'|} |E(t(v_1 - v_1') + 1)| \Big(\int_0^1 \sup_{\tau \in [0,t]} f^{\#}(\tau, y, v') dy \Big) \\ &\qquad \sup_{(\tau, X) \in [0,t] \times [0,1]} f^{\#}(\tau, X, v_*') dv_* dn. \end{aligned}$$

Performing the change of variables $(v, v_*, n) \to (v', v'_*, -n)$,

$$\begin{aligned}
&\int \sup_{x\in[0,1]} A_1(t,x,v)dv \\
&\le \int_{t|v_1-v_1'|>\delta} \frac{B\chi_j}{|v_1-v_1'|}|E(t(v_1'-v_1)+1)|\Big(\int_0^1 \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy\Big) \\
&\qquad\qquad\qquad\qquad \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*dn \\
&\le t(1+\frac{1}{\delta})\int B\chi_j\Big(\int_0^1 \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy\Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*dn \\
&\le 4\pi B_0\, t(1+\frac{1}{\delta})\Big(\int \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dydv\Big)M_j(t).
\end{aligned}$$

Apply Lemma 4.1, so that

$$\int \sup_{x\in[0,1]} A_1(t,x,v)dv \le 4\pi B_0 c_1' t(1+\frac{1}{\delta})\Big((1+\frac{1}{\epsilon^2})(1+t)+\epsilon t M_j(t)\Big)M_j(t). \tag{4.8}$$

Moreover,

$$\begin{aligned}
c\epsilon\int \sup_{x\in[0,1]} A_2(t,x,v)dv &\le \frac{\delta}{\alpha}\int\int_{|v'|<\lambda} B\chi_j \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*')dvdv_*dn \\
&= \frac{\delta}{\alpha}\int_{|v|<\lambda}\int B\chi_j \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*dn \\
&\qquad\qquad \text{by the change of variables } (v,v_*,n)\to(v',v_*',-n) \\
&\le \frac{c\delta\lambda^3}{\alpha}M_j(t),
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&c\epsilon\int \sup_{x\in[0,1]} A_3(t,x,v)dv \\
&\le \int_{|v'|>\lambda} B\chi_j\Big(\int_0^1 \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v')dy\Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*')dvdv_*dn \\
&\le c\Big(\int_0^1\int_{|v|>\lambda} \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dvdy\Big)\int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dv_* \\
&\qquad\qquad \text{by the change of variables } (v,v_*,n)\to(v',v_*',-n) \\
&\le \frac{c}{\lambda^2}M_j^2(t) \quad \text{by Lemma 3.2.}
\end{aligned} \tag{4.10}$$

Finally, with the change of variables $(v, v_*, n) \to (v', v'_*, -n)$,

$$\begin{aligned}
\int \sup_{x\in[0,1]} A_4(t,x,v)dv &\le B_0 t\Big(\int_{|n_1|<\epsilon} dn\Big)\Big(\int \sup_{(\tau,x)\in[0,t]\times[0,1]} f^{\#}(\tau,x,v)dv\Big)^2 \\
&\le 2\pi B_0\epsilon\, t M_j^2(t).
\end{aligned} \tag{4.11}$$

It follows from (4.7), (4.8), (4.9), (4.10) and (4.11) that

$$a(t)M_j^2(t) - b(t)M_j(t) + c_0 \ge 0, \quad t \le 1, \tag{4.12}$$

where for some positive constants $(c'_l)_{2\leq l\leq 4}$ independent on ϵ, δ and λ,

$$a(t) = c'_2\big(\epsilon t(1+\delta^{-1}) + \epsilon^{-1}\lambda^{-2}\big), \quad b(t) = 1 - c'_3 t(1+\delta^{-1})(1+\epsilon^{-2}) - c'_4\epsilon^{-1}\delta\lambda^3.$$

Choose $\lambda = \epsilon^{-1}$, $\delta = \epsilon^5$ and $\epsilon = \frac{1}{16}\min\{\frac{1}{c'_4}, \frac{1}{c_0}\}$. For T small enough, it holds that

$$b(t) \in \Big]\frac{3}{4}, 1\Big[\quad \text{and } c_0 a(t) < \frac{1}{8}, \quad t \in [0,T], \tag{4.13}$$

which is sufficient for the polynomial in (4.12) to have two nonnegative roots and take a negative value at $2c_0$. Recalling that $M_j(0) = c_0$ and M_j is continuous by the continuity in time and space of f_j, it follows that

$$M_j(t) \leq 2c_0, \quad t \in [0,T].$$

■

5 Control of the mass density under the assumptions of Theorem 2.2.

Let $k \in \{1,2,3\}$. Under the supplementary assumption (2.11), we prove a uniform control with respect to j of the mass density $M_j(t)$ defined in (3.5). It relies on the two following lemmas.

Lemma 5.1
Given $\epsilon > 0$, there exists a constant c'_5 only depending on $\int f_0(x,v)dxdv$, such that

$$\int \sup_{s\in[0,t]} f_j^{\sharp}(s,x,v)dxdv \leq c'_5(1+t), \quad t > 0, \quad j \in \mathbb{N}^*. \tag{5.1}$$

<u>Proof of Lemma 5.1</u>
Denote f_j by f for simplicity. By the non-negativity of f, it holds

$$\begin{aligned} f^{\#}(s,x,v) \leq & f_0(x,v) + \int_0^s \int f^{\sharp}(\tau, x+\tau(\bar{v}-\overline{v'}), v')f^{\sharp}(\tau, x+\tau(\bar{v}-\overline{v'_*}), v'_*)\times \\ & \times F_j(f^{\sharp}(\tau,x,v))F_j(f(\tau, x+\tau(\bar{v}-\overline{v_*}), v_*))B_1(v-v_*)B_2(\theta)dv_*dnd\tau. \end{aligned}$$

