On the Cauchy problem with large data for a space-dependent Boltzmann-Nordheim boson equation.

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Abstract. This paper studies a Boltzmann Nordheim equation in a slab with two-dimensional velocity space and pseudo-Maxwellian forces. Strong solutions are obtained for the Cauchy problem with large initial data in an $L^1 \cap L^\infty$ setting. The main results are existence, uniqueness and stability of solutions conserving mass, momentum and energy that explode in L^∞ if they are only local in time. The solutions are obtained as limits of solutions to corresponding anyon equations.

1 Introduction and main result.

In a previous paper [1], we have studied the Cauchy problem for a space-dependent anyon Boltzmann equation,

$$\partial_t f(t,x,v)+v_1\partial_x f(t,x,v) = Q_\alpha(f)(t,x,v), \quad f(0,x,v) = f_0(x,v),\ (t,x) \in \mathbb{R}_+ \times [0,1],\ v = (v_1,v_2) \in \mathbb{R}^2. \tag{1.1}$$

The collision operator Q_α in [1] depends on a parameter $\alpha \in]0,1[$ and is given by

$$Q_\alpha(f)(v) = \int_{I\!R^2\times S^1} B(|v-v_*|,n)[f'f'_*F_\alpha(f)F_\alpha(f_*) - ff_*F_\alpha(f')F_\alpha(f'_*)]dv_*dn,$$

with the kernel B of Maxwellian type, f', f'_*, f, f_* the values of f at v', v'_*, v and v_* respectively, where

$$v' = v-(v-v_*,n)n, \quad v'_* = v_* + (v-v_*,n)n\,,$$

and the filling factor F_α

$$F_\alpha(f) = (1-\alpha f)^\alpha(1+(1-\alpha)f)^{1-\alpha}\,.$$

Anyons are other types of particles that occur in one and two-dimensions besides fermions and bosons. The exchange of two identical anyons may cause a phase shift different from π (fermions) and 2π (bosons). In [1], also the limiting case $\alpha = 1$ is discussed, a Boltzmann-Nordheim (BN) equation [11] for fermions. In the present paper we shall consider the other limiting case, $\alpha = 0$,

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which is a BN equation for bosons.

For the bosonic BN equation general existence results were first obtained by X. Lu in [7] in the space-homogeneous isotropic boson large data case. It was followed by a number of interesting studies in the same isotropic setting, by X. Lu [8, 9, 10], and by M. Escobedo and J.L. Velázquez [5, 6]. Results with the isotropy assumption removed, were recently obtained by M. Briant and A. Einav [3]. Finally a space-dependent case close to equilibrium has been studied by G. Royat in [12]. The papers [7, 8, 9, 10] by Lu, study the isotropic, space-homogeneous BN equation both for Cauchy data leading to mass and energy conservation, and for data leading to mass loss when time tends to infinity. Escobedo and Velásquez in [5, 6], again in the isotropic space-homogeneous case, study initial data leading to concentration phenomena and blow-up in finite time of the L^∞-norm of the solutions. The paper [3] by Briant and Einav removes the isotropy restriction and obtain in polynomially weighted spaces of $L^1 \cap L^\infty$ type, existence and uniqueness on a time interval $[0, T_0)$. In [3] either $T_0 = \infty$, or for finite T_0 the L^∞-norm of the solution tends to infinity, when time tends to T_0. Finally the paper [12] considers the space-dependent problem, for a particular setting close to equilibrium, and proves well-posedness and convergence to equilibrium.
In the papers cited above, the velocity space is $\mathbb{R}^3$. The present paper on the other hand studies a space-dependent, large data problem for the BN equation with velocities in $\mathbb{R}^2$. The analysis is based on the anyon results in [1], which are restricted to a slab set-up, since the proofs in [1] use an estimate for the Bony functional only valid in one space dimension. Due to the filling factor $F_\alpha(f)$, those proofs also in an essential way depend on the two-dimensional velocity frame. By a limiting procedure relying on the anyon case when $\alpha \to 0$, well-posedness and conservation laws are obtained in the present paper for the BN problen.
With

$$\cos\theta = n \cdot \frac{v - v_*}{|v - v_*|}\,,$$

the kernel $B(|v - v_*|, n)$ will from now on be written $B(|v - v_*|, \theta)$ and assumed measurable with

$$0 \leq B \leq B_0, \tag{1.2}$$

for some $B_0 > 0$. It is also assumed for some $\gamma, \gamma', c_B > 0$, that

$$B(|v - v_*|, \theta) = 0 \text{ for } |\cos\theta| < \gamma', \quad \text{for } 1 - |\cos\theta| < \gamma', \quad \text{and for } |v - v_*| < \gamma, \tag{1.3}$$

and that

$$\int B(|v - v_*|, \theta) d\theta \geq c_B > 0 \quad \text{for } |v - v_*| \geq \gamma. \tag{1.4}$$

These strong cut-off conditions on B are made for mathematical reasons and assumed throughout the paper. For a more general discussion of cut-offs in the collision kernel B, see [8]. Notice that contrary to the classical Boltzmann operator where rigorous derivations of B from various potentials have been made, little is known about collision kernels in quantum kinetic theory (cf [13]).
With v_1 denoting the component of v in the x-direction, the initial value problem for the Boltzmann Nordheim equation in a periodic in space setting is

$$\partial_t f(t, x, v) + v_1 \partial_x f(t, x, v) = Q(f)(t, x, v), \tag{1.5}$$

where

$$Q(f)(v) = \int_{I\!R^2 \times [0,\pi]} B(|v - v_*|, \theta)[f' f'_* F(f) F(f_*) - f f_* F(f') F(f'_*)] dv_* d\theta, \tag{1.6}$$

and

$$F(f) = 1 + f. \tag{1.7}$$

Denote by

$$f^\sharp(t,x,v) = f(t, x+tv_1, v) \quad (t,x,v) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}^2. \tag{1.8}$$

Strong solutions to the Boltzmann Nordheim paper are considered in the following sense.

Definition 1.1 *f is a strong solution to (1.5) on the time interval I if*

$$f \in \mathcal{C}^1(I; L^1([0,1] \times \mathbb{R}^2)),$$

and

$$\frac{d}{dt} f^\sharp = \big(Q(f)\big)^\sharp, \quad \textit{on } I \times [0,1] \times \mathbb{R}^2. \tag{1.9}$$

The main result of this paper is the following.

Theorem 1.1 *Assume (1.2)-(1.3)-(1.4). Let $f_0 \in L^\infty([0,1] \times \mathbb{R}^2)$ and satisfy*

$$(1+|v|^2) f_0(x,v) \in L^1([0,1] \times \mathbb{R}^2), \ \int \sup_{x \in [0,1]} f_0(x,v) dv = c_0 < \infty, \ \inf_{x \in [0,1]} f_0(x,v) > 0, \ a.a. v \in \mathbb{R}^2. \tag{1.10}$$

There exist a time $T_\infty > 0$ and a strong solution f to (1.5) on $[0, T_\infty)$ with initial value f_0. For $0 < T < T_\infty$, it holds

$$f^\sharp \in \mathcal{C}^1([0,T_\infty); L^1([0,1] \times \mathbb{R}^2)) \cap L^\infty([0,T] \times [0,1] \times \mathbb{R}^2). \tag{1.11}$$

If $T_\infty < +\infty$ then

$$\overline{\lim}_{t \to T_\infty} \parallel f(t,\cdot,\cdot) \parallel_{L^\infty([0,1]\times\mathbb{R}^2)} = +\infty. \tag{1.12}$$

The solution is unique, and conserves mass, momentum, and energy. For equibounded families in $L^\infty([0,1] \times \mathbb{R}^2)$ of initial values, the solution depends continuously in L^1 on the initial value f_0.