Using the $\frac{1}{\alpha}$ bound for $f^{\sharp}(\tau, x+\tau(\bar{v}-\overline{v'_*}), v'_*)$, and (2.11) leads to

$$\sup_{s\in[0,t]} f^{\#}(s,x,v) \leq f_0(x,v) + c\int_0^t \int f^{\#}(s, x+s(\bar{v}-\overline{v'}), v')B_1(v-v_*)B_2(\theta)dv_*dnds. \tag{5.2}$$

Hence,

$$\begin{aligned}
&\int \sup_{s\in[0,t]} f^{\#}(t,x,v)dxdv\\
&\leq \int f_0(x,v)dxdv + c\int_0^t \int f^{\#}(s,x+s(\bar{v}-\overline{v'}),v')B_1(v-v_*)B_2(\theta)dxdv_*dvdnds\\
&= \int f_0(x,v)dxdv + c\int_0^t \int f(s,x,v)B_1(v-v_*)B_2(\theta)dxdv_*dvdnds\\
&\leq \int f_0(x,v)dxdv + c\int_0^t \int_\gamma^\infty \int f(s,x,v)r^{-(1+\eta)}dxdvdrds \quad \text{by (2.11)}\\
&= \int f_0(x,v)dxdv + \frac{c}{\eta\gamma^\eta}\int_0^t \int f(s,x,v)dxdvds\\
&=: c_5'(1+t),
\end{aligned}$$

by the mass conservation. ■

Lemma 5.2
Given $T>0$*, the solutions* f_j *of (3.4) satisfy*

$$M_j(T)\leq c_1(T), \quad j\in\mathbb{N}^*,$$

where $c_1(T)$ *only depends on* T *and* c_0*.*

Proof of Lemma 5.2.
By (5.2), for any $(t,x)\in[0,T]\times\mathbb{R}^3$,

$$\sup_{(s,x)\in[0,t]\times[0,1]^k} f^{\#}(s,x,v) \leq \sup_{x\in[0,1]^k} f_0(x,v) + c\int_0^t \int \sup_{x\in[0,1]^k} f(s,x,v')B_1(v-v_*)B_2(\theta)dv_*dnds.$$

Consequently,

$$\begin{aligned}
\int \sup_{(s,x)\in[0,t]\times[0,1]^k} f(s,x,v)dv &\leq c_0 + c\int_0^t \int \sup_{x\in[0,1]^k} f(s,x,v')B_1(v-v_*)B_2(\theta)dv_*dvdnds\\
&= c_0 + c\int_0^t \int \sup_{x\in[0,1]^k} f(s,x,v)B_1(v-v_*)B_2(\theta)dv_*dvdnds\\
&\leq c_0 + \frac{c}{\eta\gamma^\eta}\int_0^t \int \sup_{x\in[0,1]^k} f(s,x,v)dvds.
\end{aligned}$$

It follows that

$$\int \sup_{(t,x)\in[0,T]\times[0,1]^k} f(t,x,v)dv \leq c_0 e^{c''T}, \quad \text{with } c''=\frac{c}{\eta\gamma^\eta}.$$

■

6 Well-posedness of the Cauchy problem.

Let T_0 be supremum of the times up to which it has been proved that the mass densities of the approximations are uniformly bounded. Recall that T_0 may be finite (resp. is infinite) under the assumptions of Theorem 2.1 (resp. 2.2). We prove in this section that for any $T \in [0, T_0[$ there is a unique solution to the Cauchy problem (1.1)-(1.2). This section is divided into three steps. In the first step, we study initial layers for the approximations. In the second step, the existence of a solution f to (1.1) on $[0, T]$ for $T \in]0, T_0[$ is shown. Finally the third step proves the uniqueness and stability result stated in Theorems 2.1 and 2.2.

First step: initial layers.

Lemma 6.1

For any $T \in [0, T_0[$, there are $j_T \in \mathbb{N}^*$, a positive time $t_m > 0$, and for $V > 0$ positive constants b_V and μ_V such that

$$f_j^\sharp(t, \cdot, v) \leq \frac{1}{\alpha} - b_V t, \quad t \in [0, t_m], \quad |v| < V, \quad j \geq j_T,$$

$$f_j^\sharp(t, \cdot, v) \leq \frac{1}{\alpha} - \mu_V, \quad t \in [t_m, T], \quad |v| < V, \quad j \geq j_T.$$

Proof of Lemma 6.1.

Denote f_j by f for simplicity. It follows from Lemmas 4.3 and 5.2 that there is $c_1(T) > 0$ such that

$$M_j(T) \leq c_1(T), \quad j \in \mathbb{N}^*. \tag{6.1}$$

Denote by

$$\tilde{\nu}_j(f) := \int B\chi_j f' f'_* F_j(f_*) dv_* dn, \quad \nu_j(f) := \int B\chi_j f_* F_j(f') F_j(f'_*) dv_* dn,$$

so that

$$Q_j(f) = F_j(f)\tilde{\nu}_j(f) - f\nu_j(f).$$

It follows from (6.1) that $\nu_j(f)^\sharp$ and $\tilde{\nu}_j(f)^\sharp$ are bounded from above uniformly with respect to j. Denote by $c_2(T)$ a bound from above of $(\tilde{\nu}_j(f)^\sharp)_{j\in\mathbb{N}}$.