Remark.
A finite T_∞ may not correspond to a condensation. In the isotropic space-homogeneous case considered in [5, 6], additional assumptions on the concentration of the initial value are considered in order to obtain condensation.

The paper is organized as follows. In the following section, solutions f_α to the Cauchy problem for the anyon Boltzmann equation in the above setting are recalled, and their Bony functionals are uniformly controlled with respect to α. In Section 3 the mass density of f_α is studied with respect to uniform control in α. Theorem 1.1 is proven in Section 4 except for the conservations of mass, momentum and energy that are proven in Section 5.

2 Preliminaries on anyons and the Bony functional.

The Cauchy problem for a space-dependent anyon Boltzmann equation in a slab was studied in [1]. That paper will be the starting point for the proof of Theorem 1.1, so we recall the main results from [1].

Theorem 2.1
Assume (1.2)-(1.3)-(1.4). Let the initial value f_0 be a measurable function on $[0,1]\times\mathbb{R}^2$ with values in $]0,\frac{1}{\alpha}]$, and satisfying (1.10). For every $\alpha\in]0,1[$, there exists a strong solution f_α of (1.1) with

$$f^\sharp_\alpha\in\mathcal{C}^1([0,\infty[;L^1([0,1]\times\mathbb{R}^2)),\qquad 0<f_\alpha(t,\cdot,\cdot)<\frac{1}{\alpha}\quad \textit{for } t>0,$$

and

$$\int\sup_{(s,x)\in[0,t]\times[0,1]}f^\sharp_\alpha(s,x,v)dv\leq c_\alpha(t),\tag{2.1}$$

for some function $c_\alpha(t)>0$ only depending on mass and energy. There is $t_m>0$ such that for any $T>t_m$, there is $\eta_T>0$ so that

$$f_\alpha(t,\cdot,\cdot)\leq\frac{1}{\alpha}-\eta_T,\quad t\in[t_m,T].$$

The solution is unique and depends continuously in $\mathcal{C}([0,T];L^1([0,1]\times\mathbb{R}^2))$ on the initial L^1-datum. It conserves mass, momentum and energy.

The conditions $f_0\in L^\infty([0,1]\times\mathbb{R}^2)$ and (1.10) are assumed throughout the paper.

To obtain Theorem 1.1 for the boson BN equation from the anyon results, we start from a fixed initial value f_0 bounded by 2^L with $L\in\mathbb{N}$. We shall prove that there is a time $T>0$ independent of $0<\alpha<2^{-L-1}$, so that the solutions are bounded by 2^{L+1} on $[0,T]$. For that, some lemmas from the anyon paper are sharpened to obtain control in terms of only mass, energy and L. We then prove that the limit f of the solutions f_α when $\alpha\to0$ solves the corresponding bosonic BN problem. Iterating the result from T on, it follows that f exists up to the first time T_∞ when $\overline{\lim}_{t\to T_\infty}\parallel f_\alpha(t,\cdot,\cdot)\parallel_{L^\infty([0,1]\times\mathbb{R}^2)}=\infty$.

We observe that

Lemma 2.2
Given $f_0\leq2^L$ and satisfying (1.10), there is for each $\alpha\in]0,2^{-L-1}[$ a time $T_\alpha>0$ so that the solution f_α to (1.1) is bounded by 2^{L+1} on $[0,T_\alpha]$.

<u>Proof of Lemma 2.2.</u>
Split the Boltzmann anyon operator Q_α into $Q_\alpha=Q^+_\alpha-Q^-_\alpha$, where the gain (resp. loss) term Q^+_α (resp. Q^-_α) is defined by

$$Q^+_\alpha(f)(v)=\int Bf'f'_*F_\alpha(f)F_\alpha(f_*)dv_*d\theta\quad(\textit{resp. } Q^-_\alpha(f)(v)=\int Bff_*F_\alpha(f')F_\alpha(f'_*)dv_*d\theta).\tag{2.2}$$

The solution f_α to (1.1) satisfies

$$f^\sharp_\alpha(t,x,v)=f_0(x,v)+\int_0^t Q_\alpha(f_\alpha)(s,x+sv_1,v)ds\leq f_0(x,v)+\int_0^t Q^+_\alpha(f_\alpha)(s,x+sv_1,v)ds.$$

Hence

$$\begin{aligned}
&\sup_{s\leq t} f^\sharp_\alpha(s,x,v)\leq f_0(x,v)+\int_0^t Q^+_\alpha(f_\alpha)(s,x+sv_1,v)ds \qquad (2.3)\\
&=f_0(x,v)+\int_0^t\int Bf_\alpha(s,x+sv_1,v')f_\alpha(s,x+sv_1,v'_*)F_\alpha(f_\alpha)(s,x+sv_1,v)F_\alpha(f_\alpha)(s,x+sv_1,v_*)dv_*d\theta ds\\
&\leq 2^L+\frac{B_0}{\alpha}\Big(\frac{1}{\alpha}-1\Big)^{2(1-2\alpha)}\int_0^t\int f_\alpha(s,x+sv_1,v')dv_*d\theta ds,
\end{aligned}$$

since the maximum of F_α on $[0,\frac{1}{\alpha}]$ is $(\frac{1}{\alpha}-1)^{1-2\alpha}$ for $\alpha\in]0,\frac{1}{2}[$. With the angular cut-off (2.2), $v_*\to v'$ is a change of variables. Using it and (2.1) for $t\leq 1$ leads to

$$\begin{aligned}
\sup_{s\leq t,x} f^\sharp_\alpha(s,x,v)&\leq 2^L+c\frac{B_0c_\alpha(1)}{\alpha}\Big(\frac{1}{\alpha}-1\Big)^{2(1-2\alpha)}t\\
&\leq 2^{L+1} \qquad\qquad \text{for } t\leq\min\{1,\frac{2^L\alpha^{3-4\alpha}(1-\alpha)^{2(2\alpha-1)}}{cB_0c_\alpha(1)}\}.
\end{aligned}$$

The lemma follows. ∎

The estimate of the Bony functional

$$\bar{B}_\alpha(t):=\int_0^1\int|v-v_*|^2Bf_\alpha f_{\alpha*}F_\alpha(f'_\alpha)F_\alpha(f'_{\alpha*})dvdv_*d\theta dx,\quad t\geq 0,$$

from the proof of Theorem 2.1 for $f_\alpha\leq 2^{L+1}$, can be sharpened.

Lemma 2.3
For $\alpha\leq 2^{-L-1}$ and $T>0$ such that $f_\alpha(t)\leq 2^{L+1}$ for $0\leq t\leq T$, it holds

$$\int_0^T\bar{B}_\alpha(t)dt\leq c'_0(1+T)\,,$$

with c'_0 independent of T and α, and only depending on $\int f_0(x,v)dxdv$, $\int|v|^2f_0(x,v)dxdv$ and L.

Proof of Lemma 2.3.
Denote f_α by f for simplicity. The proof is an extension of the classical one (cf [2], [4]), together with the control of the filling factor F_α when $v\in\mathbb{R}^2$, as follows.
The integral over time of the momentum $\int v_1f(t,0,v)dv$ (resp. the momentum flux $\int v_1^2f(t,0,v)dv$) is first controlled. Let $\beta\in C^1([0,1])$ be such that $\beta(0)=-1$ and $\beta(1)=1$. Multiply (1.1) by $\beta(x)$ (resp. $v_1\beta(x)$) and integrate over $[0,t]\times[0,1]\times\mathbb{R}^2$. It gives

$$\begin{aligned}
\int_0^t\int v_1f(\tau,0,v)dvd\tau=\frac{1}{2}\Big(&\int\beta(x)f_0(x,v)dxdv-\int\beta(x)f(t,x,v)dxdv\\
&+\int_0^t\int\beta'(x)v_1f(\tau,x,v)dxdvd\tau\Big),
\end{aligned}$$

$\Big($ resp.

$$\int_0^t \int v_1^2 f(\tau,0,v)dvd\tau = \frac{1}{2}\Big(\int \beta(x)v_1 f_0(x,v)dxdv - \int \beta(x)v_1 f(t,x,v)dxdv + \int_0^t \int \beta'(x)v_1^2 f(\tau,x,v)dxdvd\tau\Big)\Big).$$

Consequently, using the conservation of mass and energy of f,

$$|\int_0^t \int v_1 f(\tau,0,v)dvd\tau| + \int_0^t \int v_1^2 f(\tau,0,v)dvd\tau \leq c(1+t). \tag{2.4}$$

Here c is of magnitude of mass plus energy uniformly in α. Let

$$\mathcal{I}(t) = \int_{x<y} (v_1 - v_{*1})f(t,x,v)f(t,y,v_*)dxdydvdv_*.$$

It results from

$$\mathcal{I}'(t) = -\int (v_1 - v_{*1})^2 f(t,x,v)f(t,x,v_*)dxdvdv_* + 2\int v_{*1}(v_{*1} - v_1)f(t,0,v_*)f(t,x,v)dxdvdv_*,$$

and the conservations of the mass, momentum and energy of f that

$$\begin{aligned}
&\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f(s,x,v)f(s,x,v_*)dvdv_*dxds \\
&\leq 2\int f_0(x,v)dxdv \int |v_1| f_0(x,v)dv + 2\int f(t,x,v)dxdv \int |v_1| f(t,x,v)dxdv \\
&+ 2\int_0^t \int v_{*1}(v_{*1} - v_1)f(\tau,0,v_*)f(\tau,x,v)dxdvdv_*d\tau \\
&\leq 2\int f_0(x,v)dxdv \int (1+|v|^2)f_0(x,v)dv + 2\int f(t,x,v)dxdv \int (1+|v|^2)f(t,x,v)dxdv \\
&+ 2\int_0^t (\int v_{*1}^2 f(\tau,0,v_*)dv_*)d\tau \int f_0(x,v)dxdv - 2\int_0^t (\int v_{*1} f(\tau,0,v_*)dv_*)d\tau \int v_1 f_0(x,v)dxdv \\
&\leq c\Big(1 + \int_0^t \int v_1^2 f(\tau,0,v)dvd\tau + |\int_0^t \int v_1 f(\tau,0,v)dvd\tau|\Big).
\end{aligned}$$

And so, by (2.4),

$$\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f(s,x,v)f(s,x,v_*)dvdv_*dxds \leq c(1+t). \tag{2.5}$$

Denote by $u_1 = \frac{\int v_1 f dv}{\int f dv}$. Recalling (1.2) it holds

$$\begin{aligned}
&\int_0^t \int_0^1 \int (v_1 - u_1)^2 B f f_* F_\alpha(f')F_\alpha(f'_*)(s,x,v,v_*,\theta)dvdv_*d\theta dxds \\
&\leq c\int_0^t \int_0^1 \int (v_1 - u_1)^2 f f_*(s,x,v,v_*)dvdv_*dxds \\
&= \frac{c}{2}\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f f_*(s,x,v,v_*)dvdv_*dxds \\
&\leq c(1+t).
\end{aligned} \tag{2.6}$$

Here c also contains $\sup F_\alpha(f')F_\alpha(f'_*)$ which is of magnitude bounded by 2^{2L}. So c is of magnitude 2^{2L}(mass+energy) and uniformly in α. Multiply equation (1.1) for f by v_1^2, integrate and use that $\int v_1^2 Q_\alpha(f)dv = \int (v_1-u_1)^2 Q_\alpha(f)dv$ and (2.6). It results

$$\begin{aligned}
&\int_0^t \int (v_1-u_1)^2 Bf'f'_*F_\alpha(f)F_\alpha(f_*)dvdv_*d\theta dxds\\
&= \int v_1^2 f(t,x,v)dxdv - \int v_1^2 f_0(x,v)dxdv + \int_0^t \int (v_1-u_1)^2 Bff_*F_\alpha(f')F_\alpha(f'_*)dxdvdv_*d\theta ds\\
&< c_0(1+t),
\end{aligned}$$

where c_0 is a constant of magnitude 2^{2L}(mass+energy).
After a change of variables the left hand side can be written

$$\begin{aligned}
&\int_0^t \int (v'_1-u_1)^2 Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&= \int_0^t \int (c_1 - n_1[(v-v_*)\cdot n])^2 Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds,
\end{aligned}$$

where $c_1 = v_1 - u_1$. And so,

$$\begin{aligned}
&\int_0^t \int n_1^2[(v-v_*)\cdot n])^2 Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&\leq c_0(1+t) + 2\int_0^t \int c_1 n_1[(v-v_*)\cdot n]Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds.
\end{aligned}$$

The term containing $n_1^2[(v-v_*)\cdot n]^2$ is estimated from below. When n is replaced by an orthogonal (direct) unit vector $n_\perp$, v' and v'_* are shifted and the product $ff_*F_\alpha(f')F_\alpha(f'_*)$ is unchanged. In $\mathbb{R}^2$ the ratio between the sum of the integrand factors $n_1^2[(v-v_*)\cdot n]^2 + n_{\perp 1}^2[(v-v_*)\cdot n_\perp]^2$ and $|v-v_*|^2$, is, outside of the angular cut-off (1.3), uniformly bounded from below by γ'^2. Indeed, if θ (resp. θ_1) denotes the angle between $\frac{v-v_*}{|v-v_*|}$ and n (resp. the angle between e_1 and n, where e_1 is a unit vector in the x-direction),

$$\begin{aligned}
n_1^2[\frac{v-v_*}{|v-v_*|}\cdot n]^2 + n_{\perp 1}^2[\frac{v-v_*}{|v-v_*|}\cdot n_\perp]^2 &= \cos^2\theta_1 \cos^2\theta + \sin^2\theta_1 \sin^2\theta\\
&\geq \gamma'^2\cos^2\theta_1 + \gamma'(2-\gamma')\sin^2\theta_1\\
&\geq \gamma'^2, \quad \gamma' < |\cos\theta| < 1-\gamma', \quad \theta_1 \in [0,2\pi].
\end{aligned}$$

This is where the condition $v \in \mathbb{R}^2$ is used.