Let us prove that $(\nu_j(f)^\sharp)$ is bounded from below for large j on $[0, T] \times [0, 1]^k \times \{v; |v| < V\}$ for any $V > 0$. By definition,

$$\nu_j(f)^\sharp(t, x, v) = \int B\chi_j f(t, x + t\bar{v}, v_*) F_j(f(t, x + t\bar{v}, v')) F_j(f(t, x + t\bar{v}, v'_*)) dv_* dn.$$

Using Duhamel's form for the solution, (6.1) and (2.8), one gets that

$$f(t, x + t\bar{v}, v_*) \geq c_3(T) f_0(x, v_*) > 0, \quad \text{a.a. } (t, x, v, v_*) \in [0, T] \times [0, 1]^k \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{6.2}$$

for some constant $c_3(T) > 0$. For any angles $(\theta, \varphi) \in [0, 2\pi] \times [0, \pi]$ defining the relative position of $v' - v$ with respect to $v_* - v$, the maps $v_* \to v'$ and $v_* \to v'_*$ are changes of variables. Indeed, consider the map $v_* \to v'$, reduce it to $v_* - v \to v' - v$ and denote it by U . Let n be the vector with polar coordinates (θ, φ) with respect to $v_* - v$. Choose a coordinates system with the first (resp. second, resp. third) axis in the direction of $v_* - v$ (resp. orthogonal to $v_* - v$ in the plane defined by v_*-v and n, resp. orthogonal to the two first axes). The map U maps the volume $d(v_{*x}-v_x)d(v_{*y}-v_y)d(v_{*z}-v_z)$ into

$$\begin{aligned}d(v'_x-v_x)d(v'_y-v_y)d(v'_z-v_z)&=(\cos\theta)^4 d(v_{*x}-v_x)d(v_{*y}-v_y)d(v_{*z}-v_z)\\&\quad+O\Big(\big(d(v_{*x}-v_x)\big)^2+\big(d(v_{*y}-v_y)\big)^2+\big(d(v_{*z}-v_z)\big)^2\Big),\end{aligned}$$

since up to second order terms with respect to $d(v_{*x}-v_x)$, $d(v_{*y}-v_y)$ and $d(v_{*z}-v_z)$, the length $d(v_{*x}-v_x)$ (resp. $d(v_{*y}-v_y)$, resp. $d(v_{*z}-v_z)$) is changed into $|\cos\theta|d(v_{*x}-v_x)$ (resp. $|\cos\theta|d(v_{*y}-v_y)$, resp. $\cos^2\theta d(v_{*z}-v_z)$). And so the Jacobian of U equals $\cos^4\theta$. Using these changes of variables and (6.1), it holds that

$$\int f(t,x+t\bar{v},v')dv_*<\frac{c_1(T)}{(\gamma')^4}\text{ and }\int f(t,x+t\bar{v},v'_*)dv_*<\frac{c_1(T)}{(\gamma')^4},$$
$$\text{a.a. }(t,x,v,\theta,\varphi)\in[0,T]\times[0,1]^k\times\mathbb{R}^3\times[0,2\pi]\times[0,\pi],\quad|\cos\theta|>\gamma'.$$

Consequently, the measure of the set

$$Z_{(j,t,x,v,\theta,\varphi)}:=\{v_*;f(t,x+t\bar{v},v')>\frac{1}{2}\quad\text{or}\quad f(t,x+t\bar{v},v'_*)>\frac{1}{2}\}\tag{6.3}$$

is bounded by $\frac{2c_1(T)}{(\gamma')^4}$, uniformly with respect to (x,v,θ,φ) with $|\cos\theta|>\gamma'$, $t\in[0,T]$, and $j\in\mathbb{N}^*$. Take j_T so large that $\frac{4}{3}\pi j_T^3$ is at least twice this uniform bound. Notice that here j_T only depends on T, $\int f_0(x,v)dxdv$ and $\int|v|^2f_0(x,v)dxdv$. Denote by $\mathcal{B}\big(0,\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}\big)$ the ball of radius $\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}$. It follows from (6.2) and the definition of j_T that

$$\begin{aligned}&\nu_j(f)^\sharp(t,x,v)\\&\geq\int_{\mathcal{S}^2}\int_{\mathcal{B}\big(0,\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}\big)\cap Z^c_{(j,t,x,v,\theta,\varphi)}}B\chi_j f(t,x+t\bar{v},v_*)F_j(f(t,x+t\bar{v},v'))\\&\qquad\qquad F_j(f(t,x+t\bar{v},v'_*))dv_*dn\\&\geq c_3(T)(1-\frac{\alpha}{2})^{2\alpha}\int_{\mathcal{S}^2}\int_{\mathcal{B}\big(0,\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}\big)\cap Z^c_{(j,t,x,v,\theta,\varphi)}}B(|v-v_*|,\theta)\inf_{x\in[0,1]^k}f_0(x,v_*)dv_*dn,\\&\qquad\qquad j\geq j_T,\quad\text{a.a. }(t,x,v)\in[0,T]\times[0,1]^k\times\{v\in\mathbb{R}^3;|v|<V\}.\end{aligned}$$

Using a median property for the restriction of $v\to\inf_{x\in[0,1]^k}f_0(x,v)$ to the ball $\mathcal{B}\big(0,\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}\big)$, which is a bounded measurable Lebesgue function, there are two disjoint sets Ω_1 and Ω_2 of equal volume, such that

$$\inf_{x\in[0,1]^k}f_0(x,v_1)\leq\inf_{x\in[0,1]^k}f_0(x,v_2)\text{ for a.a. }v_1\in\Omega_1,\quad v_2\in\Omega_2.$$

Denote by $\Gamma=V+\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}$.
For $j\geq j_T$ and a.a. $(n,t,x,v)\in\mathcal{S}^2\times[0,T]\times[0,1]^k\times\{v\in\mathbb{R}^3;|v|<V\}$,

$$\begin{aligned}&\int_{\mathcal{B}\big(0,\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}\big)\cap Z^c_{(j,t,x,v,\theta,\varphi)}}B(|v-v_*|,\theta)\inf_{x\in[0,1]^k}f_0(x,v_*)dv_*\\&\geq\inf_{u\in[\gamma,\Gamma]}B(u,\theta)\inf_{\overline{\Omega}\subset\mathcal{B}\big(0,\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}\big);\,|\overline{\Omega}|=\frac{2c_1(T)}{(\gamma')^4}}\int_{\overline{\Omega}}\inf_{x\in[0,1]^k}f_0(x,v_*)dv_*\\&=\inf_{u\in[\gamma,\Gamma]}B(u,\theta)\int_{\Omega_1}\inf_{x\in[0,1]^k}f_0(x,v_*)dv_*.\end{aligned}$$