That leads to the lower bound

$$\begin{aligned}
&\int_0^t \int n_1^2[(v-v_*)\cdot n]^2 Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&\geq \frac{\gamma'^2}{2}\int_0^t \int |v-v_*|^2 Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds.
\end{aligned}$$

And so,

$$\begin{aligned}
&\gamma'^2 \int_0^t \int |v-v_*|^2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds \\
&\leq 2c_0(1+t) + 4\int_0^t \int (v_1-u_1) n_1 [(v-v_*)\cdot n] B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds \\
&\leq 2c_0(1+t) + 4\int_0^t \int \Big(v_1 (v_2 - v_{*2}) n_1 n_2 \Big) B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds,
\end{aligned}$$

since

$$\begin{aligned}
&\int u_1 (v_1 - v_{*1}) n_1^2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx \\
&\quad = \int u_1 (v_2 - v_{*2}) n_1 n_2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx = 0,
\end{aligned}$$

by an exchange of the variables v and v_*. Moreover, exchanging first the variables v and v_*,

$$\begin{aligned}
&2\int_0^t \int v_1 (v_2 - v_{*2}) n_1 n_2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds \\
&\quad = \int_0^t \int (v_1 - v_{*1})(v_2 - v_{*2}) n_1 n_2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds \\
&\quad \leq \frac{8}{\gamma'^2} \int_0^t \int (v_1 - v_{*1})^2 n_1^2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds \\
&\qquad + \frac{\gamma'^2}{8} \int_0^t \int (v_2 - v_{*2})^2 n_2^2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds \\
&\quad \leq \frac{8\pi c_0}{\gamma'^2}(1+t) + \frac{\gamma'^2}{8} \int_0^t \int (v_2 - v_{*2})^2 n_2^2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds.
\end{aligned}$$

It follows that

$$\int_0^t \int |v-v_*|^2 B f f_* F_\alpha(f') F_\alpha(f'_*) dv dv_* d\theta dx ds \leq c'_0 (1+t),$$

with c'_0 uniformly with respect to α, of the same magnitude as c_0, only depending on $\int f_0(x,v) dx dv$, $\int |v|^2 f_0(x,v) dx dv$ and L. This completes the proof of the lemma. ■

3 Control of phase space density.

This section is devoted to obtaining a time $T > 0$, such that

$$\sup_{t\in[0,T],\, x\in[0,1]} f_\alpha^\sharp(t,x,v) \leq 2^{L+1},$$

uniformly with respect to $\alpha \in]0, 2^{-L-1}[$. We start from the case of a fixed $\alpha \leq 2^{-L-1}$. Up to Lemma 3.3 the time interval when the solution does not exceed 2^{L+1}, may be α-dependent. Lemma 3.4 implies that this time interval can be chosen independent of α.

Lemma 3.1
Given $T>0$ such that $f_\alpha(t)\leq 2^{L+1}$ for $0\leq t\leq T$, the solution f_α of (1.1) satisfies

$$\int \sup_{t\in[0,T]} f^\sharp_\alpha(t,x,v)dxdv < c'_1 + c'_2 T, \quad \alpha\in]0,2^{-L-1}[,$$

where c'_1 and c'_2 are independent of T and α, and only depend on $\int f_0(x,v)dxdv$, $\int |v|^2 f_0(x,v)dxdv$ and L.

Proof of Lemma 3.1.
Denote f_α by f for simplicity. By (2.3),

$$\sup_{t\in[0,T]} f^\sharp(t,x,v) \leq f_0(x,v) + \int_0^T Q^+_\alpha(f)(t,x+tv_1,v)dt.$$

Integrating the previous inequality with respect to (x,v) and using Lemma 2.3, gives

$$\begin{aligned}\int \sup_{0\leq t\leq T} f^\sharp(t,x,v)dxdv &\leq \int f_0(x,v)dxdv + \int_0^T \int B \\ &f(t,x+tv_1,v')f(t,x+tv_1,v'_*)F_\alpha(f)(t,x+tv_1,v)F_\alpha(f)(t,x+tv_1,v_*)dvdv_*d\theta dxdt \\ &\leq \int f_0(x,v)dxdv + \frac{1}{\gamma^2}\int_0^T \int B|v-v_*|^2 \\ &f(t,x,v')f(t,x,v'_*)F_\alpha(f)(t,x,v)F_\alpha(f)(t,x,v_*)dvdv_*d\theta dxdt \\ &\leq \int f_0(x,v)dxdv + \frac{c'_0(1+T)}{\gamma^2} := \frac{C_1+C_2T}{\gamma^2}. \qquad \blacksquare\end{aligned}$$

Lemma 3.2
Given $T>0$ such that $f(t)\leq 2^{L+1}$ for $0\leq t\leq T$, and $\delta_1>0$, there exist $\delta_2>0$ and $t_0>0$ independent of T and α and only depending on $\int f_0(x,v)dxdv$, $\int |v|^2 f_0(x,v)dxdv$ and L, such that

$$\sup_{x_0\in[0,1]} \int_{|x-x_0|<\delta_2} \sup_{t\leq s\leq t+t_0} f^\sharp_\alpha(s,x,v)dxdv < \delta_1, \quad \alpha\in]0,2^{-L-1}[, \quad t\in[0,T].$$

Proof of Lemma 3.2.
Denote f_α by f for simplicity. For $s\in[t,t+t_0]$ it holds,

$$\begin{aligned}f^\sharp(s,x,v) &= f^\sharp(t+t_0,x,v) - \int_s^{t+t_0} Q_\alpha(f)(\tau,x+\tau v_1,v)d\tau \\ &\leq f^\sharp(t+t_0,x,v) + \int_s^{t+t_0} Q^-_\alpha(f)(\tau,x+\tau v_1,v)d\tau.\end{aligned}$$

And so

$$\sup_{t\leq s\leq t+t_0} f^\sharp(s,x,v) \leq f^\sharp(t+t_0,x,v) + \int_t^{t+t_0} Q^-_\alpha(f)(s,x+sv_1,v)ds.$$

Integrating with respect to (x,v), using Lemma 2.3 and the bound 2^{L+1} from above for f, gives

$$\begin{aligned}
&\int_{|x-x_0|<\delta_2} \sup_{t\leq s\leq t+t_0} f^\sharp(s,x,v)dxdv \\
&\leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv \\
&+\int_t^{t+t_0}\int Bf^\sharp(s,x,v)f(s,x+sv_1,v_*)F_\alpha(f)(s,x+sv_1,v')F_\alpha(f)(s,x+sv_1,v'_*)dvdv_*d\theta dxds \\
&\leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv+\frac{1}{\lambda^2}\int_t^{t+t_0}\int_{|v-v_*|\geq\lambda} B|v-v_*|^2 f^\sharp(s,x,v)f(s,x+sv_1,v_*) \\
&F_\alpha(f)(s,x+sv_1,v')F_\alpha(f)(s,x+sv_1,v'_*)dvdv_*d\theta dxds \\
&+c2^{2L}\int_t^{t+t_0}\int_{|v-v_*|<\lambda} Bf^\sharp(s,x,v)f(s,x+sv_1,v_*)dvdv_*d\theta dxds \\
&\leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv+\frac{c'_0(1+t_0)}{\lambda^2}+c2^{3L}t_0\lambda^2\int f_0(x,v)dxdv \\
&\leq \frac{1}{\Lambda^2}\int v^2 f_0dxdv+c\delta_2 2^L\Lambda^2+\frac{c'_0(1+t_0)}{\lambda^2}+c2^{3L}t_0\lambda^2\int f_0(x,v)dxdv.
\end{aligned}$$

Depending on δ_1, suitably choosing Λ and then δ_2, λ and then t_0, the lemma follows. ■

The previous lemmas imply for fixed $\alpha\leq 2^{-L-1}$ a bound for the v-integral of $f^\#_\alpha$ only depending on $\int f_0(x,v)dxdv$, $\int|v|^2 f_0(x,v)dxdv$ and L.

Lemma 3.3
With T'_α defined as the maximum time for which $f_\alpha(t)\leq 2^{L+1}$, $t\in[0,T'_\alpha]$, take $T_\alpha=\min\{1,T'_\alpha\}$. The solution f_α of (1.1) satisfies

$$\int \sup_{(t,x)\in[0,T_\alpha[\times[0,1]} f^\sharp_\alpha(t,x,v)dv\leq c_1, \tag{3.1}$$

where c_1 is independent of $\alpha\leq 2^{-L-1}$ and only depends on $\int f_0(x,v)dxdv$, $\int|v|^2 f_0(x,v)dxdv$ and L.

Proof of Lemma 3.3.
Denote by $E(x)$ the integer part of $x\in\mathbb{R}$, $E(x)\leq x<E(x)+1$.
By (2.3),

$$\begin{aligned}
&\sup_{s\leq t} f^\sharp(s,x,v)\leq f_0(x,v)+\int_0^t Q^+_\alpha(f)(s,x+sv_1,v)ds \\
&=f_0(x,v)+\int_0^t\int Bf(s,x+sv_1,v')f(s,x+sv_1,v'_*)F_\alpha(f)(s,x+sv_1,v)F_\alpha(f)(s,x+sv_1,v_*)dv_*d\theta ds \\
&\leq f_0(x,v)+c2^{2L}A, \qquad (3.2)
\end{aligned}$$

where

$$A=\int_0^t\int B\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v'_1),v')\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v'_{*1}),v'_*)dv_*d\theta ds.$$

For θ outside of the angular cutoff (2.2), let n be the unit vector in the direction $v-v'$, and $n_\perp$ the orthogonal unit vector in the direction $v-v'_*$. With e_1 a unit vector in the x-direction,

$$\max(|n\cdot e_1|,|n_\perp\cdot e_1|)\geq \frac{1}{\sqrt{2}}.$$

For $\delta_2>0$ that will be fixed later, split A into $A_1+A_2+A_3+A_4$, where

$$A_1=\int_0^t\int_{|n\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v'_1|>\delta_2} B\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v'_1),v')\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-{v'}_{*1}),v'_*)dv_*d\theta ds,$$

$$A_2=\int_0^t\int_{|n\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v'_1|<\delta_2} B\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v'_1),v')\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-{v'}_{*1}),v'_*)dv_*d\theta ds,$$

$$A_3=\int_0^t\int_{|n_\perp\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v'_1|>\delta_2} B\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v'_1),v')\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-{v'}_{*1}),v'_*)dv_*d\theta ds,$$

$$A_4=\int_0^t\int_{|n_\perp\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v'_1|<\delta_2} B\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v'_1),v')\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-{v'}_{*1}),v'_*)dv_*d\theta ds.$$

In A_1 and A_2, bound the factor $\sup_{\tau\in[0,t]} f^\sharp(\tau,x+s(v_1-v'_{*1}),v'_*)$ by its supremum over $x\in[0,1]$, and make the change of variables

$$s\to y=x+s(v_1-v'_1).$$

with Jacobian

$$\frac{Ds}{Dy}=\frac{1}{|v_1-v'_1|}=\frac{1}{|v-v_*|\,|(n,\frac{v-v_*}{|v-v_*|})|\,|n_1|}\leq\frac{\sqrt{2}}{\gamma\gamma'}.$$

It holds that

$$A_1\leq\int_{t|v_1-v'_1|>\delta_2}\frac{B}{|v_1-v'_1|}\Big(\int_{y\in(x,x+t(v_1-v'_1))}\sup_{\tau\in[0,t]} f^\#(\tau,y,v')dy\Big)\sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v'_*)dv_*d\theta,$$

and

$$A_2\leq\frac{\sqrt{2}}{\gamma\gamma'}\int_{|n\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v'_1|<\delta_2} B\Big(\int_{|y-x|<\delta_2}\sup_{\tau\in[0,t]} f^\#(\tau,y,v')dy\Big)\sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v'_*)dv_*d\theta.$$

Then, performing the change of variables $(v,v_*,n)\to(v',v'_*,-n)$,

$$\begin{aligned}&\int\sup_{x\in[0,1]}A_1dv\\ &\leq\int_{t|v_1-v'_1|>\delta_2}\frac{B}{|v_1-v'_1|}\sup_{x\in[0,1]}\Big(\int_{y\in(x,x+t(v'_1-v_1))}\sup_{\tau\in[0,t]} f^\#(\tau,y,v)dy\Big)\sup_{(\tau,X)\in[0,t]\times[0,1]} f^\#(\tau,X,v_*)dvdv_*d\theta,\end{aligned}$$

so that

$$
\begin{aligned}
&\int \sup_{x\in[0,1]} A_1 dv\\
&\leq \int_{t|v_1-v_1'|>\delta_2} \frac{B}{|v_1-v_1'|} \sup_{x\in[0,1]} \Big(\int_{y\in(x,x+E(t(v_1'-v_1)+1))} \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy\Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*d\theta\\
&= \int_{t|v_1-v_1'|>\delta_2} \frac{B}{|v_1-v_1'|} |E(t(v_1'-v_1)+1)| \Big(\int_0^1 \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy\Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*d\theta\\
&\leq t(1+\frac{1}{\delta_2}) \int B\Big(\int_0^1 \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy\Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*d\theta\\
&\leq B_0\pi t(1+\frac{1}{\delta_2}) \int \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dydv \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dv_*.
\end{aligned}
$$

Apply Lemma 3.1, so that

$$
\int \sup_{x\in[0,1]} A_1 dv \leq B_0\pi t(1+\frac{1}{\delta_2})(c_1'+c_2') \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dv_*. \tag{3.3}
$$

Moreover, performing the change of variables $(v,v_*,n)\to(v_*',v',-n)$,

$$
\int \sup_{x\in[0,1]} A_2 dv \leq \frac{B_0\pi\sqrt{2}}{\gamma\gamma'} \sup_{x\in[0,1]} \Big(\int_{|y-x|<\delta_2} \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v_*)dydv_*\Big) \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v)dv.
$$

Given $\delta_1=\frac{\gamma\gamma'}{4B_0\pi\sqrt{2}}$, apply Lemma 3.2 with the corresponding δ_2 and t_0, so that for $t\leq\min\{T,t_0\}$,

$$
\int \sup_{x\in[0,1]} A_2 dv \leq \frac{1}{4} \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v)dv. \tag{3.4}
$$

The terms A_3 and A_4 are treated similarly, with the change of variables $s\to y=x+s(v_1-v_{*1}')$. Using (3.3)-(3.4) and the corresponding bounds obtained for A_3 and A_4 leads to

$$
\begin{aligned}
\int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv &\leq 2\int \sup_{x\in[0,1]} f_0(x,v)dv\\
&\quad + 4B_0\pi t(1+\frac{1}{\delta_2})(c_1'+c_2') \int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv, \quad t\leq\min\{T,t_0\}.
\end{aligned}
$$

Hence

$$
\int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv \leq 4\int \sup_{x\in[0,1]} f_0(x,v)dv, \quad t\leq\min\{t_0,\frac{\delta_2}{8B_0\pi(\delta_2+1)(c_1'+c_2')}\}.
$$

Since t_0, c_1' and c_2' are independent of $\alpha\leq 2^{-L-1}$ and only depend on $\int f_0(x,v)dxdv$, $\int |v|^2 f_0(x,v)dxdv$ and L, it follows that the argument can be repeated up to $t=T_\alpha$ with the number of steps uniformly bounded with respect to $\alpha\leq 2^{-L-1}$. This completes the proof of the lemma. ∎

We now prove that the positive time T_α used above, such that $f_\alpha(t)\leq 2^{L+1}$ for $t\in[0,T_\alpha]$, can be taken independent of α.