Hence, by (2.3), for $j \geq j_T$ and a.a. $(t, x, v) \in [0, T] \times [0, 1]^k \times \{v \in \mathbb{R}^3; |v| < V\}$,

$$\begin{aligned} \nu_j(f)^\sharp(t,x,v) &\geq c_3(T)(1-\frac{\alpha}{2})^{2\alpha}\Big(\int_{\mathcal{S}^2} \inf_{u\in[\gamma,\Gamma]} B(u,\theta)dn\Big)\int_{\Omega_1} \inf_{x\in[0,1]^k} f_0(x,v_*)dv_* \\ &\geq c_\Gamma c_3(T)(1-\frac{\alpha}{2})^{2\alpha}\int_{\Omega_1} \inf_{x\in[0,1]^k} f_0(x,v_*)dv_*. \end{aligned} \tag{6.4}$$

Applying (2.8) to Ω_1, this is a positive bound from below of $\big(\nu_j(f)^\sharp(t,x,v)\big)_{j\geq j_T}$ on $[0, T] \times [0, 1]^k \times \{v \in \mathbb{R}^3; |v| < V\}$.
The functions defined on $]0, \frac{1}{\alpha}]$ by $x \to \frac{F_j(x)}{x}$ are uniformly bounded from above with respect to j by

$$x \to \alpha^{\alpha-1}\frac{(1-\alpha x)^\alpha}{x},$$

that is continuous and decreasing to zero at $x = \frac{1}{\alpha}$. Hence there is $\tilde{\mu}_V = \min\{\frac{1}{2\alpha}, (\frac{c_4(T)c_\Gamma}{2c_2(T)})^{\frac{1}{\alpha}}\}$ such that

$$x \in [\frac{1}{\alpha} - \tilde{\mu}_V, \frac{1}{\alpha}] \quad \text{implies} \quad \frac{F_j(x)}{x} \leq \frac{c_4(T)c_\Gamma}{4c_2(T)}, \quad j \geq j_T.$$

Consequently, for $j \geq j_T$ and $|v| < V$,

$$\begin{aligned} f^\sharp(t,x,v) \in [\frac{1}{\alpha}-\tilde{\mu}_V, \frac{1}{\alpha}] \quad \Rightarrow \quad D_t f^\sharp(t,x,v) &= \big(F_j(f^\sharp)\tilde{\nu}_j^\sharp - \frac{1}{2}f^\sharp\nu_j^\sharp\big)(t,x,v) - \frac{1}{2}f^\sharp\nu_j^\sharp(t,x,v) \\ &< -\frac{1}{2}f^\sharp\nu_j^\sharp(t,x,v) \\ &< -\frac{c_4(T)c_\Gamma}{4\alpha} := -b_V. \end{aligned} \tag{6.5}$$

This gives a maximum time $t_1 = \frac{\tilde{\mu}_V}{b}$ for $f^{\#}$ to reach $\frac{1}{\alpha} - \tilde{\mu}_V$ from an initial value $f_0(x,v) \in]\frac{1}{\alpha} - \tilde{\mu}_V, \frac{1}{\alpha}]$. On this time interval $D_t f^\sharp \leq -b_V$. If $t_1 \geq T$, then at $t = T$ the value of $f^{\#}$ is bounded from above by $\frac{1}{\alpha} - b_V T := \frac{1}{\alpha} - \mu'_V$ with $0 < \mu' \leq \tilde{\mu}_V$. Let

$$t_m = \min\{t_1, T\}, \quad \mu_V = \min\{\tilde{\mu}_V, \mu'_V\}.$$

For any (x, v) with $|v| < V$, if $f(0, x, v) < \frac{1}{\alpha} - \mu_V$ were to reach $\frac{1}{\alpha} - \mu_V$ at (t, x, v) with $t \leq t_m$, then $D_t f^{\#}(t, x, v) \leq -b_V$, which excludes such a possibility. It follows that

$$\begin{aligned} &f^\sharp(t,x,v) \leq \frac{1}{\alpha} - \mu_V \text{ for } j \geq j_T, (t,x) \in [t_m, T] \times [0,1]^k, \quad |v| < V, \\ &f^\sharp(t,x,v) \leq \frac{1}{\alpha} - b_V t \quad \text{for } j \geq j_T, (t,x) \in [0, t_m] \times [0,1]^k, |v| < V. \end{aligned} \tag{6.6}$$

The previous estimates leading to the definition of t_m are independent of $j \geq j_T$.

■

Second step: existence of a solution f to (1.1).