Lemma 3.4
Given $f_0 \leq 2^L$ and satisfying (1.10), there is $T \in]0,1]$ so that for all $\alpha \in]0, 2^{-L-1}[$, the solution f_α to (1.1) is bounded by 2^{L+1} on $[0,T]$.

Proof of Lemma 3.4.
Given $\alpha \leq 2^{-L-1}$, it follows from Lemma 2.2 that the maximum time T'_α for which $f_\alpha \leq 2^{L+1}$ on $[0, T'_\alpha]$ is positive. By (2.3),

$$\sup_{s \leq t} f^\sharp_\alpha(s,x,v) \leq f_0(x,v) + \int_0^t Q^+_\alpha(f_\alpha)(s, x+sv_1, v)ds = f_0(x,v)$$
$$+ \int_0^t \int B f_\alpha(s, x+sv_1, v') f_\alpha(s, x+sv_1, v'_*) F_\alpha(f_\alpha)(s, x+sv_1, v) F_\alpha(f_\alpha)(s, x+sv_1, v_*) dv_* d\theta ds.$$

With the angular cut-off (2.2), $v_* \to v'$ and $v_* \to v'_*$ are changes of variables, and so using Lemma 3.3, the functions f_α for $\alpha \in]0, 2^{-L-1}[$satisfy

$$\begin{aligned}
\sup_{(s,x) \in [0,t] \times [0,1]} f^\sharp_\alpha(s,x,v) &\leq f_0(x,v) + cB_0 2^{3L} t \int \sup_{(s,x) \in [0,t] \times [0,1]} f_\alpha(s,x,v') dv' \\
&\leq 2^L + cB_0 2^{3L} t c_1 \\
&\leq 3(2^{L-1}) \qquad\qquad , t \in [0, \min\{T'_\alpha, \frac{1}{cc_1 B_0 2^{2L+1}}\}] \, .
\end{aligned}$$

For all $\alpha \leq 2^{-L-1}$, it holds that $T'_\alpha \geq \frac{1}{cc_1 B_0 2^{2L+1}}$, else T'_α would not be the maximum time such that $f_\alpha(t) \leq 2^{L+1}$ on $[0, T'_\alpha]$. Denote by $T = \min\{1, \frac{1}{cc_1 B_0 2^{2L+1}}\}$. The lemma follows since T does not depend on α. ∎

4 Proof of Theorem 1.1.

After the above preparations we can now prove Theorem 1.1. The conservations of mass, momentum and energy will be proven in Section 5.

Proof of Theorem 1.1.
Let us first prove that (f_α) is a Cauchy sequence in $C([0,T]; L^1([0,1] \times \mathbb{R}^2))$ with T of Lemma 3.4.

For any $(\alpha_1,\alpha_2)\in]0,1[^2$, the function $g=f_{\alpha_1}-f_{\alpha_2}$ satisfies the equation

$$\begin{aligned}
\partial_t g+v_1\partial_x g=&\int B(f'_{\alpha_1}f'_{\alpha_1*}-f'_{\alpha_2}f'_{\alpha_2*})F_{\alpha_1}(f_{\alpha_1})F_{\alpha_1}(f_{\alpha_1*})dv_*d\theta\\
&-\int B(f_{\alpha_1}f_{\alpha_1*}-f_{\alpha_2}f_{\alpha_2*})F_{\alpha_1}(f'_{\alpha_1})F_{\alpha_1}(f'_{\alpha_1*})dv_*d\theta\\
&+\int Bf'_{\alpha_2}f'_{\alpha_2*}\Big(F_{\alpha_1}(f_{\alpha_1*})\big(F_{\alpha_1}(f_{\alpha_1})-F_{\alpha_1}(f_{\alpha_2})\big)+F_{\alpha_2}(f_{\alpha_2})\big(F_{\alpha_1}(f_{\alpha_1*})-F_{\alpha_1}(f_{\alpha_2*})\big)\Big)dv_*d\theta\\
&+\int Bf'_{\alpha_2}f'_{\alpha_2*}\Big(F_{\alpha_1}(f_{\alpha_1*})\big(F_{\alpha_1}(f_{\alpha_2})-F_{\alpha_2}(f_{\alpha_2})\big)+F_{\alpha_2}(f_{\alpha_2})\big(F_{\alpha_1}(f_{\alpha_2*})-F_{\alpha_2}(f_{\alpha_2*})\big)\Big)dv_*d\theta\\
&-\int Bf_{\alpha_2}f_{\alpha_2*}\Big(F_{\alpha_1}(f'_{\alpha_1*})\big(F_{\alpha_1}(f'_{\alpha_1})-F_{\alpha_1}(f'_{\alpha_2})\big)+F_{\alpha_2}(f'_{\alpha_2})\big(F_{\alpha_1}(f'_{\alpha_1*})-F_{\alpha_1}(f'_{\alpha_2*})\big)\Big)dv_*d\theta\\
&-\int Bf_{\alpha_2}f_{\alpha_2*}\Big(F_{\alpha_1}(f'_{\alpha_1*})\big(F_{\alpha_1}(f'_{\alpha_2})-F_{\alpha_2}(f'_{\alpha_2})\big)+F_{\alpha_2}(f'_{\alpha_2})\big(F_{\alpha_1}(f'_{\alpha_2*})-F_{\alpha_2}(f'_{\alpha_2*})\big)\Big)dv_*d\theta. \quad (4.8)
\end{aligned}$$

Using Lemma 3.3 and taking $\alpha_1,\alpha_2<2^{-L-1}$,

$$\begin{aligned}
&\int B\Big(|f_{\alpha_1}f_{\alpha_1*}-f_{\alpha_2}f_{\alpha_2*}|F_{\alpha_1}(f'_{\alpha_1})F_{\alpha_1}(f'_{\alpha_1*})\Big)^\sharp dxdvdv_*d\theta\\
&\leq c2^{2L}\Big(\int\sup_{x\in[0,1]}f^\sharp_{\alpha_1}(t,x,v)dv+\int\sup_{x\in[0,1]}f^\sharp_{\alpha_2}(t,x,v)dv\Big)\int|(f_{\alpha_1}-f_{\alpha_2})^\sharp(t,x,v)|dxdv\\
&\leq cc_1 2^{2L}\int|g^\sharp(t,x,v)|dxdv.
\end{aligned}$$