Let $T \in [0, T_0[$ where T_0, defined at the beginning of this section, may be finite under the hypothesis of Theorem 2.1 and is infinite under those of Theorem 2.2. We shall prove the convergence in $L^1([0, T] \times [0, 1]^k \times \mathbb{R}^3))$ of the sequence (f_j) to a solution f of (1.1) by proving that it is a Cauchy sequence. Let us first prove that it is a Cauchy sequence in $L^1([0,T_0]\times[0,1]^k\times\mathbb{R}^3))$ for some $T_0\in]0,T[$, i.e. for any $\beta>0$, there exists $a\geq\max\{1,j_T\}$ such that

$$\sup_{t\in[0,T_0]}\int|g_j(t,x,v)|dxdv<\beta,\quad j>a,\tag{6.7}$$

where $g_j=f_j-f_a$. The sequence (f_j) will be proven to be a Cauchy sequence in $L^1([T_0,2T_0]\times[0,1]^k\times\mathbb{R}^3))$ etc. in an analogous way.
By the uniform boundedness of energy of (g_j), there is $V>0$ such that

$$\sup_{t\in[0,T]}\int_{|v|\geq V}|g_j(t,x,v)|dxdv<\frac{\beta}{2},\quad j>a,\tag{6.8}$$

The function g_j satisfies the equation

$$\begin{aligned}
&\partial_t g_j+\bar{v}\cdot\nabla_x g_j\\
&=\int(\chi_j-\chi_a)B\Big(f'_jf'_{j*}F_j(f_j)F_j(f_{j*})-f_jf_{j*}F_j(f'_j)F_j(f'_{j*})\Big)dv_*dn\\
&+\int\chi_aB(f'_jf'_{j*}-f'_af'_{a*})F_j(f_j)F_j(f_{j*})dv_*dn\\
&-\int\chi_aB(f_jf_{j*}-f_af_{a*})F_j(f'_j)F_j(f'_{j*})dv_*dn\\
&+\int\chi_aBf'_af'_{a*}\Big(F_j(f_{j*})\big(F_j(f_j)-F_j(f_a)\big)+F_a(f_a)\big(F_j(f_{j*})-F_j(f_{a*})\big)\Big)dv_*dn\\
&+\int\chi_aBf'_af'_{a*}\Big(F_j(f_{j*})\big(F_j(f_a)-F_a(f_a)\big)+F_a(f_a)\big(F_j(f_{a*})-F_a(f_{a*})\big)\Big)dv_*dn\\
&-\int\chi_aBf_af_{a*}\Big(F_j(f'_{j*})\big(F_j(f'_j)-F_j(f'_a)\big)+F_a(f'_a)\big(F_j(f'_{j*})-F_j(f'_{a*})\big)\Big)dv_*dn\\
&-\int\chi_aBf_af_{a*}\Big(F_j(f'_{j*})\big(F_j(f'_a)-F_a(f'_a)\big)+F_a(f'_a)\big(F_j(f'_{a*})-F_a(f'_{a*})\big)\Big)dv_*dn.
\end{aligned}\tag{6.9}$$

Using Lemmas 4.3 and 5.2 and the conservation of energy of f_j,

$$\begin{aligned}
\int(\chi_j-\chi_a)B\Big(f'_jf'_{j*}F_j(f_j)F_j(f_{j*})+f_jf_{j*}F_j(f'_j)F_j(f'_{j*})\Big)dxdvdv_*dn&\\
\leq c\int_{|v|>\frac{a}{\sqrt{2}}}f_j(t,x,v)dxdv&\\
\leq\frac{c}{a^2}.&
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int\chi_aB|f_jf_{j*}-f_af_{a*}|F_j(f'_j)F_j(f'_{j*})dxdvdv_*dn\\
&\leq c\Big(\int\sup_{(t,x)\in[0,T]\times[0,1]^k}f^\sharp_j(t,x,v)dv+\int\sup_{(t,x)\in[0,T]\times[0,1]^k}f^\sharp_a(t,x,v)dv\Big)\\
&\qquad\qquad\times\Big(\int|(f^\sharp_j-f^\sharp_a)(t,x,v)|dxdv\Big)\\
&\leq c\int|(f^\sharp_j-f^\sharp_a)(t,x,v)|dxdv,\quad\text{ by Lemmas 4.3 and 5.2,}
\end{aligned}\tag{6.10}$$

and

$$\int \chi_a B\Big(f'_a f'_{a*} F_j(f_{j*})|F_j(f_a) - F_a(f_a)|\Big)^\sharp dx dv dv_* dn = \int \chi_a B f'_a f'_{a*} F_j(f_{j*})$$
$$(1-\alpha f_a)(1+(1-\alpha)f_a)^{1-\alpha}|(\frac{1}{j}+1-\alpha f_a)^{\alpha-1} - (\frac{1}{a}+1-\alpha f_a)^{\alpha-1}|dx dv dv_* dn.$$

By Lemmas 4.1, 4.3 and 5.1, 5.2, this integral restricted to the set where $1-\alpha f_a(t,x,v)) \leq \frac{2}{a}$, hence where

$$(1-\alpha f_a)|(\frac{1}{j}+1-\alpha f_a)^{\alpha-1} - (\frac{1}{a}+1-\alpha f_a)^{\alpha-1}| \leq \frac{2^{\alpha+1}}{a^\alpha},$$

is bounded by $\frac{c}{a^\alpha}$ for some constant $c > 0$.
For the remaining domain of integration where $1-\alpha f_a(t,x,v)) \geq \frac{2}{a}$, it holds

$$\begin{aligned}|F_j(f_a) - F_a(f_a)| &\leq c(1-\alpha f_a)^\alpha|(\frac{1}{j(1-\alpha f_a)}+1)^{\alpha-1} - (\frac{1}{a(1-\alpha f_a)}+1)^{\alpha-1}| \\ &= c(\frac{1}{j}-\frac{1}{a})(1-\alpha f_a)^{\alpha-1}\lambda^{\alpha-2} \quad \text{where } \lambda \in [1, \frac{3}{2}] \\ &\leq \frac{2^{\alpha-1}c}{a^\alpha}.\end{aligned}$$

And so,

$$\int \chi_a B\Big(f'_a f'_{a*} F_j(f_{j*})|F_j(f_a) - F_a(f_a)|\Big)^\sharp dx dv dv_* dn \leq \frac{c}{a^\alpha}.$$

Finally

$$\int_{|v|<V} \chi_a B\Big(f'_a f'_{a*} F_j(f_{j*})|F_j(f_j) - F_j(f_a)|\Big)^\sharp (t,x,v) dx dv dv_* dn$$
$$\leq c\int_{|v|<V} |F_j(f_j) - F_j(f_a)|^\sharp (t,x,v) dx dv + c\beta.$$

Split the (x,v)-domain of integration of the latest integral into

$$\begin{aligned}D_1 &:= \{(x,v); |v| < V \text{ and } (f_j^\sharp(t,x,v), f_a^\sharp(t,x,v)) \in [0, \frac{1}{\alpha} - \mu_V]^2\}, \\ D_2 &:= \{(x,v); |v| < V \text{ and } (f_j^\sharp(t,x,v), f_a^\sharp(t,x,v)) \in [\frac{1}{\alpha} - \mu_V, \frac{1}{\alpha}]^2\}, \\ D_3 &:= \{(x,v); |v| < V, (f_j^\sharp, f_a^\sharp)(t,x,v) \in [\frac{1}{\alpha} - \mu_V, \frac{1}{\alpha}] \times [0, \frac{1}{\alpha} - \mu_V] \\ &\qquad \text{or } (f_j^\sharp, f_a^\sharp) \in [0, \frac{1}{\alpha} - \mu_V] \times [\frac{1}{\alpha} - \mu_V, \frac{1}{\alpha}]\}.\end{aligned}$$

It holds that

$$\begin{aligned}&\int_{D_1} |F_j(f_j) - F_j(f_a)|^\sharp (t,x,v) dx dv \leq c(\alpha\mu_V)^{\alpha-1}\int_{D_1} |g_j^\sharp(t,x,v)| dx dv, \\ &\int_{D_2} |F_j(f_j) - F_j(f_a)|^\sharp (t,x,v) dx dv \leq c(b_V t)^{\alpha-1}\int_{D_2} |g_j^\sharp(t,x,v)| dx dv, \quad \text{by (6.6)}, \\ &\int_{D_3} |F_j(f_j) - F_j(f_a)|^\sharp (t,x,v) dx dv \leq c\big((\alpha\mu_V)^{\alpha-1} + (b_V t)^{\alpha-1}\big)\int_{D_3} |g_j^\sharp(t,x,v)| dx dv.\end{aligned}$$

The remaining terms to the right in (6.9) are of the same types as the ones just estimated. Consequently,

$$\frac{d}{dt}\int_{|v|<V}|g_j^\sharp(t,x,v)|dxdv \leq \frac{c}{a^\alpha}+c\beta+c(1+\mu_V^{\alpha-1}+(b_Vt)^{\alpha-1})\Big(\int_{|v|<V}|g_j^\sharp(t,x,v)|dxdv\Big). \quad (6.11)$$

And so,

$$\begin{aligned}&\sup_{t\in[0,T_0]}\int_{|v|<V}|g_j^\sharp(t,x,v)|dxdv\\ &\leq \Big(\int_{|v|<V}\mid(f_{0,j}-f_{0,a})(x,v)\mid dxdv+\frac{cT}{a^\alpha}+c\beta T_0\Big)e^{c((1+\mu_V^{\alpha-1})T+\frac{b_V^{\alpha-1}T^\alpha}{\alpha})},\end{aligned} \quad (6.12)$$

with $f_{0,j}$ (resp. $f_{0,a}$) defined in (3.3). For a (resp. T_0) large (resp. small) enough, the right-hand side of (6.12) is smaller than $\frac{\beta}{2}$, uniformly w.r.t. $j \geq a$. This proves that $(f_j)_{j\in\mathbb{N}^*}$ is a Cauchy sequence in $L^1([0,T_0]\times[0,1]^k\times\mathbb{R}^3)$ and ends the proof of the existence of a solution f to (1.1). It follows from the boundedness of $\frac{d}{dt}f^\sharp$ that $f\in C([0,T];L^1([0,1]^k\times\mathbb{R}^3))$, which in turn implies that $Q(f)\in C([0,T];L^1([0,1]^k\times\mathbb{R}^3))$ and $f\in C^1([0,T];L^1([0,1]^k\times\mathbb{R}^3))$.

Third step: uniqueness of the solution to (1.1) and stability results.

The previous line of arguments can be followed to obtain that the solution is unique. Namely, assuming the existence of two possibly local solutions f_1 and f_2 to (1.1) with the same initial datum and bounded energy, Lemma 6.1 holds for both solutions. The difference $g = f_1 - f_2$ satisfies

$$\begin{aligned}&\partial_t g+\bar{v}\cdot\nabla_x g\\ &=\int B(f_1'f_{1*}'-f_2'f_{2*}')F(f_1)F(f_{1*})dv_*dn-\int B(f_1f_{1*}-f_2f_{2*})F(f_1')F(f_{1*}')dv_*dn\\ &+\int Bf_2'f_{2*}'\Big(F(f_{1*})\big(F(f_1)-F(f_2)\big)+F(f_2)\big(F(f_{1*})-F(f_{2*})\big)\Big)dv_*dn\\ &-\int Bf_2f_{2*}\Big(F(f_{1*}')\big(F(f_1')-F(f_2')\big)+F(f_2')\big(F(f_{1*}')-F(f_{2*}')\big)\Big)dv_*dn.\end{aligned}$$

The first line in the r.h.s. of the former equation gives rise to $c\int|g^\sharp(t,x,v)|dxdv$ in the bound from above of $\frac{d}{dt}|g^\sharp(t,x,v)|dxdv$, whereas the two last lines in the r.h.s of the former equation give rise to the bound $c(1+t^{\alpha-1})\int|g^\sharp(t,x,v)|dxdv$. Consequently,

$$\frac{d}{dt}\int|g^\sharp(t,x,v)|dxdv \leq c(1+t^{\alpha-1})\int|g^\sharp(t,x,v)|dxdv.$$

This implies that $\int|g^\sharp(t,x,v)|dxdv$ is identically zero, since it is zero initially.