We similarly obtain

$$\int B\Big(f'_{\alpha_2}f'_{\alpha_2*}F_{\alpha_1}(f_{\alpha_1*})|(F_{\alpha_1}(f_{\alpha_2})-F_{\alpha_2}(f_{\alpha_2})|)\Big)^\sharp dxdvdv_*d\theta\leq cc_1 2^{2L}|\alpha_1-\alpha_2|,$$

and

$$\int B\Big(f_{\alpha_2}f_{\alpha_2*}F_{\alpha_1}(f'_{\alpha_1*})|F_{\alpha_1}(f'_{\alpha_1})-F_{\alpha_1}(f'_{\alpha_2})|\Big)^\sharp dxdvdv_*d\theta\leq cc_1 2^{L}\int|g^\sharp(t,x,v)|dxdv.$$

The remaining terms are estimated in the same way. It follows

$$\frac{d}{dt}\int|g^\sharp(t,x,v)|dxdv\leq cc_1 2^{2L}\Big(\int|g^\sharp(t,x,v)|dxdv+|\alpha_1-\alpha_2|\Big).$$

Hence

$$\lim_{(\alpha_1,\alpha_2)\to(0,0)}\sup_{t\in[0,T]}\int|g^\sharp(t,x,v)|dxdv=0.$$

And so (f_α) is a Cauchy sequence in $C([0,T];L^1([0,1]\times\mathbb{R}^2))$. Denote by f its limit. With analogous arguments to the previous ones in the proof of this lemma, it holds that

$$\lim_{\alpha\to 0}\int|Q(f)-Q(f_\alpha)|(t,x,v)dtdxdv=0.$$

Hence f is a strong solution to (1.5) on $[0,T]$ with initial value f_0. If there were two solutions, their difference denoted by G would with similar arguments satisfy

$$\frac{d}{dt}\int|G^\sharp(t,x,v)|dxdv\leq cc_1 2^{2L}\int|G^\sharp(t.x.v)|dxdv,$$

hence be identically equal to its initial value zero.
Denote by $\mathcal{F}$ a given equibounded family of initial values bounded by 2^L. Let f_1 resp. f_2 be the solution to (1.5) with initial value $f_{10} \in \mathcal{F}$ resp. $f_{20} \in \mathcal{F}$. The equation for $\bar{g} = f_1 - f_2$ can be written analogously to (4.8). Similar arguments lead to

$$\frac{d}{dt}\int |(f_1 - f_2)^\sharp(t,x,v)|dxdv \leq cc_1 2^{2L}\int |(f_1 - f_2)^\sharp(t,x,v)|dxdv,$$

so that

$$\| (f_1 - f_2)(t,\cdot,\cdot) \|_{L^1([0,1]\times\mathbb{R}^2)} \leq e^{cc_1 T 2^{2L}} \| f_{10} - f_{20} \|_{L^1([0,1]\times\mathbb{R}^2)}, \quad t \in [0,T].$$

This proves the stability statement of Theorem 1.1 .
If $\sup_{(x,v)\in[0,1]\times\mathbb{R}^2} f(T,x,v) < 2^{L+1}$, then the procedure can be repeated, i.e. the same proof can be carried out from the initial value $f(T)$. It leads to a maximal interval denoted by $[0,\tilde{T}_1]$ on which $f(t,\cdot,\cdot) \leq 2^{L+1}$. By induction there exists an increasing sequence of times $(\tilde{T}_n)$ such that $f(t,\cdot,\cdot) \leq 2^{L+n}$ on $[0,\tilde{T}_n]$. Let $T_\infty = \lim_{n\to+\infty}\tilde{T}_n$. Either $\tilde{T}_\infty = +\infty$ and the solution f is global in time, or T_∞ is finite and $\overline{\lim}_{t\to T_0} \| f(t) \|_\infty = \infty$. ∎

5 Conservations of mass, momentum and energy.

The following two preliminary lemmas are needed for the control of large velocities.

Lemma 5.1
The solution f of (1.5) with initial value f_0, satisfies

$$\int_0^1 \int_{|v|>\lambda} |v| \sup_{t\in[0,T]} f^\sharp(t,x,v)dvdx \leq \frac{c_T}{\lambda}, \quad t \in [0,T],$$

where c_T only depends on T, $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.

Proof of Lemma 5.1.
As in (2.3),

$$\sup_{t\in[0,T]} f^\sharp(t,x,v) \leq f_0(x,v) + \int_0^T Q^+(f)(s,x+sv_1,v)ds.$$

Integration with respect to (x,v) for $|v| > \lambda$, gives

$$\int_0^1 \int_{|v|>\lambda} |v| \sup_{t\in[0,T]} f^\sharp(t,x,v)dvdx \leq \int\int_{|v|>\lambda} |v| f_0(x,v)dvdx + \int_0^T \int_{|v|>\lambda} B$$
$$|v| f(s,x+sv_1,v') f(s,x+sv_1,v'_*) F(f)(s,x+sv_1,v) F(f)(s,x+sv_1,v_*)dvdv_*d\theta dxds.$$

Here in the last integral, either $|v'|$ or $|v'_*|$ is the largest and larger than $\frac{\lambda}{\sqrt{2}}$. The two cases are symmetric, and we discuss the case $|v'| \geq |v'_*|$. After a translation in x, the integrand is estimated from above by

$$c|v'| f^\#(s,x,v') \sup_{(t,x)\in[0,T]\times[0,1]} f^\#(t,x,v'_*).$$

The change of variables $(v, v_*, n) \to (v', v'_*, -n)$, the integration over

$$(s, x, v, v_*, \omega) \in [0, T] \times [0, 1] \times \{v \in \mathbb{R}^2; |v| > \frac{\lambda}{\sqrt{2}}\} \times \mathbb{R}^2 \times [-\frac{\pi}{2}, \frac{\pi}{2}],$$

and Lemma 3.3 give the bound

$$\frac{c}{\lambda}\Big(\int_0^T \int |v|^2 f^{\#}(s,x,v)dxdvds\Big)\Big(\int \sup_{(t,x)\in[0,T]\times[0,1]} f^{\#}(t,x,v_*)dv_*\Big) \leq \frac{cTc_1(T)}{\lambda}\int |v|^2 f_0(x,v)dxdv.$$

The lemma follows. ■

Lemma 5.2
The solution f of (1.5) with initial val;ue f_0 satisfies

$$\int_{|v|>\lambda} \sup_{(t,x)\in[0,T]\times[0,1]} f^{\sharp}(t,x,v)dv \leq \frac{c'_T}{\sqrt{\lambda}}, \quad t \in [0,T],$$

where c'_T only depends on T, $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.