The proof of stability is similar.

7 Conservations of mass, momentum and energy.

The conservation of mass and momentum of f follow from the boundedness of the total energy. The energy is non-increasing by the construction of f. Energy conservation will follow if the energy is non-decreasing. This requires the preliminary control of the mass density over large velocities, performed in the following lemma.

Lemma 7.1
Given $t \in [0,T]$, there is a constant $c'_t > 0$ such that for every $\lambda > 2$ the solution f of (1.1)-(1.2) satisfies

$$\int_{|v|>\lambda} \sup_{(s,x)\in[0,t]\times[0,1]^k} f^\sharp(s,x,v)dv \leq \frac{c'_t}{\sqrt{\lambda}}.$$

<u>Proof of Lemma 7.1.</u>
Take $\lambda > 2$. First consider the case $k = 1$. It follows from (3.6) that

$$\int_{|v|>\lambda} \sup_{(s,x)\in[0,t]\times[0,1]} f^\sharp(s,x,v)dv \leq \int_{|v|>\lambda} \sup_{x\in[0,1]} f_0(x,v)dv + \parallel F_\alpha \parallel_\infty^2 C, \tag{7.1}$$

where

$$C = \int_{|v|>\lambda} \sup_{x\in[0,1]} \int_0^t \int Bf^{\#}(s, x + s(v_1 - v'_1), v')f^{\#}(s, x + s(v_1 - v'_{*1}), v'_*)dvdv_*dnds.$$

For v', v'_* outside of the angular cutoff (2.2), let n be the unit vector in the direction $v - v'$ and $n_\perp$ its orthogonal unit vector in the direction $v - v'_*$. Split C into $C = \sum_{0\leq i\leq 2} C_i$, where

$$C_0 = \int_{|v|>\lambda} \sup_{x\in[0,1]} \Big(\int_0^t \int_{|n_1|<\epsilon \text{ or } |n_{\perp 1}|<\epsilon} Bf^{\#}(s, x + s(v_1 - v'_1), v')f^{\#}(s, x + s(v_1 - v'_{*1}), v'_*)dv_*dnds\Big)dv,$$

and C_1 (resp. C_2) refers to integration with respect to (v_*, n) on

$$\{(v_*, n); \quad |n_1| \geq \epsilon, \quad |n_{\perp 1}| \geq \epsilon, \quad |v'| \geq |v'_*|\},$$
$$\big(\text{resp. } \{(v_*, n); \quad |n_1| \geq \epsilon, \quad |n_{\perp 1}| \geq \epsilon, \quad |v'| \leq |v'_*|\}\big).$$

By Lemma 4.3 and the change of variables $(v, v_*, n) \rightarrow (v_*, v, n_\perp)$,

$$C_0 \leq c\epsilon t, \tag{7.2}$$

for some constant $c > 0$. Analogously to the control of A_1 in the proof of Lemma 4.3 and using Lemma 3.2, it holds that

$$\begin{aligned}
C_1 &\leq \int_{|v|\geq\lambda} \sup_{x\in[0,1]} \int_{|v'|>|v'_*|} B\Big(\int_0^t \sup_{\tau\in[0,t]} f^\sharp(\tau, x+s(v_1-v'_1), v')ds\Big) \\
&\qquad\qquad\qquad \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\sharp(\tau, X, v'_*)dvdv_*dn \\
&= \int_{|v|\geq\lambda} \sup_{x\in[0,1]} \int_{|v'|>|v'_*|} \frac{B}{|v_1-v'_1|}\Big(\int_{y\in(x,x+t(v_1-v'_1)} \sup_{\tau\in[0,t]} f^\sharp(\tau, y, v')dy\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\sharp(\tau, X, v'_*)dvdv_*dn \\
&\leq \int_{|v|\geq\lambda, |v'|>|v'_*|} B\frac{E(t|v_1-v'_1|)+1}{|v_1-v'_1|}\Big(\int_0^1 \sup_{\tau\in[0,t]} f^\sharp(\tau, y, v')dy\Big) \\
&\qquad\qquad\qquad\qquad\qquad \sup_{(\tau,X)\in[0,t]\times[0,1]} f^\sharp(\tau, X, v'_*)dvdv_*dn \\
&\leq c(t+\frac{1}{\epsilon\gamma\gamma'})\int_{|v|\geq\frac{\lambda}{\sqrt{2}}} \int_0^1 \sup_{\tau\in[0,t]} f^\sharp(\tau, y, v)dydv \\
&\leq \frac{c}{\lambda}(1+\frac{1}{\epsilon}), \quad t\leq\max\{1,T\}.
\end{aligned}$$

The term C_2 can be controlled similarly to C_1 with the change of variables $s \to y = x + s(v_1 - v'_{*1})$. And so,

$$C \leq c\big(\epsilon + \frac{1}{\lambda} + \frac{1}{\epsilon\lambda}\big), \quad t \leq \max\{1,T\}.$$

Choosing $\epsilon = \frac{1}{\sqrt{\lambda}}$ leads to

$$C \leq \frac{c}{\sqrt{\lambda}}, \quad t \leq \max\{1,T\}.$$

Repeating the previous proof up to time T, the lemma follows.
In the case of Theorem 2.2 where in particular $k \in \{1,2,3\}$ and (2.7) is assumed, analogously to the proof of Lemma 5.2 we obtain

$$\int \sup_{(s,x)\in[0,t]\times[0,1]^k} |v|^2 f(s,x,v)dv \leq \tilde{c}_0 e^{ct},$$

for some constant c. It follows that

$$\begin{aligned}
\int_{|v|>\lambda} \sup_{(s,x)\in[0,t]\times[0,1]^k} f^\sharp(s,x,v)dv &\leq \frac{1}{\lambda^2}\int |v|^2 \sup_{(s,x)\in[0,t]\times[0,1]^k} f^\sharp(s,x,v)dv \\
&\leq \frac{\tilde{c}_0 e^{ct}}{\lambda^2}.
\end{aligned}$$

■

Lemma 7.2 *The solution f to the Cauchy problem (1.1)-(1.2) conserves energy.*

Proof of Lemma 7.2.
It remains to prove that the energy is non-decreasing. Taking $\psi_\epsilon = \frac{|v^2|}{1+\epsilon|v|^2}$ as approximation for $|v|^2$, it is enough to bound

$$\int Q(f)(t,x,v)\psi_\epsilon(v)dxdv = \int B\psi_\epsilon\Big(f'f'_*F(f)F(f_*) - ff_*F(f')F(f'_*)\Big)dxdvdv_*dn$$

from below by zero in the limit $\epsilon \to 0$. Similarly to [12],

$$\begin{aligned}\int Q(f)\psi_\epsilon dxdv &= \frac{1}{2}\int Bff_*F(f')F(f'_*\Big(\psi_\epsilon(v') + \psi_\epsilon(v'_*) - \psi_\epsilon(v) - \psi_\epsilon(v_*)\Big)dxdvdv_*dn\\ &\geq -\int Bff_*F(f')F(f'_*)\frac{\epsilon|v|^2|v_*|^2}{(1+\epsilon|v|^2)(1+\epsilon|v_*|^2)}dxdvdv_*dn.\end{aligned}$$

The previous line, with the integral taken over a bounded set in (v, v_*), converges to zero when $\epsilon \to 0$. In integrating over $|v|^2 + |v_*|^2 \geq 2\lambda^2$, there is symmetry between the subset of the domain with $|v|^2 > \lambda^2$ and the one with $|v_*|^2 > \lambda^2$. We discuss the first sub-domain, for which the integral in the last line is bounded from below by

$$\begin{aligned}&-c\int |v_*|^2 f(t,x,v_*)dxdv_* \int_{|v|\geq\lambda} B \sup_{(s,x)\in[0,t]\times[0,1]^k} f^\#(s,x,v)dvdn\\ &\geq -c\int_{|v|\geq\lambda} \sup_{(s,x)\in[0,t]\times[0,1]^k} f^\#(s,x,v)dv.\end{aligned}$$

It follows from Lemma 7.1 that the right hand side tends to zero when $\lambda \to \infty$.
This implies that the energy is non-decreasing, and bounded from below by its initial value.
That completes the proof of the lemma. ■

Acknowledgement. The authors wish to thank the anonymous referees for several important suggestions to improve the manuscript.

References

[1] L. Arkeryd, A quantum Boltzmann equation for Haldane statistics and hard forces; the space-homogeneous initial value problem, Comm. Math. Phys., 298 (2010), 573-583.

[2] L. Arkeryd, A. Nouri, Well-posedness of the Cauchy problem for a space-dependent anyon Boltzmann equation, SIAM J. Math. Anal., 47-6 (2015), 4720-4742.

[3] L. Arkeryd, A. Nouri, On the Cauchy problem with large data for a space-dependent Boltzmann-Nordheim boson equation, Comm. Math. Sci., 15-5 (2017), 1247-1264.

[4] L. Arkeryd, A. Nouri, On the Cauchy problem with large data for the space-dependent Boltzmann-Nordheim equation III, Preprint (2018), arXiv:1801.02494.

[5] R. K. Bhaduri, R. S. Bhalero, M. V. Murthy, Haldane exclusion statistics and the Boltzmann equation, J. Stat. Phys., 82 (1996), 1659-1668.

[6] J.-M. Bony, Solutions globales bornées pour les modèles discrets de l'équation de Boltzmann, en dimension 1 d'espace, Journées "Équations aux dérivées partielles ", Exp. XVI, École Polytech., Palaiseau, (1987), 1-10.

[7] M. Briant, A. Einav, On the Cauchy problem for the homogeneous Boltzmann-Nordheim equation for bosons: local existence, uniqueness and creation of moments, J. Stat. Phys., 163-5 (2016), 1108-1156.

[8] C. Cercignani, R. Illner, Global weak solutions of the Boltzmann equation in a slab with diffusive boundary conditions, Arch Rat. Mech. Anal., 134 (1996), 1-16.

[9] M. Escobedo, J.L. Velazquez, Finite time blow-up and condensation for the bosonic Nordheim equation, Inv. Math. 200 (2015), 761-847.

[10] F. D. Haldane, Fractional statistics in arbitrary dimensions: a generalization of the Pauli principle, Phys. Rev. Lett., 67 (1991), 937-940.

[11] J. M. Leinaas, J. Myrheim, On the theory of identical particles, Nuovo Cim. B, 37-1 (1977), 1-23.

[12] X. LU, *On isotropic distributional solutions to the Boltzmann equation for Bose-Einstein particles*, J. Stat. Phys., 116 (2004), pp. 1597-1649.

[13] X. Lu, The Boltzmann equation for Bose-Einstein particles: condensation in finite time, J. Stat. Phys., 150 (2013), 1138-1176.

[14] L. W. Nordheim, On the kinetic methods in the new statistics and its applications in the electron theory of conductivity, Proc. Roy. Soc. London Ser. A, 119 (1928), 689-698.

[15] C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations, Arch. Rat. Mech. Anal., 143 (1998), 273-307.

Leif Arkeryd, arkeryd@chalmers.se
Anne Nouri, anne.nouri@univ-amu.fr

Running head: On a Boltzmann equation for Haldane statistics.

Name and mailing address of the author to whom proofs should be sent.
Anne Nouri,
anne.nouri@univ-amu.fr