Proof of Lemma 5.2.
Take $\lambda > 2$. As above,

$$\int_{|v|>\lambda} \sup_{(t,x)\in[0,T]\times[0,1]} f^{\sharp}(t,x,v)dv \leq \int_{|v|>\lambda} \sup_{x\in[0,1]} f_0(x,v)dv + cC, \tag{5.1}$$

where

$$C = \int_{|v|>\lambda} \sup_{x\in[0,1]} \int_0^T \int Bf^{\#}(s, x + s(v_1 - v'_1), v')f^{\#}(s, x + s(v_1 - v'_{*1}), v'_*)dvdv_*d\theta ds.$$

For v', v'_* outside of the angular cutoff (1.3), let n be the unit vector in the direction $v - v'$, and $n_\perp$ the orthogonal unit vector in the direction $v - v'_*$. Let e_1 be a unit vector in the x-direction. Split C as $C = \sum_{1\leq i\leq 6} C_i$, where C_1 (resp. C_2, C_3) refers to integration with respect to (v_*, θ) on

$$\{(v_*, \theta); \quad n \cdot e_1 \geq \frac{1}{\sqrt{2}}, \quad |v'| \geq |v'_*|\},$$

$$\Big(\text{resp. } \{(v_*, \theta); n \cdot e_1 \geq \sqrt{1 - \frac{1}{\lambda}},\ |v'| \leq |v'_*|\}, \quad \{(v_*, \theta); n \cdot e_1 \in [\frac{1}{\sqrt{2}}, \sqrt{1 - \frac{1}{\lambda}}],\ |v'| \leq |v'_*|\}\Big),$$

and analogously for C_i, $4 \leq i \leq 6$, with n replaced by $n_\perp$. By symmetry, C_i, $4 \leq i \leq 6$ can be treated as C_i, $1 \leq i \leq 3$, so we only discuss the control of C_i, $1 \leq i \leq 3$.
By the change of variables $(v, v_*, n) \to (v', v'_*, -n)$, and noticing that $|v'| \geq \frac{\lambda}{\sqrt{2}}$ in the domain of integration of C_1, it holds that

$$\begin{aligned} C_1 &\leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{x\in[0,1]} \int_0^T \int_{n\cdot e_1 \geq \frac{1}{\sqrt{2}}} Bf^{\#}(s, x + s(v'_1 - v_1), v)f^{\#}(s, x + s(v'_1 - v_{*1}), v_*)dv_*d\theta dsdv \\ &\leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{x\in[0,1]} \int_0^T \int_{n\cdot e_1 \geq \frac{1}{\sqrt{2}}} B \sup_{\tau\in[0,T]} f^{\#}(\tau, x + s(v'_1 - v_1), v) \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau, X, v_*)dv_*d\theta dsdv. \end{aligned}$$

With the change of variables $s \to y = x + s(v_1' - v_1)$,

$$C_1 \leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{x\in[0,1]} \int_{n\cdot e_1 \geq \frac{1}{\sqrt{2}}} \int_{y\in(x,x+T(v_1'-v_1))} \frac{B}{|v_1'-v_1|} \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v) \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v_*) dy dv_* d\theta dv$$
$$\leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \int_{n\cdot e_1 \geq \frac{1}{\sqrt{2}}} \frac{|E(T(v_1'-v_1))+1)|}{|v_1'-v_1|} \int_0^1 B \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v) \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v_*) dy dv_* d\theta dv.$$

Moreover,

$$|E(T(v_1'-v_1))+1)| \leq T|v_1'-v_1| + 1 \leq \big(T + \frac{\sqrt{2}}{\gamma\gamma'}\big)|v_1'-v_1|,$$

where γ and γ' were defined in (2.2). Consequently,

$$C_1 \leq c(T+1) \int_0^1 \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v) dy dv \int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v_*) dv_*$$
$$\leq \frac{c(T+1)}{\lambda} \int_0^1 \int_{|v|>\frac{\lambda}{\sqrt{2}}} |v| \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v) dy dv \int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v_*) dv_*.$$

By Lemmas 3.3 and 5.1,

$$C_1 \leq \frac{c}{\lambda^2}(T+1) c_T c_1(T).$$

Moreover,

$$C_2 \leq \int_{|v'|>\lambda, |v_*|>|v|, n\cdot e_1 \geq \sqrt{1-\frac{1}{\lambda}}} \frac{B}{|v_1'-v_1|}$$
$$\sup_{x\in[0,1]} \int_{y\in(x,x+T(v_1'-v_1))} \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v) \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v_*) dy dv dv_* d\theta$$
$$\leq c(T+1) \int_{n\cdot e_1 \geq \sqrt{1-\frac{1}{\lambda}}} d\theta \int \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v) dy dv \int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v_*) dv_*$$
$$\leq \frac{c}{\sqrt{\lambda}}(T+1)^2 c_1(T),$$

by Lemmas 3.1 and 3.3. Finally,

$$C_3 \leq \int_{|v_*|>\frac{\lambda}{\sqrt{2}}, \frac{1}{\sqrt{\lambda}} \leq n_\perp\cdot e_1 \leq \frac{1}{\sqrt{2}}} \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v) \frac{B}{|v_1'-v_{*1}|}$$
$$\sup_{x\in[0,1]} \Big(\int_{y\in(x,x+T(v_1'-v_{*1}))} \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v_*) dy \Big) dv dv_* d\theta$$
$$\leq c(T+1)\sqrt{\lambda} \Big(\int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v) dv \Big) \Big(\int_{|v_*|>\frac{\lambda}{\sqrt{2}}} \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v_*) dy dv_* \Big).$$

By Lemmas 3.3 and 5.1,

$$C_3 \leq \frac{c}{\sqrt{\lambda}}(T+1) c_1(T) c_T.$$

The lemma follows. ■

Lemma 5.3 *The solution f to (1.5) with initial value f_0 conserves mass, momentum and energy.*

Proof of Lemma 5.3.
The conservation of mass and first momentum of f will follow from the boundedness of the total energy. The energy is non-increasing since the approximations f_α conserve energy and

$$\lim_{\alpha\to 0}\int_0^1\int_{|v|<V}|(f-f_\alpha)(t,x,v)||v|^2dxdv=0,\quad \text{for all } t\in[0,T] \text{ and positive } V.$$

Energy conservation will be satisfied if the energy is non-decreasing. Taking $\psi_\epsilon=\frac{|v^2|}{1+\epsilon|v|^2}$ as approximation for $|v|^2$, it is enough to bound

$$\int Q(f)(t,x,v)\psi_\epsilon(v)dxdv=\int B\psi_\epsilon\Big(f'f'_*F(f)F(f_*)-ff_*F(f')F(f'_*)\Big)dxdvdv_*d\theta$$

from below by zero in the limit $\epsilon\to 0$. Similarly to [8],

$$\begin{aligned}\int Q(f)\psi_\epsilon dxdv&=\frac{1}{2}\int Bff_*F(f')F(f'_*\Big(\psi_\epsilon(v')+\psi_\epsilon(v'_*)-\psi_\epsilon(v)-\psi_\epsilon(v_*)\Big)dxdvdv_*d\theta\\&\geq-\int Bff_*F(f')F(f'_*)\frac{\epsilon|v|^2|v_*|^2}{(1+\epsilon|v|^2)(1+\epsilon|v_*|^2)}dxdvdv_*d\theta.\end{aligned}$$

The previous line, with the integral taken over a bounded set in (v,v_*), converges to zero when $\epsilon\to 0$. In integrating over $|v|^2+|v_*|^2\geq 2\lambda^2$, there is symmetry between the subset of the domain with $|v|^2>\lambda^2$ and the one with $|v_*|^2>\lambda^2$. We discuss the first sub-domain, for which the integral in the last line is bounded from below by

$$\begin{aligned}&-c\int|v_*|^2f(t,x,v_*)dxdv_*\int_{|v|\geq\lambda}B\sup_{(s,x)\in[0,t]\times[0,1]}f^\#(s,x,v)dvd\theta\\&\geq -c\int_{|v|\geq\lambda}\sup_{0\leq(s,x)\in[0,t]\times[0,1]}f^\#(s,x,v)dv.\end{aligned}$$

It follows from Lemma 5.2 that the right hand side tends to zero when $\lambda\to\infty$. This implies that the energy is non-decreasing, and bounded from below by its initial value. That completes the proof of the lemma. ∎

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